Stanford Exploration Project, Report 93, November 11, 1997, pages 179–192

Range of the P-wave anisotropy parameter for finely layered VTI media

James G. Berryman[1]

keywords: *anisotropy, elastic, velocity, dispersion*

ABSTRACT

Since the work of Postma (1955) and Backus (1962), much is known about elastic constants in vertical transversely isotropic (VTI) media when the anisotropy is due to fine layering of isotropic elastic materials. I review earlier work and then show that the P-wave anisotropy parameter c_{11}/c_{33} lies in the range $\frac{1}{4} \le c_{11}/c_{33} \le \langle\lambda + 2\mu\rangle \langle 1/(\lambda + 2\mu)\rangle$, when the layers are themselves composed of isotropic elastic materials with Lamé constants λ and μ and the vertical average of the layers is symbolized by $\langle\rangle$. The upper bound is true for all finely layered media of this type, while the lower bound is the best possible for fine layering with two constituents. This lower bound corrects an error in Postma's (1955) paper on two-component VTI media. The method used here shows in general terms that the P-wave anisotropy parameter is smallest when the variation in the layer λ Lamé parameter is large, independent of the variation in the shear modulus μ. This result is therefore important for applications to porous layers where the effects of fluids influence only the λ Lamé constants, not μ. Thus, the P-wave anisotropy parameter may be a useful hydrocarbon indicator in some situations.

INTRODUCTION

Two primary goals of seismic reflection processing are: (1) to image geologic structure and (2) to provide information about lithology for interpretation. The process used to achieve the second goal is made complex by the fact that the same seismic velocity may result from several different combinations/mixtures of materials in the earth. It is therefore necessary to explore the possible range of seismic velocities that can occur within the set of circumstances deemed most likely to occur in the earth at the site of interest.

Fine horizontal layering (*i.e.*, layers with thickness small compared to the wavelength of the seismic wave) is known to result in vertical transverse isotropy (VTI)

[1]**email:** berryman@sep.stanford.edu

– wherein wave speeds vary with angle in such media, but are uniquely determined by the angle from the vertical. There has continued to be some doubt about the range of anisotropy parameters possible in such media. Here I will correct an error of Postma (1955) and show that the P-wave anisotropy parameter c_{11}/c_{33} can be a factor of 2 smaller than previously supposed. I also obtain a simple upper bound on this parameter in terms of layer elastic parameters.

WAVE PROPAGATION IN ANISOTROPIC ELASTIC MEDIA

In tensor notation, the relationship between components of stress σ_{ij} and strain $u_{k,l}$ is given by

$$\sigma_{ij} = c_{ijkl}u_{k,l}, \tag{1}$$

where c_{ijkl} is the adiabatic stiffness tensor, and repeated indices on the right hand side of (1) are summed. In (1), u_k is the kth Cartesian component of the displacement vector $\mathbf{u}$, and $u_{k,l} = \partial u_k/\partial x_l$. Whereas for an isotropic elastic medium the stiffness tensor has the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{2}$$

depending on only two parameters (the Lamé constants, λ and μ), this tensor can have up to 21 independent constants for general anisotropic elastic media. The stiffness tensor has pairwise symmetry in its indices such that $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{ijlk}$, which will be used later to simplify the resulting equations.

The general equation of motion for wave propagation through an anisotropic elastic medium is given by

$$\rho\ddot{u}_i = \sigma_{ij,j} = c_{ijkl}u_{k,lj}, \tag{3}$$

where $\ddot{u}_i$ is the second time derivative of the ith Cartesian component of the displacement vector $\mathbf{u}$ and ρ is the density (assumed constant). Equation (3) is a statement that the product of mass times acceleration of a particle is determined by the internal stress force $\sigma_{ij,j}$.

