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# An acoustic wave equation for orthorhombic anisotropy

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***keywords:*** *Anisotropy, finite difference, modeling*


## ABSTRACT

Using a dispersion relation derived under the acoustic medium assumption for $P$-waves in orthorhombic anisotropic media, I obtain an acoustic wave equation valid under the same assumption. Although this assumption is physically impossible for anisotropic media, it results in wave equations that are kinematically and dynamically accurate for elastic media. The orthorhombic acoustic wave equation, unlike the transversely isotropic (TI) one, is a six-order equation with three sets of complex conjugate solutions. Only one set of these solutions are perturbations of the familiar acoustic wavefield solution in isotropic media for in-coming and out-going $P$-waves, and thus, are of interest here. The other two sets of solutions are simplify the result of this artificially derived sixth order equation, and thus, represent unwanted artifacts. Like in the TI case, these artifacts can be eliminated by placing the source in an isotropic layer, where such artifacts do not exist.


## INTRODUCTION

Compared with the elastic wave equation, the acoustic wave equation has two features: it is simpler, and thus, more efficient to use, and it does not yield Shear waves, and as a result, it can be used for zero-offset modeling of $P$-waves. Though in anisotropic media, an acoustic wave equation does not, physically, exit, Alkhalifah (1997b) derived such an acoustic wave equation for transversely isotropic media with a vertical symmetry axis (VTI media). If we ignore the physical aspects of the problem, an acoustic equation for TI media can be extracted by simply setting the shear wave velocity to zero. Kinematically such equations yield good approximations to the elastic ones. Orthorhombic anisotropic media are more complicated than VTI ones. They represent models where we can have vertical fractures along with the general VTI preference (i.e. horizontal thin layering). Orthorhombic media have three mutually orthogonal planes of mirror symmetry; for the model with a single system of vertical cracks in a VTI background, the symmetry planes are determined by the crack orientation. Numerous equations have been derived lately for orthorhombic media

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including the normal moveout (NMO) equation for horizontal and dipping reflectors derived by Grechka and Tsvankin (1997). These equations are generally complicated and are often solved numerically. The complexity of these equations stem from the complexity of the phase velocity and the dispersion relation for orthorhombic media. However, the phase velocity and dispersion relation, as we will see later, can be simplified considerably using the acoustic medium assumption. Thus, practical analytical solutions for the NMO equation in orthorhombic media is possible. In this paper, I derive a dispersion relation for orthorhombic anisotropic media based on the acoustic approximation. I then look into the accuracy of such an equation, and subsequently use it to derive an acoustic wave equation for orthorhombic anisotropic media. Finally, I solve the acoustic wave equations analytically. This is a preliminary study of the subject and a follow up paper will include details and experiments left out of this paper.

## ANISOTROPIC MEDIA PARAMETERS IN ORTHORHOMBIC MEDIA

Unlike in VTI media, where the model is fully characterized by 5 parameters, in orthorhombic media we need 9 parameters for full characterization. However, like in VTI media, not all parameters are expected to influence $P$-wave propagation to a detectable degree. Therefore, alternative parameter representation is important to simplify the problem to a level where key parameter dependencies are recognizable. The stiffness tensor $c_{ijkl}$ for orthorhombic media can be represented in a compressed two-index notation (the so-called "Voigt recipe") as follows:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \tag{1}$$

In VTI media, $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{44} = c_{55}$, $c_{12} = c_{11} - 2c_{44}$, and thus the number of independent parameters reduce from 9 to 5. With additional constraints given by the isotropic model the number of independent parameters will ultimately reduce to two. Significant progress, however, can be made by combining the stiffnesses in such a way that will simplify analytic description of seismic velocities. Tsvankin (1997) suggested a parameterization similar to what Thomsen (1986) used for VTI media and to what Alkhalifah and Tsvankin (1995) added for processing purposes, namely the $\eta$ parameter. I will list the nine parameters needed to characterize orthorhombic media below. However, for convenience in later derivations, I will replace the $\epsilon$ parameters with $\eta$ parameters and use slightly different notations than those given by Tsvankin (1997). In summary, these nine parameters are related to the elastic coefficients as follows:

- the $P$-wave vertical velocity:

$$v_v \equiv \sqrt{\frac{c_{33}}{\rho}} \;\; (\rho \text{ is the density}) \,, \tag{2}$$

- the vertical velocity of the $S$-wave polarized in the $x_1$-direction:

$$v_{s1} \equiv \sqrt{\frac{c_{55}}{\rho}} \,, \tag{3}$$

- the vertical velocity of the $S$-wave polarized in the $x_2$-direction:

$$v_{s2} \equiv \sqrt{\frac{c_{66}}{\rho}} \,, \tag{4}$$

- the horizontal velocity of the $S$-wave polarized in the $x_3$-direction:

$$v_{s3} \equiv \sqrt{\frac{c_{44}}{\rho}} \,, \tag{5}$$

- the NMO $P$-wave velocity for horizontal reflectors in the $[x_1, x_3]$ plane:

$$v_1 \equiv \sqrt{\frac{c_{13}(c_{13} + 2c_{55}) + c_{33}c_{55}}{\rho(c_{33} - c_{55})}} \,, \tag{6}$$

