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Stacking operators: Adjoint versus asymptotic inverse

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ABSTRACT

The paper addresses the theory of stacking operators used in seismic data processing. I compare the notion of asymptotically inverse operators with the notion of adjoint operators. These two classes of operators share the same kinematic properties, but their amplitudes (weighting functions) are defined differently. I introduce the notion of the *asymptotic pseudo-unitary* operator, which possesses both the property of being adjoint and the property of being asymptotically inverse. The weighting function of the asymptotic pseudo-unitary stacking operator is completely defined by its kinematics. I exemplify the general theory by considering such stacking operators as Kirchhoff datuming, migration, offset continuation, DMO, and velocity transform.

INTRODUCTION

Integral (stacking) operators play a very important role in seismic data processing. The most common applications are common midpoint stacking, Kirchhoff-type migration, and dip moveout. Other examples include (listed in random order) Kirchhoff-type datuming, back-projection tomography, slant stack, velocity transform, offset continuation, and azimuth moveout (AMO). The role of the integral methods increases with the development of prestack three-dimensional processing because they appear flexible toward irregularities in the data geometry.

Often an integral operator represents the forward modeling problem, and we need to invert it to solve for the model. In this paper, I consider two different approaches to inversion. The first is least-square inversion, which requires constructing the adjoint counterpart of the modeling operator. The second approach is asymptotic inversion, which aims to reconstruct the high-frequency (discontinuous) parts of the model. I compare the two approaches and introduce the notion of what I call the *asymptotic pseudo-unitary operator* to tie them together.

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The first part of this paper contains a formal definition of a stacking operator and reviews the theory of asymptotic inversion, following the fundamental results of Beylkin (1985) and Goldin (?). According to this theory, the high-frequency asymptotic inverse of a stacking operator is also a stacking operator with a different summation path and weighting. To connect this theory with the theory of adjoint operators, I prove that the adjoint of a stacking operator can also be included in the class of stacking operators. The stacking ("pull") adjoint has the same summation path as the asymptotic inverse but a different weighting function. These two results combine together to form the theory of asymptotic pseudo-unitary integral operators. I apply this theory to define a general preconditioning operator for least-square inversion.

Finally, I consider such examples of commonly used stacking operators as wave-equation datuming, migration, velocity transform, and offset continuation.

THEORETICAL DEFINITION OF A STACKING OPERATOR

In practice, integration of discrete data is performed by stacking, which requires special caution in the case of spatial aliasing (Claerbout, 1992). In theory, it is convenient to represent a stacking operator in the form of a continuous integral:

$$S(t,y) = \mathbf{A}\left[M(z,x)\right] = \int w(x;t,y)\,M(\theta(x;t,y),x)\,dx\;. \tag{1}$$

Function $M(z,x)$ is the input of the operator, $S(t,y)$ is the output, θ represents the summation path, and w stands for the weighting function. The range of integration (the operator aperture) may also depend on t and y. Allowing x to be a two-dimensional variable, we can use definition (1) to represent an operator applied to three-dimensional data. Throughout this paper, I assume that t and z belong to a one-dimensional space, and that x and y have the same number of dimensions.

The goal of inversion is to reconstruct some function $\widehat{M}(z,x)$ for a given $S(t,y)$, so that $\widehat{M}$ is in a particular sense close to M in equation (1).

ASYMPTOTIC INVERSION: RECONSTRUCTING THE DISCONTINUITIES

Mathematical analysis of the inverse problem for operator (1) shows that only in rare cases can we obtain a theoretically exact inversion. A well-known example is the Radon transform, which has acquired a lot of different aliases in geophysical literature: slant stack, tau-p transform, plane wave decomposition, and controlled directional reception (CDR) transform (Gardner and Lu, 1991). In this case,

$$\begin{aligned} \theta(x;t,y) &= t + x\,y\;, && (2)\\ w(x;t,y) &= 1\;. && (3) \end{aligned}$$

Radon obtained a result similar to the theoretical inversion of operator (1) with the summation path (2) and the weighting function (3) in 1917, but this result was not widely known until the development of computer tomography. According to Radon (1917), the inverse operator has the form

$$M(z,x) = \mathbf{A^{-1}}[S(t,y)] = |\mathbf{D}|^m \int \widehat{w}\, S(\widehat{\theta}(y;z,x),y)\, dy \;, \tag{4}$$

where

$$\begin{aligned} \widehat{\theta}(y;z,x) &= z - x\,y \;, & (5)\\ \widehat{w} &= \frac{1}{(2\,\pi)^m} \;, & (6) \end{aligned}$$

$|\mathbf{D}|$ is a one-dimensional convolution operator with the spectrum $|\omega|$ (the *rho filter*), and m is the dimensionality of x and y (usually 1 or 2). In Russian geophysical literature, a similar result for the inversion of the CDR transform was published by Nakhamkin (1969).