A commonly used simplification of the notation for elastic analysis is given by introducing the strain tensor, where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

Then, using one version of the Voigt convention, in which the pairwise symmetries of the stiffness tensor indices are used to reduce the number of indices from 4 to 2 using the rules $11 \to 1$, $22 \to 2$, $33 \to 3$, 23 or $32 \to 4$, 13 or $31 \to 5$, and 12 or $21 \to 6$, we have

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & 2c_{44} & & \\ & & & & 2c_{55} & \\ & & & & & 2c_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \tag{5}$$

Although the Voigt convention introduces no restrictions on the stiffness tensor, I have chosen to limit discussion to the form in (5), which is not completely general. Of the 36 coefficients (of which 21 are generally independent), we choose to treat only those cases for which the 12 coefficients shown (of which nine are generally independent) are nonzero. This form includes all orthorhombic, cubic, hexagonal, and isotropic systems, while excluding triclinic, monoclinic, trigonal, and some tetragonal systems, since each of the latter contains additional off-diagonal constants that may be nonzero. Nevertheless, I will restrict the discussion to (5) or to the still simpler case of transversely isotropic (TI) materials.

For TI materials, $c_{11} = c_{22} \equiv a$, $c_{12} \equiv b$, $c_{13} = c_{23} \equiv f$, $c_{33} \equiv c$, $c_{44} = c_{55} \equiv l$, and $c_{66} \equiv m$. There is also one further constraint on the constants that $a = b + 2m$, following from rotational symmetry in the x_1x_2-plane. In such materials, (5) may be replaced by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \tag{6}$$

in which the matrix has the same symmetry as hexagonal systems and of which isotropic symmetry is a special case (having $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$).

Recall that the equation of motion may be written as

$$\rho \ddot{u}_i = c_{ijkl} u_{k,lj}. \tag{7}$$

After Fourier transforming in both space and time [*i.e.*, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u} \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$, where $\mathbf{k}$ is the wavevector and ω is the angular frequency], I find

$$(\rho\omega^2 \delta_{ik} - c_{ijkl} k_j k_l) u_k = 0, \tag{8}$$

which provides three equations for the components of displacement u_1, u_2, u_3. These equations can be solved if and only if the determinant of the coefficients vanishes, which implies

$$det(\rho\omega^2 \delta_{il} - c_{ijkl} k_j k_l) = 0. \tag{9}$$

Equation (9) is known as the dispersion relation, giving ω as a function of $\mathbf{k}$. The magnitude of the phase velocity of a mode of propagation is given by $v = \omega/k$, where $k = |\mathbf{k}|$. Then, if the stiffness tensor is assumed to have the symmetry of a transversely isotropic material, the equation of motion can be written in the following form

$$\rho\omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & k_3 \end{pmatrix} \begin{pmatrix} a & b & f \\ b & a & f \\ f & f & c \end{pmatrix} \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & k_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} l & & \\ & l & \\ & & m \end{pmatrix} \begin{pmatrix} 0 & k_3 & k_2 \\ k_3 & 0 & k_1 \\ k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \quad (10)$$

It is not difficult to solve (10) in a completely general fashion, but for simplicity I will consider the equations under the assumption that the direction of propagation lies in the x_1x_3-plane. Then, $k_2 = 0$ and the equations become

$$\left(ak_1^2 + lk_3^2 - \rho\omega^2\right) u_1 + (f+l)k_1k_3u_3 = 0, \quad (11)$$

$$(f+l)k_1k_3u_1 + \left(ck_3^2 + lk_1^2 - \rho\omega^2\right) u_3 = 0, \quad (12)$$

and

$$\left(mk_1^2 + lk_3^2 - \rho\omega^2\right) u_2 = 0. \quad (13)$$

The first two equations are coupled and can be solved for the dispersion relation of the propagation modes given by

$$\rho\omega_\pm^2 = \tfrac{1}{2}\left\{(a+l)k_1^2 + (c+l)k_3^2 \pm \sqrt{[(a-l)k_1^2 - (c-l)k_3^2]^2 + 4(f+l)^2k_1^2k_3^2}\right\}. \quad (14)$$