- the NMO $P$-wave velocity for horizontal reflectors in the $[x_2, x_3]$ plane:

$$v_2 \equiv \sqrt{\frac{c_{23}(c_{23} + 2c_{66}) + c_{33}c_{66}}{\rho(c_{33} - c_{66})}} \,, \tag{7}$$

- the $\eta$ parameter in the $[x_1, x_3]$ symmetry plane:

$$\eta_1 \equiv \frac{c_{11}(c_{33} - c_{55})}{2c_{13}(c_{13} + 2c_{55}) + 2c_{33}c_{55}} - \frac{1}{2} \,, \tag{8}$$

- the $\eta$ parameter in the $[x_2, x_3]$ symmetry plane:

$$\eta_2 \equiv \frac{c_{22}(c_{33} - c_{44})}{2c_{23}(c_{23} + 2c_{44}) + 2c_{33}c_{44}} - \frac{1}{2} \,, \tag{9}$$

- the $\delta$ parameter in the $[x_1, x_2]$ plane ($x_1$ plays the role of the symmetry axis):

$$\delta \equiv \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2\,c_{11}\,(c_{11} - c_{66})} \,. \tag{10}$$

This notation preserves the attractive features of Thomsen parameters in describing velocities, and traveltimes. They also provide a simple way to measure anisotropy, since the dimensionless parameters in the new representation equal zero when the medium is isotropic. The notation used above, also, simplify description of time-related processing equations. To ease some of the derivations later in this paper I will also use the horizontal velocity in the $x_1$ direction:

$$V_1 \equiv v_1\sqrt{1+2\eta_1}, \tag{11}$$

and the horizontal velocity in the $x_2$ direction:

$$V_2 \equiv v_2\sqrt{1+2\eta_2}, \tag{12}$$

Both horizontal velocities are given in terms of the above parameters and thus do not add to the number of independent parameters required to represent orthorhombic media. Finally, I will use $\gamma$ as follows

$$\gamma \equiv \sqrt{1+2\delta}. \tag{13}$$

Thus, for isotropic media $\gamma = 1$.

## THE DISPERSION RELATION

Seismic reflection data are often recorded on the Earth surface. Therefore, an equation that describes the vertical slowness as a function of the horizontal one, the dispersion relation, is a key equation for imaging such data. In fact, reflection seismic data explicitly provides horizontal slowness information given by the slope of the reflections. However, a simple $P$-wave analytical equation that describes the vertical slowness as a function of the horizontal one does not exist in a practical form for orthorhombic media. Because obtaining such an equation requires solving for the roots of a cubic polynomial as a function of the squared vertical slowness. On the other hand, setting all three shear wave velocities to zero will reduce the cubic equation to a linear one. The influence of the shear-wave velocities on $P$-wave propagation is small. This is a general statement that holds for most anisotropies, but have been proven extensively for transversely isotropic media. The setting of the shear velocity to zero, as we will see later, will not compromise the accuracy of the equations for kinematic or dynamic uses. To obtain the dispersion relation for orthorhombic anisotropy we must first derive the Christoffel equation for such media. A general form for the Christoffel equation in 3-D anisotropic media is given by

$$\Gamma_{ik}(x_s, p_i) = a_{ijkl}(x_s)p_j p_l,$$

with

$$p_i = \frac{\partial \tau}{\partial x_i},$$

$$a_{ijkl} = c_{ijkl}/\rho,$$

where $p_i$ are the components of the phase vector, $\tau$ is the traveltime along the ray, $\rho$ is the bulk density, $x_s$ are the Cartesian coordinates for position along the ray, s=1,2,3. For orthorhombic media, the Christoffel equation slightly simplifies, and in its matrix form (using $A_{ij}$ instead of $a_{ijkl}$) is given by

$$\begin{matrix} A_{11}\,{p_1}^2 + A_{66}\,{p_2}^2 + A_{55}\,{p_3}^2 - 1 & (A_{12}+A_{66})\;p_1\,p_2 & (A_{13}+A_{55})\;p_1\,p_3 \\ (A_{12}+A_{66})\;p_1\,p_2 & A_{66}\,{p_1}^2 + A_{12}\,{p_2}^2 + A_{44}\,{p_3}^2 - 1 & (A_{23}+A_{44})\;p_2\,p_3 \\ (A_{13}+A_{55})\;p_1\,p_3 & (A_{23}+A_{44})\;p_2\,p_3 & A_{55}\,{p_1}^2 + A_{44}\,{p_2}^2 + A_{33}\,{p_3}^2 - 1 \end{matrix} \tag{14}$$

where $A_{ij}$ are the density normalized elastic coefficients ($A_{ij} = \frac{C_{ij}}{\rho}$). Setting all shear wave velocities ($v_{s1}$, $v_{s2}$, and $v_{s3}$) to zero and using Tsvankin's (1997) parameter representation, the Christoffel equation reduces to

$$\begin{pmatrix} {p_x}^2\,{v_1}^2\;(1+2\,\eta_1) - 1 & \gamma\,p_x\,p_y\,{v_1}^2\;(1+2\,\eta_1) & p_x\,p_z\,v_1\,v_v \\ \gamma\,p_x\,p_y\,{v_1}^2\;(1+2\,\eta_1) & {p_y}^2\,{v_2}^2\;(1+2\,\eta_2) - 1 & p_y\,p_z\,v_2\,v_v \\ p_x\,p_z\,v_1\,v_v & p_y\,p_z\,v_2\,v_v & {p_z}^2\,{v_v}^2 - 1 \end{pmatrix} \tag{15}$$