Extension of Radon's result to the general form of integral operator (1) (*generalized Radon transform*) is possible through asymptotic analysis of the inverse problem. In the general case, it was shown (Beylkin, 1985; Goldin, 1988) that asymptotic inversion can reconstruct discontinuous parts of the model. These are the parts responsible for the asymptotic behavior of the model at high frequencies. Since the discontinuities are associated with wavefronts and reflection events at seismic sections, there is a certain correspondence between asymptotic inversion and such standard goals of seismic data processing as kinematic equivalence and amplitude preservation.

The main theorem of asymptotic inversion can be formulated as follows (Goldin, 1988). Main (leading-order) discontinuities in M are reconstructed by an integral operator of the form

$$\widehat{M}(z,x) = \widehat{\mathbf{A}}[S(t,y)] = |\mathbf{D}|^m \int \widehat{w}(y;z,x)\, S(\widehat{\theta}(y;z,x),y)\, dy \;, \tag{7}$$

where the summation path $\widehat{\theta}$ is obtained simply by solving the equation

$$z = \theta(x;t,y) \tag{8}$$

for t (if such an explicit solution is possible). The correctly chosen summation path reconstructs the geometry of the discontinuities. To recover the amplitude, we must choose the correct weighting function, which is constrained by the equation

$$w\,\widehat{w} = \frac{1}{(2\,\pi)^m} \sqrt{\left|F\,\widehat{F}\right| \left|\frac{\partial \widehat{\theta}}{\partial z}\right|^m} \;, \tag{9}$$

where

$$\begin{aligned} F &= \frac{\partial \theta}{\partial t}\,\frac{\partial^2 \theta}{\partial x\,\partial y} - \frac{\partial \theta}{\partial y}\,\frac{\partial^2 \theta}{\partial x\,\partial t} \;, & (10)\\ \widehat{F} &= \frac{\partial \widehat{\theta}}{\partial z}\,\frac{\partial^2 \widehat{\theta}}{\partial x\,\partial y} - \frac{\partial \widehat{\theta}}{\partial x}\,\frac{\partial^2 \widehat{\theta}}{\partial y\,\partial z} \;. & (11) \end{aligned}$$

The solution assumes that differential forms F and $\widehat{F}$ exist and are bounded and non-vanishing. In the multi-dimensional case ($m \geq 2$), they are replaced by the determinants of the corresponding matrices. To ensure the asymptotic inversion, equation (9) must be satisfied at least in the vicinity of the *stationary points* of integral (1). Those are the points where the summation path of the form (8) is tangent to the traveltimes of the actual events on the transformed model.

In the case of the Radon transform, $\left|F\,\widehat{F}\right| = \left|\frac{\partial \widehat{\theta}}{\partial z}\right| = 1$, and the asymptotic inverse coincides with the exact inversion.

PULL ADJOINTS

The least-square (generalized) inverse of operator (1) has the famous form

$$\widetilde{M}(z,x) = \widetilde{\mathbf{A}}[S(t,y)] = \left(\mathbf{A^T}\,\mathbf{A}\right)^{-1}\,\mathbf{A^T}[S(t,y)]\;, \tag{12}$$

where the adjoint operator $\mathbf{A^T}$ is defined by the dot-product test:

$$(S(t,y), \mathbf{A}[M(z,x)]) \equiv \left(\mathbf{A^T}[S(t,y)], M(z,x)\right)\;. \tag{13}$$

With a specified definition of the dot-product, the generalized inverse minimizes the following quantity, which is the squared L_2 norm of the residual:

$$(S(t,y) - \mathbf{A}[M(z,x)], S(t,y) - \mathbf{A}[M(z,x)])\;. \tag{14}$$

In the case of integral operators, a natural definition of the dot-product is the double integral

$$(S_1(t,y), S_2(t,y)) = \int\!\!\int S_1(t,y)\,S_2(t,y)\,dy\,dt\;, \tag{15}$$

$$(M_1(z,x), M_2(z,x)) = \int\!\!\int M_1(z,x)\,M_2(z,x)\,dx\,dz\;. \tag{16}$$

What is the adjoint of the integral operator (1) in this case? In the discrete world, where stacking is represented by a row vector, the adjoint (transpose) of a summation matrix is a column vector. In other words, the adjoint of collecting the input data along the stacking curve trajectory and summing it into an individual output bin is dividing the output bin into a number of portions sprayed along the specified trajectory. Claerbout (1995) calls the stacking operator a “pull” and its adjoint a “push”.