These modes of propagation are called quasi-P (or qP) and quasi-SV (or qSV) because the particle motion does not generally satisfy the requirements for pure compressional waves ($\mathbf{k} \times \mathbf{u} = 0$) or pure shear waves ($\mathbf{k} \cdot \mathbf{u} = 0$), although they may satisfy these conditions at a few special angles of propagation. Since compressional waves generally have higher wave speeds than shear waves, it is easy to see that ω_+ is the expression for the quasi-P-wave dispersion and that ω_- is the expression for the quasi-SV-wave dispersion. The third equation gives the dispersion relation for shear waves polarized in the horizontal direction (*i.e.*, SH waves) given by

$$\rho\omega_{sh}^2 = mk_1^2 + lk_3^2. \quad (15)$$

The argument of the radical in (14) can be rewritten (uniquely) into the form

$$\left[(a-l)k_1^2 - (c-l)k_3^2\right]^2 + 4(f+l)^2k_1^2k_3^2 =$$
$$\left[(a-l)k_1^2 + (c-l)k_3^2\right]^2 + 4\left[(f+l)^2 - (a-l)(c-l)\right]k_1^2k_3^2. \quad (16)$$

The left hand side of this equality would be a perfect square for all values of k_1^2 and k_3^2 if $f + l \equiv 0$, while the right hand side would be a perfect square if the quantity defined as $A \equiv (a-l)(c-l) - (f+l)^2 = 0$. The first case with $f + l = 0$ will virtually never happen because both f and l are normally positive quantities. The second case with $A = 0$ can occur for some types of anisotropic media, but I show later that this cannot occur for finely layered media. Nevertheless, if $A = 0$, then the dispersion relations of (14) reduce to

$$\rho\omega_p^2 = ak_1^2 + ck_3^2 \tag{17}$$

for $\omega_p = \omega_+$ and

$$\rho\omega_{sv}^2 = l(k_1^2 + k_3^2) = lk^2 \tag{18}$$

for $\omega_{sv} = \omega_-$, showing that the P-wave surface for velocity squared is elliptical if $a \neq c$, while the SV-wave surface for velocity squared is circular and therefore isotropic. The dispersion relation (15) shows that the SH-wave surface for velocity squared is always an ellipse as long as $l \neq m$. I call A the anellipticity parameter because, if $A \neq 0$, then the dispersion relations for qP- and qSV-waves are anelliptical (*i.e.*, something other than elliptical) in shape.

Phase velocities are obtained as a function of angle from these expressions by first defining the wavevector angle θ such that

$$\mathbf{k} = k(\sin\theta\hat{x}_1 + \cos\theta\hat{x}_3). \tag{19}$$

Then, the phase velocity vector for each type of wave is given in general by

$$\mathbf{v}_{ph} = v_{ph}(\sin\theta\hat{x}_1 + \cos\theta\hat{x}_3), \tag{20}$$

where $v_{ph} \equiv \omega/k$.

The group velocity is defined by

$$\mathbf{v}_{gr} = \frac{\partial\omega}{\partial k_1}\hat{x}_1 + \frac{\partial\omega}{\partial k_3}\hat{x}_3 = v_{gr}(\sin\phi\hat{x}_1 + \cos\phi\hat{x}_3), \tag{21}$$

where the group angle ϕ is determined by

$$\tan\phi = \frac{\partial\omega/\partial k_1}{\partial\omega/\partial k_3}. \tag{22}$$

One other angle is particularly important, since it is the one that is most easily measured, and that is the angle of particle motion ψ for a wave passing a particular point in space. The particle motion is given by the displacement vector $\mathbf{u}$, so