Here we have replaced $p_1$ with $p_x$, $p_2$ with $p_y$, and $p_3$ with $p_z$, for convenience. Taking the determinant of (15) gives a linear equation in $p_z^2$, as follows

$$\begin{aligned} \mathbf{Det} = {} & -1 + {p_z}^2\,{v_v}^2 + {p_y}^2\,{v_2}^2\,\left(1 + 2\,\eta_2 - 2\,{p_z}^2\,{v_v}^2\,\eta_2\right) \\ & -{p_x}^2\,{v_1}^2\,\left(-1 - 2\,\eta_1 + 2\,{p_z}^2\,{v_v}^2\,\eta_1\right) \\ & -{p_x}^2\,{v_1}^2\,{p_y}^2\,\left(-\left(\gamma^2\,{v_1}^2\right) + {v_2}^2 + \gamma^2\,{p_z}^2\,{v_1}^2\,{v_v}^2 - 2\,\gamma\,{p_z}^2\,v_1\,v_2\,{v_v}^2\right) - \\ & {p_x}^2\,{v_1}^2\,{p_y}^2\,\left({p_z}^2\,{v_2}^2\,{v_v}^2 - 4\,\gamma^2\,{v_1}^2\,\eta_1 + 2\,{v_2}^2\,\eta_1 + 4\,\gamma^2\,{p_z}^2\,{v_1}^2\,{v_v}^2\,\eta_1 - 4\,\gamma\,{p_z}^2\,v_1\,v_2\,{v_v}^2\,\eta_1\right) \\ & -{p_x}^2\,{v_1}^2\,{p_y}^2\,\left(-4\,\gamma^2\,{v_1}^2\,{\eta_1}^2 + 4\,\gamma^2\,{p_z}^2\,{v_1}^2\,{v_v}^2\,{\eta_1}^2 + 2\,{v_2}^2\,\eta_2 + 4\,{v_2}^2\,\eta_1\,\eta_2 - 4\,{p_z}^2\,{v_2}^2\,{v_v}^2\,\eta_1\,\eta_2\right) \end{aligned} \tag{16}$$

Setting equation (16) to zero and solving for $p_z$ provides the dispersion relation for orthorhombic media as follow $p_z^2 =$

$$\frac{1 - {p_y}^2\,{V_2}^2 - {p_x}^2\,{V_1}^2\;(1 + {p_y}^2\;(\gamma^2\,{v_1}^2 - {v_2}^2 + 2\,\gamma^2\,{v_1}^2\,\eta_1 - 2\,{v_2}^2\,\eta_2))}{{v_v}^2\,\left(1 - 2\,{p_y}^2\,{v_2}^2\,\eta_2 - {p_x}^2\,{v_1}^2\,\left(2\,\eta_1 + {p_y}^2\,\left(\gamma^2\,{v_1}^2(1+4\eta_1) - 2\,\gamma\,\frac{V_1}{v1}\,v_2 + {v_2}^2(1 - 4\,\eta_1\,\eta_2)\right)\right)\right)} \tag{17}$$

Each principal plane of the orthorhombic model is VTI in nature, thus setting $p_y$=0 in equation (17) gives

$$p_z^2 = \frac{1}{v_v^2}\left(1 - \frac{v_1^2 p_x^2}{1 - 2\eta_1 v_1^2 p_x^2}\right),$$

which is simply the dispersion relation for the VTI case, shown by Alkhalifah (1997a). Note that the vertical velocity appears only once in equation (17) and thus the vertical-time based dispersion relation, like in the case of VTI media, become vertical velocity independent. This feature is shown approximately by Grechka and Tsvankin (1997) using numerical methods, and is shown exactly here.

## ACCURACY TESTS

Before we use the new dispersion relation to derive an acoustic wave equation we must first investigate how accurate is this dispersion equation in representing elastic media. Specifically, I will measure the error associated with equation 17 with respect to the elastic equation. The dispersion relation for elastic media is evaluated numerically. Since the isotropic dispersion relation is independent of the shear wave velocity, and it's acoustic version is exact, the errors corresponding to the acoustic dispersion equation is expected to be dependent on the strength of anisotropy. Figure 1 shows