The relationship between forward and adjoint operators is different in the continuous world. Let us substitute the definition of the stacking operator (1) into the dot product (13), as follows:

$$(S(t,y), \mathbf{A}[M(z,x)]) = \int\!\!\int\!\!\int w(x;t,y)\,M(\theta(x;t,y),x)\,S(t,y)\,dx\,dy\,dt\;. \tag{17}$$

Changing the integration variable t to $z = \theta(x;t,y)$, we can rewrite (17) in the form

$$(S(t,y), \mathbf{A}[M(z,x)]) = \int\int\int \widetilde{w}(y;z,x)\, M(z,x)\, S(\widehat{\theta}(y;z,x),x)\, dy\, dx\, dz \;, \tag{18}$$

where $\widehat{\theta}$ has the same meaning as in equation (7), and

$$\widetilde{w}(y;z,x) = w(x;\widehat{\theta}(y;z,x),y) \left|\frac{\partial\widehat{\theta}}{\partial z}\right| \;. \tag{19}$$

Comparing formulas (18) and (13), we conclude that the adjoint operator $\mathbf{A^T}$ is defined by the equality

$$\mathbf{A^T}[S(t,y)] = \int \widetilde{w}(y;z,x)\, S(\widehat{\theta}(y;z,x),y)\, dy \;. \tag{20}$$

Thus we have proven that in the continuous world the adjoint of a stacking operator is another stacking operator. The adjoint operator has the same summation path as the asymptotic inverse (7), which guarantees the correct reconstruction of the kinematics of the input wavefield. The amplitude (weighting function) of the adjoint operator is directly proportional to the forward weighting according to equation (19). The coefficient of proportionality is the Jacobian of the transformation of the variables z and t.

Similar results have been published for particular cases of stacking operators: velocity transform (Thorson, 1984; Jedlicka, 1989), Kirchhoff constant-velocity migration (Ji, 1994b), and NMO (Crawley, 1995).

To exemplify the application of a "pull" adjoint to inversion, let us consider the case of the Radon transform from the preceding section. Forming the product $\mathbf{A^T\,A}$ for this case leads to the double integral

$$\begin{aligned} H(z,x) &= (\mathbf{A^T\,A})[M(z,x)] = \\ &= \int\int \widehat{w}(y;z,x)\, w\left(\xi;\widehat{\theta}(y;z,x),y\right)\, M\left(\theta\left(\xi;\widehat{\theta}(y;z,x),y\right),\xi\right)\, d\xi\, dy = \\ &= \int\int M\left(z + y\,(\xi - x)\right)\, d\xi\, dy \;. \end{aligned} \tag{21}$$

Applying Fourier transform with respect to z, we can rewrite equation (21) in the frequency domain as

$$\check{H}(\omega,x) = \int \check{M}(\omega,\xi) \int e^{i\omega\,y\,(\xi-x)}\, dy\, d\xi \;, \tag{22}$$

where

$$\check{H}(\omega,x) = \int H(z,x)\, e^{-i\omega\,z}\, dz \;, \tag{23}$$

$$\check{M}(\omega,x) = \int M(z,x)\, e^{-i\omega\,z}\, dz \;. \tag{24}$$

The inner integral in equation (22) reduces to the m-dimensional delta function:

$$\check{H}(\omega,x) = (2\,\pi)^m \int \check{M}(\omega,\xi)\,\delta\left(\omega^m\,(\xi - x)\right)\,d\xi\;. \tag{25}$$

As follows from the properties of delta function,

$$\check{H}(\omega,x) = \frac{(2\,\pi)^m}{|\omega|^m} \int \check{M}(\omega,\xi)\,\delta(\xi - x)\,d\xi \;=\; \frac{(2\,\pi)^m}{|\omega|^m}\,\check{M}(\omega,x)\;. \tag{26}$$

Inverting (26) for M, we conclude that

$$\left(\mathbf{A^T}\,\mathbf{A}\right)^{\mathbf{-1}} = \frac{|\mathbf{D}|^m}{(2\,\pi)^m}\;. \tag{27}$$

Substituting equation (27) into (12) produces the result precisely equivalent to Radon's inversion (4).

The SEPlib canonical library contains various examples of stacking operators coupled with their adjoint counterparts. In practice, discrete "push" adjoints provide the machine-precise accuracy of the discrete dot-product test. The "pull" adjoints defined in this section cannot compete in precision because of round-off errors. However, their practical use can be justified for the purpose of a "smoother" output. Claerbout (1995) and Crawley (1995) discuss this possibility in more detail.

The notion of the adjoint operator completely depends on the arbitrarily chosen definition of the dot product and norm in the model and data spaces. A simple way to change those definitions is to find some positive weights $W_M(z,x)$ in the model space and $W_S(t,y)$ in the data space that define the dot products as follows:

$$\left(S_1(t,y), S_2(t,y)\right) \;=\; \int\!\!\int W_S(t,y)\,S_1(t,y)\,S_2(t,y)\,dy\,dt\;, \tag{28}$$

$$\left(M_1(z,x), M_2(z,x)\right) \;=\; \int\!\!\int W_M(z,x)\,M_1(z,x)\,M_2(z,x)\,dx\,dz\;. \tag{29}$$

