Calculation of Diffracted Multiple Reflections

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Prerequisite reading material is "Two Techniques for Wave Equation Migration" also contained in this report.

For downward continuing a downgoing wave we have the equation

$$D_{zt'} = \frac{c}{2} D_{xx} \tag{1}$$

which is equations (8) or (15) in the Claerbout & Johnson paper. In equation (1) the independent variable x is a receiver location. Equation (1) can be converted to a form for use on zero offset sections if we define the shot-receiver midpoint y as $y = x/2$. This gives

$$D_{zt'} = \frac{c}{8} D_{yy} \tag{2}$$

Let T be the tri-diagonal matrix with $- c\ \Delta z\ \Delta t\ /\ 32\ \Delta y^2$ times the second difference operator δ_{yy} on its main diagonal. Then equation (2) with the boundary condition that D vanishes before $t' = 0$ can be written in the tabular form

$$\left[\begin{array}{c|c} T+I & T-I \\ \hline T-I & T+I \end{array}\right] * \begin{array}{c|c|c|c|c|c} & & & z \rightarrow & & \\ t'=-1 & 0 & 0 & 0 & 0 & 0 \\ \hline & d_0^0 & d_0^1 & d_0^2 & d_0^3 & d_0^4 \\ \hline & d_1^0 & d_1^1 & d_1^2 & d_1^3 & d_1^4 \\ \hline t' \downarrow & d_2^0 & d_2^1 & d_2^2 & d_2^3 & d_2^4 \\ \hline & d_3^0 & d_3^1 & d_3^2 & & \end{array} = 0 \tag{3}$$

Given that the initial conditions are that the downgoing waveform is observed at the surface $z = 0$ for times $t' = 0, 1, 2,$ and 3 (but not $t' = 4$ and beyond) we can fill in table (3) in the order indicated below (g denotes something given as an initial condition)

z →, t' ↓

0	0	0	0	0
g	1	3	6	10
g	2	5	9	
g	4	8		
g	7			
?				

(4)

Down-shifting successive columns in the table (4) we are converting t' to a t'' coordinate.

z=0, z →, t'' ↓

	0	0	0		
$t''=0$	d_0^0	0	0	0	0
	d_1^0	d_0^1	0	0	0
	d_2^0	d_1^1	d_0^2	0	0
	d_3^0	d_2^1	d_1^2	d_0^3	0
t''	?	d_3^1	d_2^2	d_1^3	d_0^4

(5)

The meaning of the t'' coordinate is that all elements on a given row of constant t'' can contribute to the upcoming received wave at a surface arrival time t'' . Now equation (11) in the section "Programming Techniques for Wave Equation Migration" modified by $y = x/2$ reads

$$\partial_{tz}\, U(\, y,z,t\,) = -\frac{\bar{c}}{8}\,\partial_{yy}\, U(\, y,z,t\,) - c(\, y,z\,)\,\partial_t\, D(\, y,z,t-2z/\bar{c}\,)$$

The right hand term which is the source term $S(\, y,z,t\,)$ acting in a region Δz for up-coming waves can be expressed in table form as

$z \rightarrow$

t'' ↓					
$t''=0$	0				
	0	s_1^1			
	0	s_2^1	s_2^2		
	0	s_3^1	s_2^2	s_3^3	
	0	s_4^1	s_3^2	s_2^3	s_4^4

=

d_0^0	0	0	0	0
d_1^0	d_0^1	0	0	0
d_2^0	d_1^1	d_0^2	0	0
d_3^0	d_2^1	d_1^2	d_0^3	0
?	d_3^1	d_2^2	d_1^3	d_0^4

(6)

*

$c_0=0$	c_1	c_2	c_3	c_4	} convolve upward

In (6) we have set the surface reflection coefficient $c_0 = 0$ because we will handle the surface boundary condition $D = -U$ separately. This means that "?" drops out of the left side of (6). The negative of equation (5) is

$$\left(\frac{\bar{c}}{8}\,\partial_{yy} + \partial_{tz}\right)(\,-U\,) = c\,\partial_t\, D \qquad (7)$$

which can be written in table form as

$$
\begin{array}{|c|c|}\hline T-I & T+I \\ \hline T+I & T-I \\ \hline \end{array}
*
\begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 0 \\ \hline +d_1^0 & & 0 & 0 & 0 \\ \hline d_2^0 & & & 0 & 0 \\ \hline d_3^0 & & & & 0 \\ \hline ? & & & & \\ \hline \end{array}
= \frac{\Delta z}{2}
\begin{array}{|c|c|}\hline -1 & -1 \\ \hline +1 & +1 \\ \hline \end{array}
*
\begin{array}{|c|c|c|c|c|}\hline d_0^0 & 0 & 0 & 0 & 0 \\ \hline d_1^0 & d_0^1 & 0 & 0 & 0 \\ \hline d_2^0 & d_1^1 & d_0^2 & 0 & 0 \\ \hline d_3^0 & d_2^1 & d_1^2 & d_0^3 & 0 \\ \hline ? & d_3^1 & d_2^2 & d_1^3 & d_0^4 \\ \hline \end{array}
\quad (8)
$$