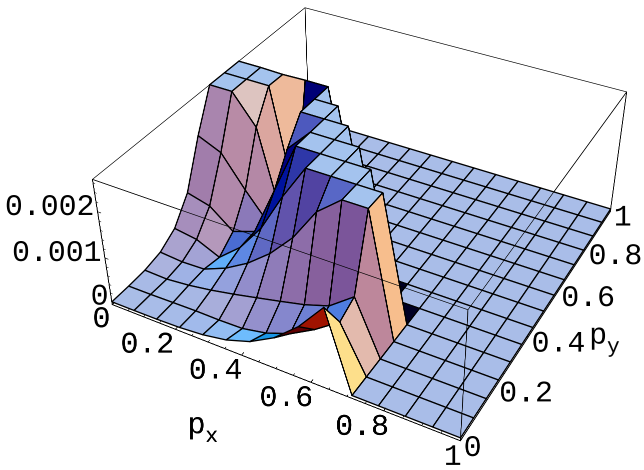



Figure 1: Errors in $p_z$ evaluated using equation 17 measured in s/km as a function of $p_x$ and $p_y$. The orthorhombic model has $v_v$=1 km/s, $v_1$ = 1.1 km/s, $\eta_1$ = 0.1, $v_2$ = 1.2 km/s, $\eta_2$ = 0.2, and $\delta$=0.1. The shear wave velocities for the elastic medium equal 0.5 km/s. $\boxed{\text{tariq2-error1}}$ [NR]

a 3-D surface plot of the error in $p_z$ evaluated using equation 17 as a function $p_x$ and $p_y$. The error is given by the difference between $p_z$ measured using equation 17 and that using the elastic equation, with non-zero shear wave velocities. In fact, the reference elastic medium has shear wave velocities equal to half the vertical $P$-wave velocity ( $v_{s1} = v_{s2} = v_{s3} = 0.5$ km/s). Since the vertical $P$-wave velocity in all the examples equal 1 km/s, $p_z$ can have a maximum value of 1 and a minimum of zero. The Orthorhombic model used in Figure 1 has $v_1 = 1.1$ km/s, $\eta_1 = 0.1$, $v_2 = 1.2$ km/s, $\eta_2 = 0.2$, and $\delta$=0.1. This model is practical with a strength of anisotropy that is considered moderate. Clearly, the errors given by a maximum value of 0.002 is extremely small suggesting that equation 17 is accurate for this case. Equation 17 is exact for zero and 90 degree dip reflectors, and therefore, most of the errors occur at angles in between. However, such errors are clearly small. From my earlier experience (Alkhalifah, 1997a), errors in the acoustic approximations increase with increasing shear wave velocity. Obviously, if shear wave velocity equals zero no errors are incurred. Figure 2 shows the same model used in Figure 1, but with higher shear wave velocities. Specifically, $v_{s1} = 0.6$ km/s, $v_{s2} = 0.7$ km/s,and $v_{s3} = 0.7$ km/s. Here, the vertical $S$-wave to $P$-wave velocity ratio equal 0.6, which can be considered as an upper limit for most practical models in the subsurface. Yet, the errors given by the acoustic approximations (maximum error equal to 0.003) is still extremely small. To test the limits of the new dispersion relation, I use an orthorhombic model

Figure 2: Errors in $p_z$ evaluated using equation 17 measured in s/km as a function of $p_x$ and $p_y$. The orthorhombic model has $v_v$=1 km/s, $v_1$ = 1.1 km/s, $\eta_1$ = 0.1, $v_2$ = 1.2 km/s, $\eta_2$ = 0.2, and $\delta$=0.1. Here, the elastic medium has $v_{s1}$ = 0.6 km/s, $v_{s2}$ = 0.7 km/s,and $v_{s3}$ = 0.7 km/s. tariq2-error1s [NR]

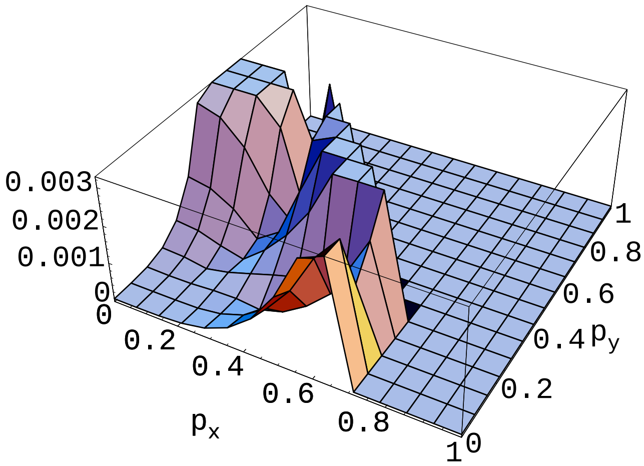
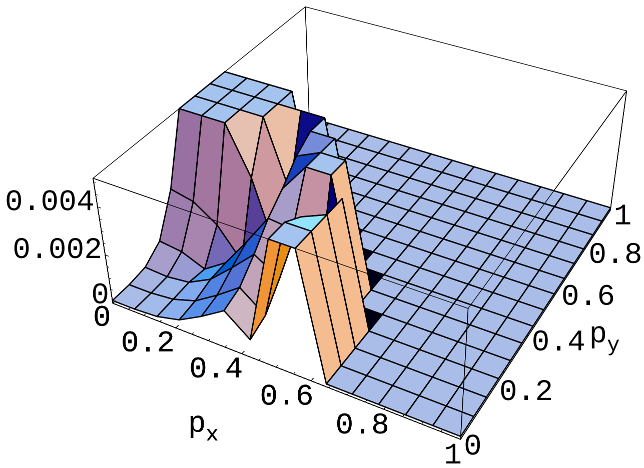