ASYMPTOTIC PSEUDO-UNITARY OPERATOR

According to the theory of asymptotic inversion, briefly reviewed in the first part of this paper, the weighting function of the asymptotically inverse operator is *inversely* proportional to the weighting of the forward operator. On the other hand, the weighting in the "pull" adjoint is *directly* proportional to the forward weighting. This difference allows us to define a hybrid type of operator, which possesses both the property of being adjoint and the property of being asymptotic inverse. It is appropriate to call a pair of operators defined in this way *asymptotic pseudo-unitary*. The definition of asymptotic pseudo-unitary operators follows directly from the combination of definitions (7) and (20). Splitting the derivative operator $|\mathbf{D}|$ in (7) into the product of two operators, we can write the forward operator as

$$S(t,y) = \mathbf{A}\left[M(z,x)\right] = \int w^{(+)}(x;t,y)\,|\mathbf{D}|^{m/2}\,M(\theta(x;t,y),x)\,dx \tag{30}$$

and its asymptotic pseudo-unitary adjoint as

$$\widetilde{M}(z,x) = \widetilde{\mathbf{A}}[S(t,y)] = |\mathbf{D}|^{m/2} \int w^{(-)}(y;z,x)\, S(\widehat{\theta}(y;z,x),y)\, dy \; . \tag{31}$$

According to equation (9),

$$w^{(+)}\, w^{(-)} = \frac{1}{(2\,\pi)^m} \sqrt{\left|F\,\widehat{F}\right| \left|\frac{\partial\widehat{\theta}}{\partial z}\right|^m} \; . \tag{32}$$

According to equation (19),

$$w^{(-)} = w^{(+)} \left|\frac{\partial\widehat{\theta}}{\partial z}\right| \; . \tag{33}$$

Combining equations (32) and (33) uniquely determines both weighting functions, as follows:

$$w^{(+)} = \frac{1}{(2\,\pi)^{m/2}} \left|F\,\widehat{F}\right|^{1/4} \left|\frac{\partial\widehat{\theta}}{\partial z}\right|^{(m-2)/4} \; , \tag{34}$$

$$w^{(-)} = \frac{1}{(2\,\pi)^{m/2}} \left|F\,\widehat{F}\right|^{1/4} \left|\frac{\partial\widehat{\theta}}{\partial z}\right|^{(m+2)/4} \; . \tag{35}$$

Equations (34) and (35) complete the definition of asymptotic pseudo-unitary operators.

The notion of pseudo-unitary operators is directly applicable in the situations where we can arbitrarily construct both forward and inverse operators. One example of such a situation is the velocity transform considered in the next section of this paper. In the more common case, the forward operator is strictly defined by the physics of a problem. In this case, we can include asymptotic inversion in the iterative least-square inversion by means of *preconditioning.* The linear preconditioning operator should transform the forward stacking-type operator to the form (30) with the weighting function (34). Theoretically, this form of preconditioning leads to the fastest convergence of the iterative least-square inversion with respect to the high-frequency parts of the model.

EXAMPLES

In this section, I consider several particular examples of stacking operators used in seismic data processing and derive their asymptotic pseudo-unitary versions.

Datuming

Let x denote a point on the surface at which the propagating wavefield is recorded. Let y denote a point on another surface, to which the wavefield is propagating. Then the summation path of the stacking operator for the forward wavefield continuation is

$$\theta(x;t,y) = t - T(x,y) \; , \tag{36}$$

where t is the time recorded at the y-surface, and $T(x,y)$ is the traveltime along the ray connecting x and y. The backward propagation reverses the sign in (36), as follows:

$$\widehat{\theta}(y;z,x) = z + T(x,y) \; . \tag{37}$$

Substituting the summation path formulas (36) and (37) into the general weighting function formulas (34) and (35), we immediately obtain

$$w^{(+)} = w^{(-)} = \frac{1}{(2\,\pi)^{m/2}} \left|\frac{\partial^2 T}{\partial x\, \partial y}\right|^{1/2} \; . \tag{38}$$

Gritsenko's formula (Gritsenko, 1984; Goldin, 1986) states that the second mixed traveltime derivative $\frac{\partial^2 T}{\partial x\, \partial y}$ is connected with the geometric spreading R along the x-y ray by the equality

$$R(x,y) = \frac{\sqrt{\cos\alpha(x)\, \cos\alpha(y)}}{v(x)} \left|\frac{\partial^2 T}{\partial x\, \partial y}\right|^{-1/2} \; , \tag{39}$$

where $v(x)$ is the velocity at the point x, and $\alpha(x)$ and $\alpha(y)$ are the angles formed by the ray with the x and y surfaces, respectively. In a constant-velocity medium,

$$R(x,y) = v^{m-1}\, T(x,y)^{m/2} \; . \tag{40}$$

Gritsenko's formula (39) allows us to rewrite equation (38) in the form (Goldin, 1988)

$$w^{(+)}(x;t,y) = \frac{1}{(2\,\pi)^{m/2}} \frac{\sqrt{\cos\alpha(x)\, \cos\alpha(y)}}{v(x)\, R(x,y)} \; , \tag{41}$$

$$w^{(-)}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}} \frac{\sqrt{\cos\alpha(x)\, \cos\alpha(y)}}{v(y)\, R(y,x)} \; . \tag{42}$$