$$\mathbf{u} = u(\sin\psi\hat{x}_1 + \cos\psi\hat{x}_3), \tag{23}$$

where

$$\tan\psi = u_1/u_3. \tag{24}$$

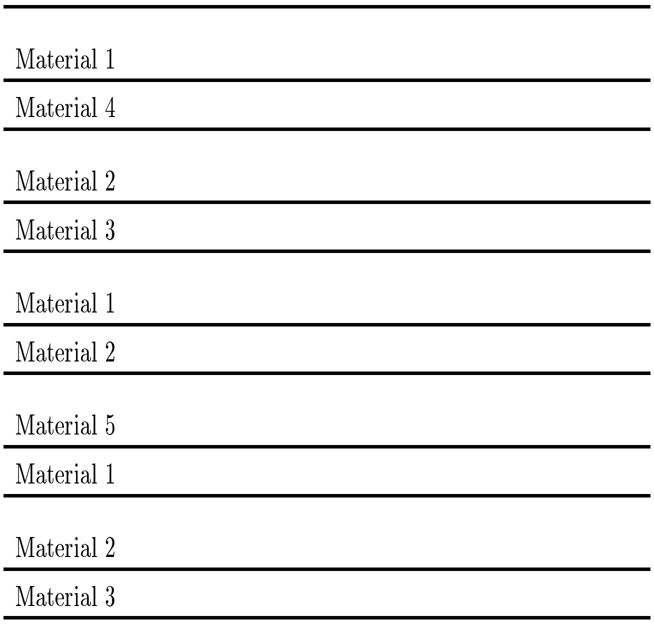

Figure 1: Either random or periodic thinly layered elastic materials give rise to an effective medium characterized by transverse isotropy when the layers are composed of isotropic elastic materials and the layer thickness is small compared to the wavelength. jim1-layercake [NR]

AVERAGING THIN LAYERS FOR LOW FREQUENCY BEHAVIOR

Backus (1962) presents an elegant method of producing the effective constants for a thinly layered medium composed of either isotropic or anisotropic elastic layers. This method applies either to spatially periodic layering or to random layering, by which we mean either that the material constants change in a nonperiodic (unpredictable) manner from layer to layer (see Figure 1) or that the layer thicknesses may also be random. For simplicity, I will assume that the layers are isotropic, in which case equation (5) becomes

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \tag{25}$$

The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying coefficients multiply slowly varying stresses or strains. By doing so, I arrive at the following equation

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ -e_{33} \\ e_{23} \\ e_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \frac{4\lambda(\lambda+\mu)}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\lambda(\lambda+\mu)}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{\lambda}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & -\frac{1}{\lambda+2\mu} & & & \\ & & & \frac{1}{2\mu} & & \\ & & & & \frac{1}{2\mu} & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}, \tag{26}$$

which can be averaged essentially by inspection. Equation (26) can be viewed as a Legendre transform of the original equation, to a different set of dependent/independent variables. Both vectors now have components with mixed physical significance, some being stresses and some being strains. Otherwise these equations are completely equivalent to the original ones.

Performing the layer average, assuming the variation is along the z or x_3 direction, I find, using the notation of (6),

$$\begin{pmatrix} <\sigma_{11}> \\ <\sigma_{22}> \\ -<e_{33}> \\ <e_{23}> \\ <e_{31}> \\ <\sigma_{12}> \end{pmatrix} = \begin{pmatrix} \left\langle\frac{4\lambda(\lambda+\mu)}{\lambda+2\mu}\right\rangle & \left\langle\frac{2\lambda\mu}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & & & \\ \left\langle\frac{2\lambda\mu}{\lambda+2\mu}\right\rangle & \left\langle\frac{4\lambda(\lambda+\mu)}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & & & \\ \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & -\left\langle\frac{1}{\lambda+2\mu}\right\rangle & & & \\ & & & \left\langle\frac{1}{2\mu}\right\rangle & & \\ & & & & \left\langle\frac{1}{2\mu}\right\rangle & \\ & & & & & \langle 2\mu\rangle \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}$$

$$
= \begin{pmatrix}
a - f^2/c & b - f^2/c & f/c & & & \\
b - f^2/c & a - f^2/c & f/c & & & \\
f/c & f/c & -1/c & & & \\
 & & & 1/2l & & \\
 & & & & 1/2l & \\
 & & & & & 2m
\end{pmatrix}
\begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}, \tag{27}
$$

which can then be solved to yield the expressions

$$
a = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} + 4 \left\langle \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} \right\rangle, \tag{28}
$$

$$
b = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} + 2 \left\langle \frac{\lambda\mu}{\lambda + 2\mu} \right\rangle, \tag{29}
$$

$$
c = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \tag{30}
$$

$$
f = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}, \tag{31}
$$

$$
l = \left\langle \frac{1}{\mu} \right\rangle^{-1} \tag{32}
$$

and

$$
m = \langle \mu \rangle . \tag{33}
$$

One very important fact that is known about these equations is that they reduce to isotropic results with $a = c$, $b = f$, and $l = m$, if the shear modulus is a constant, regardless of the behavior of λ.