Figure 3: Errors in $p_z$ evaluated using equation 17 measured in s/km as a function of $p_x$ and $p_y$. The orthorhombic model has $v_v$=1 km/s, $v_1$ = 0.9 km/s, $\eta_1$ = 0.6, $v_2$ = 1.2 km/s, $\eta_2$ = 0.4, and $\delta$=0.3. Here, the elastic medium has $v_{s1}$ = 0.7 km/s, $v_{s2}$ = 0.8 km/s, and $v_{s3}$ = 0.8 km/s. tariq2-error2 [NR]



of strong anisotropy. Specifically, $v_1 = 0.9$ km/s, $\eta_1 = 0.6$, $v_2 = 1.2$ km/s, $\eta_2 = 0.4$, and $\delta$=0.3. The strength of anisotropy in this test is given by the high $\eta_1$, $\eta_2$, and $\delta$ values. I also use for the elastic equation high shear wave velocities given by $v_{s1} = 0.7$ km/s, $v_{s2} = 0.8$ km/s,and $v_{s3} = 0.8$ km/s. Figure 3 shows a 3-D surface plot of the error in $p_z$ evaluated using equation 17 as a function $p_x$ and $p_y$. The errors are slightly larger than those in Figures 1 and 2, but overall acceptable. The maximum error of about 0.004 is much smaller than the possible range of $p_z$. However, there is

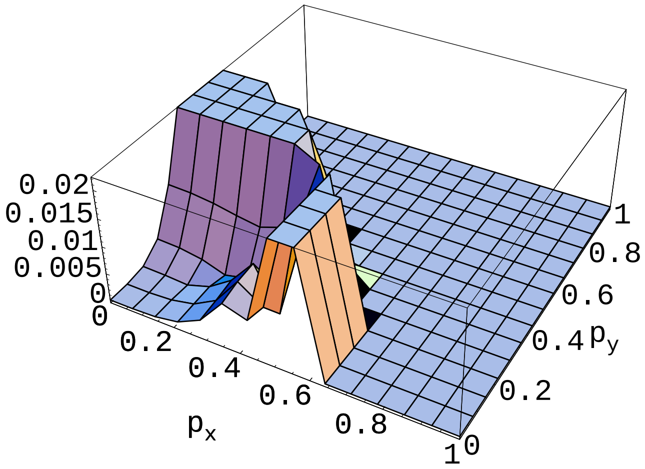


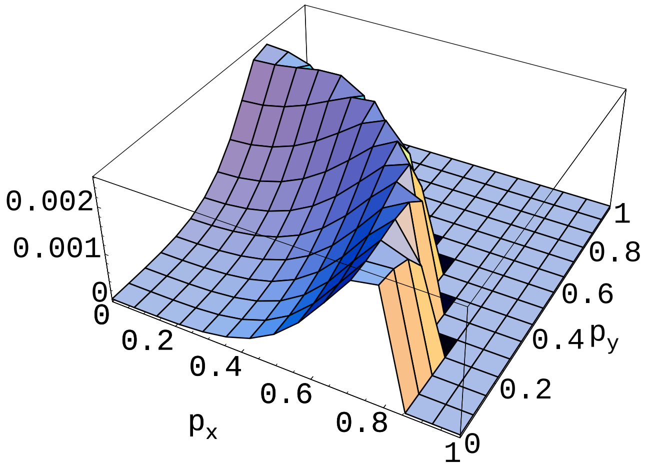

Figure 4: Errors in $p_z$ evaluated using equation 17 measured in s/km as a function of $p_x$ and $p_y$. The orthorhombic model has $v_v$=1 km/s, $v_1 = 0.9$ km/s, $\eta_1 = 0.6$, $v_2 = 1.2$ km/s, $\eta_2 = 0.4$, and $\delta$=0.3. Here, the elastic medium has $v_{s1} = 0.9$ km/s, $v_{s2} = 0.9$ km/s, and $v_{s3} = 0.9$ km/s. tariq2-error2s [NR]

a limit to what kind of shear wave velocities the elastic media can have before this acoustic approximation breaks down. Figure 4 shows the errors for the same model in Figure 3 with even higher shear wave velocities. Specifically, $v_{s1} = v_{s2} = v_{s3} = 0.9$ km/s. Suddenly the acoustic approximation incurs large errors, up to 0.02 in the value of $p_z$ over a possible range of 1. However, note that such a model given by $v_{s1} = v_{s2} = v_{s3} = 0.9$ km/s is highly unlikely considering that the vertical $P$-wave velocity equal 1 km/s. This constitutes an extreme orthorhombic anisotropy model that probably does not exist in the subsurface. For comparison, Figure 5 shows this same error test, however for a VTI model. The model is given by $v_1 = 1.1$ km/s, $\eta_1 = 0.1$, $v_2 = 1.1$ km/s, $\eta_2 = 0.1$, and $\delta$=0. The error size is very similar to that in Figure 1, but more symmetric since the VTI model exerts symmetry on the horizontal plane. In summary, the dispersion relation given by equation 17 is, for all practical purposes, exact. Thus, the acoustic wave equation extracted from this dispersion relation is expected to be accurate as well.