The weighting functions commonly used in Kirchhoff datuming (Berryhill, 1979; Wiggins, 1984; Goldin, 1985) are defined as

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}} \frac{\cos\alpha(x)}{v(x)\, R(x,y)} \; , \tag{43}$$

$$\widehat{w}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}} \frac{\cos\alpha(y)}{v(y)\, R(y,x)} \; . \tag{44}$$

These two operators appear to be asymptotically inverse according to formula (9). They coincide with the asymptotic pseudo-unitary operators if the velocity v is constant ($v(x) = v(y)$), and the two datum surfaces are parallel ($\alpha(x) = \alpha(y)$).

Migration

As recognized recently by Tygel et al. (1994), *true-amplitude* migration (Goldin, 1992; Schleicher et al., 1993) is the asymptotic inversion of seismic modeling represented by the Kirchhoff high-frequency approximation. The Kirchhoff approximation for a reflected wave (Haddon and Buchen, 1981; Bleistein, 1984) belongs to the class of stacking-type operators (1) with the summation path

$$\theta(x;t,y) = t - T\left(s(y),x\right) - T\left(x,r(y)\right) \ , \tag{45}$$

the weighting function

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{C\left(s(y),x,r(y)\right)}{R\left(s(y),x\right)\;R\left(x,r(y)\right)} \ , \tag{46}$$

and the additional time filter $\left(\frac{\partial}{\partial z}\right)^{m/2}$. Here x denotes a point at the reflector surface, s is the source location, and r is the receiver location at the observation surface. The parameter y corresponds to the configuration of observation. That is, $s(y) = s\,,\ r(y) = y$ for the common-shot configuration, $s(y) = r(y) = y$ for the zero-offset configuration, and $s(y) = y - h\,,\ r(y) = y + h$ for the common-offset configuration (where h is the half-offset). The functions T and R have the same meaning as in the datuming example, representing the one-way traveltime and the one-way geometric spreading, respectively. The function $C(s,x,r)$ is known as the *obliquity factor*. Its definition is

$$C(s,x,r) = \frac{1}{2}\left(\frac{\cos\alpha_s(x)}{v_s(x)} + \frac{\cos\alpha_r(x)}{v_r(x)}\right) \ , \tag{47}$$

where the angles $\alpha_s(x)$ and $\alpha_r(x)$ are formed by the incident and reflected waves with the normal to the reflector at the point x, and $v_s(x)$ and $v_r(x)$ are the corresponding velocities in the vicinity of this point. In this paper, I leave the case of converted (e.g., P-SV) waves outside the scope of consideration and assume that $v_s(x)$ equals $v_r(x)$ (e.g., in P-P reflection). In this case, it is important to notice that at the stationary point of the Kirchhoff integral, $\alpha_s(x) = \alpha_r(x) = \alpha(x)$ (the law of reflection), and therefore

$$C(s,x,r) = \frac{\cos\alpha(x)}{v(x)} \ . \tag{48}$$

The stationary point of the Kirchhoff integral is the point where the stacking curve (45) is tangent to the actual reflection traveltime curve. When our goal is asymptotic inversion, it is appropriate to use equation (48) in place of (47) to construct the inverse operator. The weighted function (46) can include other factors affecting the leading-order (WKBJ) ray amplitude, such as the source signature, caustics counter (the KMAH-index), and transmission coefficient for the interfaces (Červený et al., 1977; Chapman and Drummond, 1982). In the following analysis, I neglect these factors for simplicity.

The model M implied by the Kirchhoff modeling integral is the wavefield with the wavelet shape of the incident wave and the amplitude proportional to the reflector coefficient along the reflector surface. The goal of true-amplitude migration is to recover M from the observed seismic data. In order to obtain the image of the reflectors, the reconstructed model is evaluated at the time z equal to zero. The Kirchhoff modeling integral requires explicit definition of the reflector surface. However, its inverse doesn't require explicit specification of the reflector location. For each point of the subsurface, one can find the normal to the hypothetical reflector by bisecting the angle between the $s-x$ and $x-r$ rays. Born scattering approximation provides a different physical model for the reflected waves. According to this approximation, the recorded waves are viewed as scattered on smooth local inhomogeneities rather than reflected from sharp reflector surfaces. The inversion of Born modeling (Miller et al., 1987; Bleistein, 1987) closely corresponds with the result of Kirchhoff integral inversion. For an unknown reflector and the correct macro-velocity model, the asymptotic inversion reconstructs the signal located at the reflector surface with the amplitude proportional to the reflector coefficient.