REVIEW OF KNOWN INEQUALITIES FOR ELASTIC CONSTANTS

Since the stress-strain relation (6) is derivable from an energy functional, it is not hard to show that the matrix must be nonnegative or the material will be mechanically unstable. Nonnegativity of the matrix implies that all its principal minors must be nonnegative, which in turn implies the following inequalities:

$$
a = b + 2m \geq 0, \qquad c \geq 0, \tag{34}
$$

$$
l \geq 0, \qquad m \geq 0, \tag{35}
$$

and

$$(a^2 - b^2)/4m = b + m \geq 0, \qquad ac - f^2 \geq 0, \tag{36}$$

and

$$\left[a(ac - 2f^2) - b(bc - 2f^2)\right]/4m = (b+m)c - f^2 \geq 0. \tag{37}$$

The second inequality in (36) follows from (37), (35), and the second inequality in (34) and is therefore often omitted from such listings. Similarly, the inequality for a follows from those for m and $b + m$. All of these inequalities must be satisfied regardless of the source of the anisotropy.

The formulas (28)-(33) can be used to derive some very simple relations among the constants. For example,

$$c \geq f \tag{38}$$

follows directly from (30) and (31), simply noting that $\lambda/(\lambda + 2\mu) \leq 1$ in every layer. The inequality

$$c \geq \frac{4}{3}l \tag{39}$$

is derived directly from the fact that

$$\left\langle \frac{1}{\lambda + 2\mu} \right\rangle \leq \left\langle \frac{1}{4\mu/3} \right\rangle = \frac{3}{4}\left\langle \frac{1}{\mu} \right\rangle, \tag{40}$$

which follows from the fact that the bulk modulus must be nonnegative in each layer so that $\lambda + 2\mu/3 \geq 0$ everywhere. Then, the two shear moduli must satisfy

$$l \leq m \tag{41}$$

since

$$1 \leq \langle \mu \rangle \left\langle \frac{1}{\mu} \right\rangle. \tag{42}$$

follows easily from the well-known Cauchy-Schwartz inequality $\langle \alpha\beta \rangle^2 \leq \langle \alpha^2 \rangle \langle \beta^2 \rangle$ by setting $\alpha = \mu^{1/2}$ and $\beta = 1/\mu^{1/2}$. Equality applies in the Cauchy-Schwartz inequalities only when $\alpha = const \times \beta$, which implies in the present circumstances that μ must be constant for $l = m$. But this is precisely the condition mentioned earlier for the layer equations to be isotropic, so I generally exclude this case from consideration.

Another inequality can be derived from the formulas obtained for finely layered media. I showed earlier that the anellipticity parameter given by

$$A \equiv (a - l)(c - l) - (f + l)^2 \tag{43}$$

has the property that the dispersion relations for both qP- and qSV-waves are simple ellipses when $A = 0$ and are anelliptic otherwise. Using the results (28)-(33), A is shown to satisfy $A > 0$ for any fine layered transversely isotropic medium by noting that

$$A = cl\left[\left\langle\mu\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\right\rangle\left\langle\frac{1}{\mu}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\right\rangle - \left\langle\frac{\lambda+\mu}{\lambda+2\mu}\right\rangle^2\right] \geq 0. \tag{44}$$

The inequality follows again from the Cauchy-Schwartz inequality in another form somewhat more transparent for the present application

$$\langle\alpha\rangle^2 = \left\langle(\mu\alpha)^{1/2}(\alpha/\mu)^{1/2}\right\rangle \leq \langle\mu\alpha\rangle\langle\alpha/\mu\rangle\,. \tag{45}$$

Equality again applies only when μ is identically constant. But, it was mentioned earlier that finely layered media are isotropic if μ does not vary, so $A > 0$ holds for all such finely layered media if their overall constants are anisotropic. The result (44) was first obtained by Postma (1955) for two-component layered materials and later by Berryman (1979) for the multicomponent case considered here.