## THE ACOUSTIC WAVE EQUATION FOR ORTHORHOMBIC MEDIA

Using the dispersion relation of equation 17, we can derive an acoustic wave equation for orthorhombic media as shown in Appendix A. The resultant wave equation is

Figure 5: Errors in $p_z$ evaluated using equation 17 measured in s/km as a function of $p_x$ and $p_y$. The VTI model has $v_v$=1 km/s, $v_1$ = 1.1 km/s, $\eta_1$ = 0.1, $v_2$ = 1.1 km/s, $\eta_2$ = 0.1, and $\delta$=0. tariq2-errorVTI [NR]




sixth-order in time given by,

$$\frac{\partial^6 F}{\partial t^6} = V_1^2 \frac{\partial^6 F}{\partial t^4 \partial x^2} + V_2^2 \frac{\partial^6 F}{\partial t^4 \partial y^2} + v_v^2 \frac{\partial^6 F}{\partial t^4 \partial z^2} - 2\eta_1 v_1^2 v_v^2 \frac{\partial^6 F}{\partial t^2 \partial x^2 \partial z^2}$$
$$-2\eta_2 v_2^2 v_v^2 \frac{\partial^6 F}{\partial t^2 \partial y^2 \partial z^2} + {V_1}^2 \left(\gamma^2 {V_1}^2 - {V_2}^2\right) \frac{\partial^6 F}{\partial t^2 \partial x^2 \partial y^2}$$
$$- \left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2 \left(1 - 4\,\eta_1\,\eta_2\right)\right)\right) \frac{\partial^6 F}{\partial x^2 \partial y^2 \partial z^2}. \quad (18)$$

This equation is two time-derivative orders higher than its VTI equivalent and 4 orders higher than the conventional isotropic acoustic wave equation. Setting $\eta_1 = \eta_2$, $v_1 = v_2$, and $\gamma = 1$, the conditions necessary for the medium to be VTI, equation (18) reduces to

$$\frac{\partial^2}{\partial t^2} \left( \frac{\partial^4 F}{\partial t^4} - (1+2\eta) v^2 \left( \frac{\partial^4 F}{\partial x^2 \partial t^2} + \frac{\partial^4 F}{\partial y^2 \partial t^2} \right) - v_v^2 \frac{\partial^4 F}{\partial z^2 \partial t^2} + 2\eta v^2 v_v^2 \left( \frac{\partial^4 F}{\partial x^2 \partial z^2} - \frac{\partial^4 F}{\partial y^2 \partial z^2} \right) \right) = 0. (19)$$

Substituting $M = \frac{\partial^2 F}{\partial t^2}$ gives us the acoustic wave equation for VTI media derived by Alkhalifah (1997a),

$$\frac{\partial^4 M}{\partial t^4} - (1+2\eta) v^2 \left( \frac{\partial^4 M}{\partial x^2 \partial t^2} + \frac{\partial^4 M}{\partial y^2 \partial t^2} \right) = v_v^2 \frac{\partial^4 M}{\partial z^2 \partial t^2} - 2\eta v^2 v_v^2 \left( \frac{\partial^4 M}{\partial x^2 \partial z^2} + \frac{\partial^4 M}{\partial y^2 \partial z^2} \right). (20)$$

Thus, as expected, equation (18) reduces to the exact VTI form when VTI model conditions are used, and subsequently it will reduce to the exact isotropic form (the conventional second-order acoustic wave equation) when isotropic model parameters are used (i.e., $\delta = 0$, $\eta_1 = \eta_2 = 0$ and $v_1 = v_2 = v_v$). Equation (18) can be solved numerically using finite difference methods. However, such solutions require complicated numerical evaluations based on sixth-order approximation of derivative in time. Substituting $M = \frac{\partial^2 F}{\partial t^2}$ into equation (18) yields,

$$\begin{aligned}\frac{\partial^4 M}{\partial t^4} = V_1^2 \frac{\partial^4 M}{\partial t^2 \partial x^2} + V_2^2 \frac{\partial^4 M}{\partial t^2 \partial y^2} + v_v^2 \frac{\partial^4 M}{\partial t^2 \partial z^2} - 2\eta_1 v_1^2 v_v^2 \frac{\partial^4 M}{\partial x^2 \partial z^2} \\ -2\eta_2 v_2^2 v_v^2 \frac{\partial^4 M}{\partial y^2 \partial z^2} + {V_1}^2 \left(\gamma^2 {V_1}^2 - {V_2}^2\right) \frac{\partial^4 M}{\partial x^2 \partial y^2} \\ - \left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1 - 4\,\eta_1\,\eta_2)\right)\right) \frac{\partial^6 F}{\partial x^2 \partial y^2 \partial z^2},\end{aligned} \tag{21}$$

which is a fourth order equation in derivates of $t$. In addition, substituting $P = \frac{\partial^2 M}{\partial t^2}$ into equation (21) yields