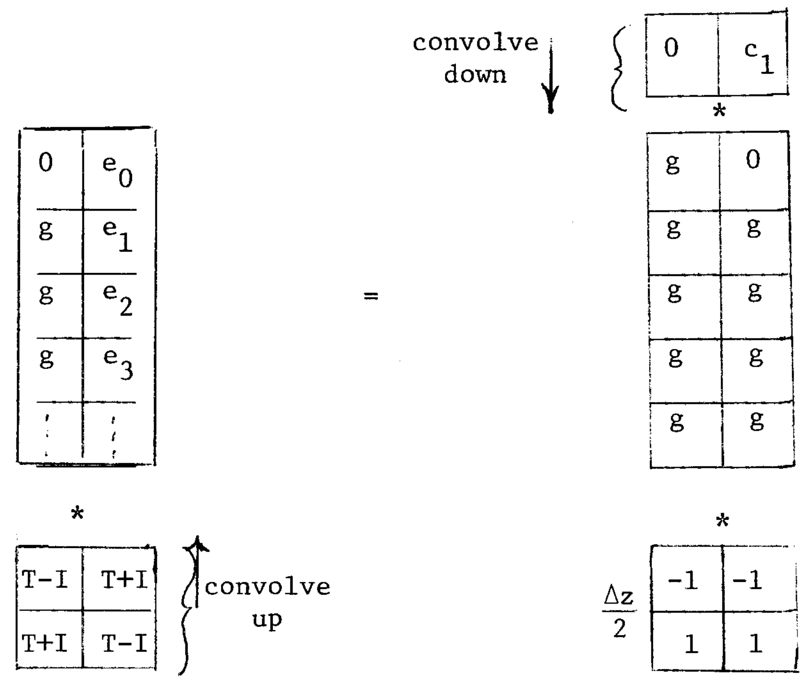
$$
*\ \begin{array}{|c|c|c|c|c|}\hline 0 & c_1 & c_2 & c_3 & c_4 \\ \hline \end{array}
$$

Now suppose the $-U$ table was completely specified for all z and all those t'' which are at and before $t'' = 3$. Now we can calculate '?' $= d_4^0$ from the bottom two rows of (8) by calculating the entire bottom row in the order indicated

$$
\begin{array}{|c|c|}\hline T-I & T+I \\ \hline T+I & T-I \\ \hline \end{array}
*
\begin{array}{|c|c|c|c|c|c|}\hline g & g & g & g & 0 & 0 \\ \hline ? & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}
= \frac{\Delta z}{2}
\begin{array}{|c|c|}\hline -1 & -1 \\ \hline +1 & +1 \\ \hline \end{array}
*
\begin{array}{|c|c|c|c|c|c|}\hline 0 & s_3^1 & s_2^2 & s_3^3 & 0 & 0 \\ \hline 0 & s_4^1 & s_3^2 & s_2^3 & s_4^4 & 0 \\ \hline \end{array}
\quad (9)
$$

All the steps up to the present have shown how knowledge of all the reflection coefficients and the surface wave $D = -U$ for times $t = 0$, $1,2,3$ can be used to calculate the surface waves at time $t = 4$, namely $d_4^0 = -u_4^0$. Obviously the process may be used recursively to get $d_t^0 = -u_t^0$ for all time.

The inverse calculation proceeds in a similar fashion. Suppose d_t^0 is known for all t and we wish to calculate c_1, c_2, etc., in a recursive fashion. It is sufficient to show how to compute c_1. Suppose we skim off the left two columns of (8). We get



(10)

In (10) all the boxes filled by g on the left are given. The boxes with g on the right are readily computable as before. If c_1 were known it would be a straightforward task to compute successively $\ldots e_3$, e_2, e_1, e_0. We would be compelled to initialize the computation with an approximation such as $e_N = 0$ for some large N. If the correct value of c_1 had been used then we should find that e_0 vanishes. Since we do not know what value of c_1 to use we try $c_1 = +1$ obtaining e_0^+ and we try $c_1 = -1$ obtaining e_0^-. The correct value of c_1 is the appropriately weighted linear combination

$$0 = \alpha e_0^+ + \beta e_0^- \tag{11}$$

where

$$1 = \alpha + \beta \tag{12}$$

$$c_1 = \alpha - \beta \tag{13}$$

which inverts to

$$2\alpha = 1 + c_1 \tag{14}$$

$$2\beta = 1 - c_1 \tag{15}$$

reducing (11) to

$$0 = (1 + c_1)\, e_0^+ + (1 - c_1)\, e_0^-$$

or

$$c_1 = \frac{e_0^+ + e_0^-}{e_0^+ - e_0^-} \tag{16}$$