As follows from the form of the summation path (45), the integral migration operator must have the summation path

$$\widehat{\theta}(y;z,x) = z + T\left(s(y),x\right) + T\left(x,r(y)\right) \tag{49}$$

to reconstruct the geometry of the reflector at the migrated section. According to (7), the asymptotic reconstruction of the wavelet requires, in addition, the derivative filter $\left(-\frac{\partial}{\partial t}\right)^{m/2}$. The asymptotic reconstruction of the amplitude defines the true-amplitude weighting function in accordance with (9), as follows:

$$\widehat{w}(y;z,x) = \frac{v(x)\,R\left(s(y),x\right)\;R\left(x,r(y)\right)}{(2\,\pi)^{m/2}\;\cos\alpha(x)}\left|\frac{\partial^2 T\left(s(y),x\right)}{\partial x\,\partial y} + \frac{\partial^2 T\left(x,r(y)\right)}{\partial x\,\partial y}\right| . \tag{50}$$

In the case of common-shot migration, we can simplify equation (50) with the help of Gritsenko's formula (39) to the form

$$\widehat{w}_{CS}(r;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(r)}{v(x)}\,\frac{R(s,x)}{R(x,r)} = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(r)}{v(r)}\,\frac{R(s,x)}{R(r,x)}\,, \tag{51}$$

where the angle $\alpha(r)$ is measured between the reflected ray and the normal to the observation surface at the reflector point r. Formula (51) coincides with the analogous result of Keho and Beydoun (1988), derived directly from Claerbout's imaging principle (Claerbout, 1970). An alternative derivation is given by Goldin (1987). Docherty (1991) points out a remarkable correspondence between this formula and the classic results of Born scattering inversion (Bleistein, 1987).

In the case of zero-offset migration, Gritsenko's formula simplifies the true-amplitude migration weighting function (50) to the form

$$\widehat{w}_{ZO}(y;z,x) = \frac{2^m}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(y)}{v(y)}\,. \tag{52}$$

In a constant-velocity medium, one can accomplish the true-amplitude zero-offset migration by premultiplying the recorded zero-offset seismic section by the factor

$\left(\frac{v}{2}\right)^{m-1} \left(\frac{t}{2}\right)^{m/2}$ [which corresponds at the stationary point to the geometric spreading $R(x,y)$] and downward continuation according to formula (44) with the effective velocity $v/2$ (Goldin, 1987; Hubral et al., 1991). This conclusion is in agreement with the analogous result of Born inversion (Bleistein et al., 1985), though derived from a different viewpoint.

In the case of common-offset migration in a general variable-velocity medium, the weighting function (50) cannot be simplified to a different form, and all its components need to be calculated explicitly by dynamic ray tracing (Červený and de Castro, 1993). In the constant-velocity case, we can differentiate the explicit expression for the summation path

$$\widehat{\theta}(y;z,x) = z + \frac{\rho_s(x,y) + \rho_r(x,y)}{v} \;, \tag{53}$$

where ρ_s and ρ_r are the lengths of the incident and reflected rays:

$$\rho_s(y,x) \;=\; \sqrt{x_3^2 + (x_1 - y_1 + h_1)^2 + (x_2 - y_2 + h_2)^2}\;, \tag{54}$$

$$\rho_r(y,x) \;=\; \sqrt{x_3^2 + (x_1 - y_1 - h_1)^2 + (x_2 - y_2 - h_2)^2}\;. \tag{55}$$

For simplicity, the vertical component of the midpoint y_3 is set here to zero. Evaluating the second derivative term in formula (50) for the common-offset geometry leads, after some heavy algebra, to the expression

$$\left|\frac{\partial^2 T\left(s(y),x\right)}{\partial x\,\partial y} + \frac{\partial^2 T\left(x,r(y)\right)}{\partial x\,\partial y}\right| = \frac{x_3\left(\rho_s^2+\rho_r^2\right)}{v\left(\rho_s\,\rho_r\right)^2}\left(\frac{\rho_s+\rho_r}{v\,\rho_s\,\rho_r}\right)^{m-1}\cos\alpha(x)\;. \tag{56}$$

Substituting (56) into the general formula (50) yields the weighting function for the common-offset true-amplitude constant-velocity migration:

$$\widehat{w}_{CO}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{x_3\left(\rho_s+\rho_r\right)^{m-1}\left(\rho_s^2+\rho_r^2\right)}{v\left(\rho_s\,\rho_r\right)^{m/2+1}}\;. \tag{57}$$