RANGE OF THE ANISOTROPY PARAMETERS

From (15), I know that the SH-wave in finely layered VTI media has an elliptical surface for velocity squared. Furthermore,

$$m/l = \langle\mu\rangle\left\langle\frac{1}{\mu}\right\rangle \geq 1 \tag{46}$$

follows from (32), (33), and (42). So the horizontal shear wave velocity for SH-waves is always greater than or equal to the vertical velocity. I define the ratio m/l to be the SH-wave anisotropy parameter, and have the simple universal result that this parameter is always greater than or equal to unity.

The qP-wave does not always have an elliptical dispersion relation, but it is nevertheless always true that if $k_3 = 0$ then $\rho\omega_+^2 = ak_1^2$ and if $k_1 = 0$ then $\rho\omega_+^2 = ck_3^2$. Thus, I may reasonably define the P-wave anisotropy parameter to be a/c and seek to determine what the range of this parameter might be. Formula (28) for a may be rewritten as

$$a = \left\langle\frac{(\lambda+2\mu)^2-\lambda^2}{\lambda+2\mu}\right\rangle + \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^2\left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}, \tag{47}$$

which can be rearranged into the convenient and illuminating form

$$a = \langle\lambda+2\mu\rangle - \left[\left\langle\frac{\lambda^2}{\lambda+2\mu}\right\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^2\right]\left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}. \tag{48}$$

This formula is very instructive because the term in square brackets is again in Cauchy-Schwartz form, so this factor is nonnegative. Furthermore, the magnitude of this term depends principally on the fluctuations in the λ Lamé constant, largely independent of μ. Clearly, if $\lambda = constant$, then this factor vanishes identically, regardless of the behavior of μ. Large fluctuations in λ will tend to make this term large. If in addition I now consider the combination

$$\frac{a}{c} - 1 = \left[\langle \lambda + 2\mu \rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - 1 \right] - \left[\left\langle \frac{\lambda^2}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \right], \quad (49)$$

I find that the first bracket on the right hand side is again in Cauchy-Schwartz form showing that it always makes a positive contribution unless $\lambda + 2\mu = constant$, in which case it vanishes. Similarly, the second term always makes a negative contribution unless $\lambda = constant$, in which case it vanishes.

If the finely layered medium is composed of only two distinct types of isotropic elastic materials and they appear in the layering sequence with equal spatial frequency, then I find that

$$\frac{a}{c} - 1 = (\mu_2 - \mu_1) \frac{(\lambda_2 - \lambda_1) + (\mu_2 - \mu_1)}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}. \quad (50)$$

This result agrees with Postma (1955) except for an obvious typographical error in the denominator of his published formula. This formula show clearly that if $\mu_1 = \mu_2$ then the P-wave anisotropy parameter is identically equal to unity as expected. Also, if $\mu_1 \neq \mu_2$ but $\lambda_1 = \lambda_2$, then (50) implies $a/c \geq 1$, as I inferred from (49).