$$\begin{aligned}\frac{\partial^2 P}{\partial t^2} = V_1^2 \frac{\partial^2 P}{\partial x^2} + V_2^2 \frac{\partial^2 P}{\partial y^2} + v_v^2 \frac{\partial^2 P}{\partial z^2} - 2\eta_1 v_1^2 v_v^2 \frac{\partial^4 M}{\partial x^2 \partial z^2} \\ -2\eta_2 v_2^2 v_v^2 \frac{\partial^4 M}{\partial y^2 \partial z^2} + {V_1}^2 \left(\gamma^2 {V_1}^2 - {V_2}^2\right) \frac{\partial^4 M}{\partial x^2 \partial y^2} \\ - \left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1 - 4\,\eta_1\,\eta_2)\right)\right) \frac{\partial^6 F}{\partial x^2 \partial y^2 \partial z^2},\end{aligned} \tag{22}$$

which is now second order in derivates of $t$. Equation (22) also clearly displays the various levels of parameter influence on the wave equation. For example, if $\eta_1 = \eta_2$ and $v_1 = v_2$ the last term in equation (22) drops.

## ANALYTICAL SOLUTIONS OF THE ANISOTROPIC EQUATION

To solve equation (18), we use the plane wave,

$$F(x, y, z, t) = A(t) \exp i(k_x x + k_y y + k_z z),$$

as a trial solution. Substituting the trial solution into the partial differential equation (18), we obtain the following linear equation for $A$,

$$\begin{aligned}\frac{d^6 A}{dt^6} + \left({k_z}^2 {v_v}^2 + {k_x}^2 {V_1}^2 + {k_y}^2 {V_2}^2\right) \frac{d^4 A}{dt^4} + \\ \left({k_x}^2 {k_y}^2 {V_1}^2 \left(\gamma^2 {V_1}^2 - {V_2}^2\right) + 2\,{k_x}^2 {k_z}^2 {v_1}^2 {v_v}^2 \eta_1 + 2\,{k_y}^2 {k_z}^2 {v_2}^2 {v_v}^2 \eta_2\right) \frac{d^2 A}{dt^2} \\ - \left({k_x}^2 {k_y}^2 {k_z}^2 {v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1 - 4\,\eta_1\,\eta_2)\right)\right) A = 0.\end{aligned} \tag{23}$$

The fact that equation (23) includes only even order derivates of $A$ implies that we have three sets of complex-conjugate solutions. These solutions are

$$A_1(t) = e^{\pm\sqrt{\frac{a_1}{6}}\,t}, \tag{24}$$

where

$$a_1 = 2\,a + \frac{2^{\frac{4}{3}}\,\left(a^2+3\,b\right)}{\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}} +$$
$$2^{\frac{2}{3}}\,\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}.$$

$$A_2(t) = e^{\pm\sqrt{\frac{a_2}{12}}\,t}, \tag{25}$$

where

$$a_2 = 4\,a - \frac{i\,2^{\frac{4}{3}}\,\left(-i+\sqrt{3}\right)\,\left(a^2+3\,b\right)}{\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}} +$$
$$i\,2^{\frac{2}{3}}\,\left(i+\sqrt{3}\right)\,\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}.$$

and

$$A_3(t) = e^{\pm\sqrt{\frac{a_3}{12}}\,t}, \tag{26}$$

where

$$a_3 = 4\,a + \frac{i\,2^{\frac{4}{3}}\,\left(i+\sqrt{3}\right)\,\left(a^2+3\,b\right)}{\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}} -$$
$$2^{\frac{2}{3}}\,\left(1+i\,\sqrt{3}\right)\,\left(2\,a^3+9\,a\,b+27\,c+\sqrt{-4\,\left(a^2+3\,b\right)^3+\left(2\,a^3+9\,a\,b+27\,c\right)^2}\right)^{\frac{1}{3}}.$$

The above are simply the three roots of the following cubic polynomial

$$x^3 + ax^2 + bx + c = 0$$

with

$$a = -\left({k_z}^2\,{v_v}^2\right) - {k_x}^2\,{V_1}^2 - {k_y}^2\,{V_2}^2,$$

$$b = \left(\gamma^2\,{k_x}^2\,{k_y}^2\,{V_1}^4\right) - {k_x}^2\,\left({k_y}^2\,{V_1}^2\,{V_2}^2 - 2\,{k_z}^2\,{v_1}^2\,{v_v}^2\,\eta_1\right) - 2\,{k_y}^2\,{k_z}^2\,{v_2}^2\,{v_v}^2\,\eta_2,$$

and

$$c = {k_x}^2\,{k_y}^2\,{k_z}^2\,{v_1}^2\,{v_v}^2\,\left(-2\,\gamma\,v_2\,V_1 + \gamma^2\,{V_1}^2 + {v_2}^2\,\left(1-4\,\eta_1\,\eta_2\right)\right),$$

as it relates to our problem. Solution (25) reduces to the isotropic medium solution when $\eta$=0. Solutions (24) and (26) are additional waves that reduces in the isotropic limit ($\eta_1 \to 0$, $\eta_2 \to 0$, and $\delta \to 0$) to 1 and, with the proper initial condition, its coefficient to zero. In other words, solutions (24) and (26) become independent of time for $\eta_1 = \eta_2 = \delta = 0$. However, these waves will prove to be harmful in orthorhombic case. The main concern here is the sign of $a_1$, $a_2$ and $a_3$. A negative sign will result in an imaginary exponential term which corresponds to wave propagation behavior. A positive sign will result in a real exponential that is either decaying or growing depending on the sign of the exponential term. Considering we have conjugate solutions, at least one the solutions will be growing exponentially and causing serious instability problems. I will leave the analysis of $a_1$, $a_2$, and $a_3$ to a follow up paper.