Formula (57) is similar to the result obtained by Sullivan and Cohen (1987). In the case of zero offset $h=0$, (57) reduces to formula (52). Note that the value of $m=1$ in (57) corresponds to the two-dimensional (cylindric) waves recorded on the seismic line. A special case, valuable in practice, is the 2.5-D inversion, when the waves are assumed to be spherical, while the recording is on a line, and the medium has cylindric symmetry. In this case, the modeling weighting function (46) transforms to (Deregowski and Brown, 1983; Bleistein, 1986)

$$w(x;t,y) = \frac{1}{(2\,\pi)^{1/2}}\,\frac{\sqrt{v}\,C\left(s(y),x,r(y)\right)}{\sqrt{\rho_s\,\rho_r\left(\rho_s+\rho_r\right)}}\;, \tag{58}$$

and the time filter is $\left(\frac{\partial}{\partial z}\right)^{1/2}$. Combining this result with formula (56) for $m=1$, we obtain the weighting function for the 2.5-D common-offset migration in a constant velocity medium (Sullivan and Cohen, 1987):

$$\widehat{w}_{CO;2.5D}(y;z,x) = \frac{1}{(2\,\pi)^{1/2}}\,\frac{x_3\sqrt{\rho_s+\rho_r}\left(\rho_s^2+\rho_r^2\right)}{\sqrt{v}\left(\rho_s\,\rho_r\right)^{3/2}}\;. \tag{59}$$

The corresponding time filter for 2.5-D migration is $\left(-\frac{\partial}{\partial t}\right)^{1/2}$.

The weighting function of the asymptotic pseudo-unitary migration is found analogously to (38) as

$$w^{(+)} = w^{(-)} = \frac{1}{(2\,\pi)^{m/2}} \left|\frac{\partial^2 T\left(s(y),x\right)}{\partial x\,\partial y} + \frac{\partial^2 T\left(x,r(y)\right)}{\partial x\,\partial y}\right|^{1/2} . \tag{60}$$

Unlike true-amplitude migration, this type of migration operator doesn't change the dimensionality of the input. For common-shot migration, pseudo-unitary weighting coincides with the weighting of datuming and corresponds to the downward continuation of the receivers. In the zero-offset case, it reduces to downward pseudo-unitary continuation with a velocity of $v/2$. In the common-offset case, the pseudo-unitary weighting is defined from (60) and (56) as follows:

$$w_{CO}^{(-)}(y;z,x) = \frac{1}{(2\,\pi\,v)^{m/2}} \frac{\sqrt{x_3\,\cos\alpha}\,(\rho_s+\rho_r)^{\frac{m-1}{2}}\,\sqrt{\rho_s^2+\rho_r^2}}{(\rho_s\,\rho_r)^{\frac{m+1}{2}}} , \tag{61}$$

where

$$\cos\alpha = \left(\frac{(x-y)^2+\rho_s\,\rho_r-h^2}{2\,\rho_s\,\rho_r}\right)^{1/2} . \tag{62}$$

Post-Stack Time Migration

An interesting example of a stacking operator is the hyperbola summation used for time migration in the post-stack domain. In this case, the summation path is defined as

$$\widehat{\theta}(y;z,x) = \sqrt{z^2+\frac{(x-y)^2}{v^2}} , \tag{63}$$

where z denotes the vertical traveltime, x and y are the horizontal coordinates on the migrated and unmigrated sections respectively, and v stands for the effectively constant root-mean-square velocity (Claerbout, 1995b). The summation path for the reverse transformation (demigration) is found from solving equation (63) for z. It has the well-known elliptic form

$$\theta(x;t,y) = \sqrt{t^2-\frac{(x-y)^2}{v^2}} . \tag{64}$$

The Jacobian of transforming z to t is

$$\left|\frac{\partial\widehat{\theta}}{\partial z}\right| = \frac{z}{t} . \tag{65}$$

If the migration weighting function is defined by conventional downward continuation (Schneider, 1978), it takes the following form, which is equivalent to equation (44):

$$\widehat{w}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}} \frac{\cos\alpha(y)}{v\,R(y,x)} = \frac{1}{(2\,\pi)^{m/2}} \frac{\cos\alpha}{v^m\,t^{m/2}} . \tag{66}$$

The simple trigonometry of the reflected ray suggests that the cosine factor in formula (66) is equal to the simple ratio between the vertical traveltime z and the zero-offset reflected traveltime t:

$$\cos\alpha = \frac{z}{t}\,. \tag{67}$$

The equivalence of the Jacobian (65) and the cosine factor (67) has important interpretations in the theory of Stolt frequency-domain migration (Stolt, 1978; Chun and Jacewitz, 1981; Levin, 1986). According to equation (19), the weighting function of the adjoint operator is the ratio of (66) and (65):

$$\widetilde{w}(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{1}{v^m\,t^{m/2}}\,. \tag{68}$$

We can see that the cosine factor z/t disappears from the adjoint weighting. This is completely analogous to the known effect of "dropping the Jacobian" in Stolt migration (Harlan, 1983; Levin, 1994). The product of the weighting functions for the time migration and its asymptotic inverse is defined according to formula (9) as

$$w\,\widehat{w} = \frac{1}{(2\,\pi)^m}\sqrt{\left|F\,\widehat{F}\right|\left|\frac{\partial\widehat{\theta}}{\partial z}\right|^m} = \frac{1}{(v^2\,t)^m}\,. \tag{69}$$

Thus, the asymptotic inverse of the conventional time migration has the weighting function determined from equations (9) and (66) as

