2-D Inhomogeneous Media Wave Calculations

by Jon Claerbout Lecture at Princeton University 3-3-73

The purpose of these few pages is to explain as simply as possible how we extrapolate waves through 2-D inhomogeneous media. This is an extraction of bare essentials from our earlier publications. Using subscripts for partial derivatives we write the wave equation as

$$P_{xx} + P_{zz} = c(x,z)^{-2} P_{tt} \tag{1}$$

and define a coordinate change

$$\begin{aligned} x' &= x \\ z' &= z \\ t' &= t - z/\bar{c} \end{aligned} \tag{2}$$

Now we state that the old coordinates contain the same wave field as the new ones.

$$P(x,z,t) = P'(x',z',t') \tag{3}$$

Using the chain rule for partial derivatives we obtain

$$P_{xx} = P'_{x'x'}$$

$$P_z = P'_{x'} x'_z + P'_{z'} z'_z + P'_{t'} t'_t$$

$$= P'_{z'} - \bar{c}^{-1} P'_{t'}$$

$$P_{zz} = P'_{z'z'} - 2 \bar{c}^{-1} P'_{z't'} + \bar{c}^{-2} P'_{t't'}$$

$$P_{tt} = P'_{t't'} \qquad (4)$$

Now insert all these into the wave equation (1) and define a new notation by dropping all the primes.

$$P_{xx} + P_{zz} - (2/\bar{c}) P_{zt} + (\bar{c}^{-2} - c(x,z)^{-2}) P_{tt} = 0 \qquad (5)$$

Equation (5) is as exact as the wave equation for any constant numerical value of $\bar{c}$. It is convenient to pick $\bar{c}$ about equal to some sort of spatial average of $c(x,z)$ and define $s(x,z) = (\bar{c}^{-2} - c(x,z)^{-2}) c/2$. Thus we have exactly

$$P_{xx} + P_{zz} - (2/\bar{c}) (P_{zt} - s(x,z) P_{tt}) = 0 \qquad (6)$$

A very useful approximation to (6) is found by dropping the P_{zz} term. This has little effect when P_{zz} (actually $P'_{z'z'}$) turns out to be small which turns out to be the case for combinations of rays propagating roughly in the direction of the +z axis. When s=0 this approximation is properly called the Fresnel approximation. The approximation is commonly considered to be useful for waves propagating within about 15 degrees of the z-axis. We have discovered that it is also useful for small s(x,z) and have developed other more accurate (though more complicated) approximations for larger angles and larger s. The restriction of constant $\bar{c}$ can also be relaxed but to do so would complicate the presentation. Dropping the P_{zz} term and integrating once with respect to t we obtain the basic equation which must be programmed.

$$P_z = s(x,z)P_t + (\bar{c}/2) \int_{-\infty}^{t} P_{xx}\, dt \qquad (7)$$

= thin lens + diffraction

By computing the right two terms at z we have found the left term dP/dz which we can use to get P at z+dz. Continuing the process indefinately amounts to extrapolating a wave field in space. Some practical simplifications occur if the smoothly inhomogeneous medium is considered to be broken up into many layers of thickness dz. Then the process of extrapolating a wave field with (7) breaks down into two separate processes, one for crossing interfaces and the other for crossing layers, i.e. alternately

$$P_z = s(x,z)\, P_t \qquad \text{(interfaces)} \qquad (8)$$

$$P_z = (\bar{c}/2) \int_{-\infty}^{t} P_{xx}\, dt \qquad \text{(layers)} \qquad (9)$$

Equation (8) says that a time function measured on one side of the interface is merely a time shifted version of the same time function seen on the other side of the interface. Computationally this is just a question of interpolating a time function defined on a grid. You can do this in many ways. Equation (9) performs the less obvious function of carrying the waves across a region of free space. It too would be simply a shift if $P(x,z,t)$ were a plane wave, but it is far more interesting because $P(x,t)$ is generally a superposition of many plane waves. Except for the integral, equation (9) looks almost like the heat flow equation

$$\frac{\partial H}{\partial t} = \frac{\partial^2}{\partial x^2} H \qquad (10)$$