## CONCLUSIONS

Equations that describe wave propagation in orthorhombic media are derived under the acoustic assumption. These include a dispersion relation, which is a key equation for imaging, and a wave equation, which is important for modeling $P$-waves. Though the Earth subsurface is always elastic in anisotropic media, these acoustic approximations yield accurate kinematic and dynamic descriptions of $P$-wave propagation in elastic media. In addition, the acoustic equations are simpler than the elastic ones, and thus, they are more efficient and easier to use.

## REFERENCES

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## APPENDIX A

## THE ACOUSTIC WAVE EQUATION FOR ORTHORHOMBIC MEDIA

In the text we derived an acoustic dispersion relation [equation 17] for orthorhombic anisotropy. This relation can be used to derive the acoustic wave equation following the same steps I took in deriving such an equation for VTI media (Alkhalifah, 1997b). First, we cast the dispersion relation in a polynomial form in terms of the slownesses and substitute these slownesses with wavenumbers as follows,

$$\begin{aligned}\omega^6 = V_1^2\omega^4 k_x^2 + V_2^2\omega^4 k_y^2 + v_v^2\omega^4 k_z^2 - 2\eta_1 v_1^2 v_v^2 \omega^2 k_x^2 k_z^2 \\ -2\eta_2 v_2^2 v_v^2 \omega^2 k_y^2 k_z^2 + {V_1}^2 \left(\gamma^2 {V_1}^2 - {V_2}^2\right)\omega^2 k_x^2 k_y^2 \\ -\left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1-4\,\eta_1\,\eta_2)\right)\right) k_x^2 k_y^2 k_z^2\end{aligned} \tag{A-1}$$

where $p_x = \frac{k_x}{\omega}$, $p_y = \frac{k_y}{\omega}$, and $p_z = \frac{k_z}{\omega}$. As a reminder, $V_1$ and $V_2$ are the horizontal velocities along the x-axis and the y-axis, respectively. Multiplying both sides of equation (A-1) by the wavefield in the Fourier domain, $F(k_x, k_y, k_z, \omega)$, as well as using inverse Fourier transform on $k_x$, $k_y$, and $k_z$ ($k_x \to -i\frac{d}{dx}$, $k_y \to -i\frac{d}{dy}$, and $k_z \to -i\frac{d}{dz}$) yields a wave equation in the space-frequency domain, given by

$$\begin{aligned}-\omega^6 F = V_1^2\omega^4\frac{\partial^2 F}{\partial x^2} + V_2^2\omega^4\frac{\partial^2 F}{\partial y^2} + v_v^2\omega^4\frac{\partial^2 F}{\partial z^2} + 2\eta_1 v_1^2 v_v^2\omega^2\frac{\partial^4 F}{\partial x^2\partial z^2} + 2\eta_2 v_2^2 v_v^2\omega^2\frac{\partial^4 F}{\partial y^2\partial z^2} \\ -{V_1}^2\left(\gamma^2 {V_1}^2 - {V_2}^2\right)\omega^2\frac{\partial^4 F}{\partial x^2\partial y^2} \\ -\left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1-4\,\eta_1\,\eta_2)\right)\right)\frac{\partial^6 F}{\partial x^2\partial y^2\partial z^2}\end{aligned} \tag{A-2}$$

Finally, applying inverse Fourier transform on $\omega$ ($\omega \to i\frac{\partial}{\partial t}$), the acoustic wave equation for VTI media is given by

$$\begin{aligned}\frac{\partial^6 F}{\partial t^6} = V_1^2\frac{\partial^6 F}{\partial t^4\partial x^2} + V_2^2\frac{\partial^6 F}{\partial t^4\partial y^2} + v_v^2\frac{\partial^6 F}{\partial t^4\partial z^2} - 2\eta_1 v_1^2 v_v^2\frac{\partial^6 F}{\partial t^2\partial x^2\partial z^2} \\ -2\eta_2 v_2^2 v_v^2\frac{\partial^6 F}{\partial t^2\partial y^2\partial z^2} + {V_1}^2\left(\gamma^2 {V_1}^2 - {V_2}^2\right)\frac{\partial^6 F}{\partial t^2\partial x^2\partial y^2} \\ -\left({v_1}^2 {v_v}^2 \left(-2\,\gamma\, v_2\, V_1 + \gamma^2 {V_1}^2 + {v_2}^2\,(1-4\,\eta_1\,\eta_2)\right)\right)\frac{\partial^6 F}{\partial x^2\partial y^2\partial z^2}.\end{aligned} \tag{A-3}$$

Unlike the acoustic wave equation for VTI media (Alkhalifah, 1997b) which is fourth order in time, equation (A-3) is sixth order in time, and thus can provide us with 6 independent solutions.