Now, I can use this formula to deduce the smallest possible value of the right hand side of (50). The shear moduli must not be equal, so without loss of generality I suppose that $\mu_2 > \mu_1$. Then, the numerator is seen to become negative by taking λ_2 towards negative values and $\lambda_1 \to +\infty$. The smallest value λ_2 can take is determined by the bulk modulus bound $\lambda_2 + \frac{2}{3}\mu_2 \geq 0$. So we may set $\lambda_2 = -\frac{2}{3}\mu_2$ in both the numerator and denominator. This choice also makes the factor $\lambda_2 + 2\mu_2 = \frac{4}{3}\mu_2$ as small as possible in the denominator, thus helping to magnify the effect of the negative numerator as much as possible. The result so far is that

$$\frac{a}{c} - 1 = \frac{3}{4} \left(\frac{\mu_2 - \mu_1}{\mu_2} \right) \left(\frac{-\lambda_1 + \mu_2/3 - \mu_1}{\lambda_1 + 2\mu_1} \right) \quad (51)$$

The parameter λ_1 may vary from $-\frac{2}{3}\mu_1$ to plus infinity. At $\lambda_1 = -\frac{2}{3}\mu_1$, the second expression in parentheses is positive, But, this expression is also a monotonically decreasing function of λ_1 and approaches -1 as $\lambda_1 \to +\infty$. Thus, the smallest value of the P-wave anisotropy is given by

$$\frac{a}{c} = 1 - \frac{3}{4} \frac{\mu_2 - \mu_1}{\mu_2} \geq \frac{1}{4}. \quad (52)$$

This result differs by a factor of 2 from the corresponding result of Postma (1955), which was obtained improperly by allowing three of the four elastic constants to vanish and also using a physically motivated but unnecessary restriction that both λ_1 and λ_2 must be nonnegative. If I had used the nonnegativity constraint on the λ's, the present result would have changed to

$$\frac{a}{c} = 1 - \frac{\mu_2 - \mu_1}{2\mu_2} \geq \frac{1}{2}, \tag{53}$$

which is the same *inequality* as that found by Postma, but his *equality* differed from that in (53) and was in fact improperly obtained.

As a final point about the formula (48), I note that it implies in general that

$$a \leq \langle \lambda + 2\mu \rangle \,, \tag{54}$$

so I have a general upper bound on the P-wave anisotropy parameter stating that

$$\frac{a}{c} \leq \langle \lambda + \mu \rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle . \tag{55}$$

Before concluding this section, I want to note one further identity for the P-wave anisotropy parameter. The general formula can be rearranged to give

$$\frac{a}{c} - 1 = 4 \left[\left\langle \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\lambda + \mu}{\lambda + 2\mu} \right\rangle \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle \right] . \tag{56}$$

This formula is not in Cauchy-Schwartz form, but is nevertheless probably the simplest form of the result for this anisotropy parameter. In particular, it is easy to see from this form that if either $\mu = constant$ or $\lambda + \mu = constant$, then the right hand side vanishes identically. The first result is well-known and the second has been known since Postma's (1955) work to be true for two-constituent periodic layered media [also see (50)]. The present result generalizes Postma's observation in this case.

DISCUSSION

The results obtained here and in particular (48) and (49) show in general terms that the P-wave anisotropy parameter is smallest when the variation in the layer λ Lamé parameter is large, independent of the variation in the shear modulus μ. This result is therefore important for applications to porous layers containing pore fluids for which Gassmann's (1951) equation shows that the effects of fluids influence only the λ Lamé constants, not μ. Thus, the P-wave anisotropy parameter may be a useful hydrocarbon indicator in some situations. This result cannot, however, be considered in isolation, as there may be other effects that produce small P-wave anisotropies, and these should be explored and enumerated before any definitive statement can be made.

ACKNOWLEDGMENTS

I thank Patricia Berge for helpful technical discussions and for numerous pointers to important references on anisotropy. I also thank Matthias Schwab and Hector Urdaneta for helpful comments on the manuscript.

REFERENCES

Backus, G. E., 1962, Long-wave elastic anisotropy produced by horizontal layering: J. Geophys. Res., **67**, 4427–4440.

Berryman, J. G., 1979, Long-wave elastic anisotropy in transversely isotropic media: Geophysics, **44**, 896–917.

Gassmann, F., 1951, Über die Elastizität poroser Medien: Veirteljahrsschrift der Naturforschenden Gesellschaft in Zürich, **96**, 1–23.

Postma, G. W., 1955, Wave propagation in a stratified medium: Geophysics, **20**, 780–806.