Let us have a quick look at a numerical method of solution to the heat flow equation because our task is very similar. Using a superscript t to denote discretized time, a centered difference arrangement is

$$H^{t+1} - H^{t} = dt\, \frac{\partial^2}{\partial x^2} \left(\frac{H^{t+1} + H^{t}}{2} \right) \qquad (11)$$

From a computational point of view it is now convenient to regard H^t as a column vector given for each time t where the index running down the column points to the x-coordinate. In this notation $\partial^2/\partial x^2$ becomes a big matrix which is empty except for the second difference operator $(1,-2,1)/dx^2$ along the main diagonal. Letting the matrix with $(1,-2,1)dt/2dx^2$ along the main diagonal be called T, the heat flow equation becomes

$$H^{t+1} - H^t = T (H^{t+1} + H^t) \tag{12}$$

solveing for the unknown column vector H^{t+1} we have

$$(I-T) H^{t+1} = (I+T) H^t \tag{13}$$

Don't be discouraged by seeing that (13) is a large set of simultaneous equations for the unknown column vector H^{t+1}. Actually the matrix of coefficients (I-T) is a sparse tri-diagonal matrix and all the textbooks contain an excellant algorithm which is nearly as quick as the matrix times vector operation on the right side of (13). The centered difference arrangement in (11) is called the Crank-Nicolson scheme and while it is not the only practical means of solving the heat flow equation it does seem to be the only practical scheme for the diffraction equation (9). Getting back to waves, now let us differentiate (9) with respect to t

$$P_{zt} = (\bar{c}/2)\ P_{xx} \tag{14}$$

Let time $t = j\ dt$ and depth $z = n\ dz$. Let δ denote a first difference operator. Let P_j^n be a vector at each value of n and j . Entries in the vector represent values of pressure along the x-axis. Let T denote a tri-diagonal matrix with the negative of the second difference operator $-(1,-2,1)$ on the diagonal. With all these definitions and conventions (14) becomes

$$\delta_z\ \delta_t \quad P_j^n = -\frac{\bar{c}\ dz\ dt}{8\ dx^2}\ T\ 4\ P_j^n \tag{15}$$

Let us define $a = \bar{c}\ dz\ dt/8\ dx^2$. Now we must decide more precisely how to set up a differenceing scheme. We will do this entirely analogously to to Crank-Nicolson scheme. First do centered time differenceing.

$$\delta_z \quad (P_{j+1}^n - P_j^n) = -\ a\ T\ 2\ (P_{j+1}^n + P_j^n) \tag{16}$$

and then do centered space differenceing

$$(P_{j+1}^{n+1} - P_j^{n+1}) - (P_{j+1}^n - P_j^n) = -\ a\ T\ (P_{j+1}^{n+1} + P_j^{n+1} + P_{j+1}^n + P_j^n) \tag{17}$$

From the point of view of computation we assume that the unknown is P_{j+1}^{n+1} and that all else is known. Bringing the unknown to the left and the known to the right we have

$$(I+aT)\, P_{j+1}^{n+1} = P_j^{n+1} + P_{j+1}^{n} - P_j^{n} - a\, T(P_j^{n+1} + P_{j+1}^{n} + P_j^{n}) \tag{18}$$

For each n and j the right side collapses to a known vector. The left side is the tri-diagonal matrix $(I+aT)$ multiplying the unknown vector P_{j+1}^{n+1}. The solution of these equations is extremely simple and may be done as with the heat flow equation. Boundaty conditions in x are contained on the ends of T. For z and t boundary conditions it is sufficient to give, at all x, P_0^n for all n and P_j^0 for all j. Other boundary arrangements are possible. It turns out that if you want to turn around and extrapolate backwards in z, this is like changing the sign of dz which is like changeing the sign of a. Actually (18) turns out to be stable for increasing n and j only if a is positive. Thus if it is desired to go backwards in the z direction then it is also necessary to simultaneously take dt negative so that increasing j corresponds to decreasing time.