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{t/z}{v^m\,t^{m/2}}\,. \tag{70}$$

The weighting functions of the asymptotic pseudo-unitary operators are obtained from formulas (34) and (35). They have the form

$$w^{(+)}(x;t,y) \;=\; \frac{1}{(2\,\pi)^{m/2}}\,\frac{\sqrt{t/z}}{v^m\,t^{m/2}}\,. \tag{71}$$

$$w^{(-)}(y;z,x) \;=\; \frac{1}{(2\,\pi)^{m/2}}\,\frac{\sqrt{z/t}}{v^m\,t^{m/2}}\,. \tag{72}$$

The square roots of the cosine factor appearing in formulas (71) and (72) correspond to the analogous terms in the pseudo-unitary Stolt migration proposed by Harlan and Sword (1986).

Post-Stack Residual Migration

In an earlier article (Fomel, 1994), I found the integral solution of the boundary problem for the velocity continuation partial differential equation (Claerbout, 1986)

$$\frac{\partial^2 P}{\partial v\,\partial z} + v\,t\,\frac{\partial^2 P}{\partial x^2} = 0 \tag{73}$$

with the boundary conditions $P|_{v=v_0} = P_0$ and $P|_{z\to\infty} = 0$. The solution has the form of the stacking operator (1), with the model M replaced by P_0, the summation path

$$\widehat{\theta}(y;z,x) = \sqrt{z^2 + \frac{(x-y)^2}{v^2 - v_0^2}}\ , \tag{74}$$

the weighting function

$$w_{(-)}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{1}{v^m\,t^{m/2}}\ , \tag{75}$$

which is coincident with (68), and the correction filter $\left(\mathrm{sign}(v_0 - v)\,\frac{d}{dt}\right)^{m/2}$. Comparing equations (74) and (63), we can see that this solution is equivalent kinematically to residual migration with the velocity $v_r = \sqrt{v^2 - v_0^2}$ (Rothman et al., 1985). The reverse operator is the solution of equation (73) with the boundary condition on v and has the reciprocal form of the summation path

$$\theta(x;t,y) = \sqrt{t^2 + \frac{(x-y)^2}{v_0^2 - v^2}} = \sqrt{t^2 - \frac{(x-y)^2}{v^2 - v_0^2}}\ , \tag{76}$$

the weighting function

$$w_{(+)}(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{1}{v^m\,z^{m/2}}\ , \tag{77}$$

and the correction filter $\left(\mathrm{sign}(v - v_0)\,\frac{d}{d\,z}\right)^{m/2}$. The derivative filters are connected by the simple asymptotic relationship

$$\left(\pm\frac{d}{d\,z}\right)^{m/2} = \left(\pm\frac{d}{d\,t}\right)^{m/2}\left(\frac{dt}{dz}\right)^{m/2} = \left(\pm\frac{d}{d\,t}\right)^{m/2}\left(\frac{z}{t}\right)^{m/2}\ , \tag{78}$$

which transforms the reversed velocity continuation operator to the familiar form (31) with the weighting function equal to (75). According to formula (69), these two operators are seen to be asymptotically inverse.

To obtain the velocity continuation operator completely equivalent to residual migration with the weighting function (66), we can divide the continued wavefield by the time t, which is equivalent to transforming equation (73) to the form

$$\frac{\partial^2 P}{\partial v\,\partial t} + v\,t\,\frac{\partial^2 P}{\partial x^2} + \frac{1}{t}\,\frac{\partial P}{\partial v} = 0\ . \tag{79}$$

The reverse continuation in this case has the weighting function (70).

Analogously, one can obtain the pseudo-unitary residual migration with the weighting functions (71) and (72) by dividing the wavefield by $\sqrt{t}$. This leads to the equation

$$\frac{\partial^2 P}{\partial v\,\partial t} + v\,t\,\frac{\partial^2 P}{\partial x^2} + \frac{1}{2\,t}\,\frac{\partial P}{\partial v} = 0\ . \tag{80}$$

It is apparent that the operators of forward and reverse continuation with equation (73) become adjoint to each other if the definition of the dot product is changed according to formulas (28) and (29) with the model weight $W_M(z) = z$ and the data weight $W_S(t) = t$. Analogously, the solutions of equation (79) are adjoint if $W_M(z) = \frac{1}{z}$ and $W_S(t) = \frac{1}{t}$. This is a simple example of how the arbitrarily chosen definition of the dot product can affect the basic properties of the inverted operators.

Velocity Transform

Velocity transform is another form of hyperbolic stacking with the summation path

$$\widehat{\theta}(h;t_0,s) = \sqrt{t_0^2 + s^2\,h^2}\;, \tag{81}$$

where h corresponds to the offset, s is the stacking slowness, and t_0 is the estimated zero-offset traveltime. Hyperbolic stacking is routinely applied for scanning velocity analysis in common-midpoint stacking. Velocity transform inversion has proved to be a powerful tool for data interpolation and amplitude-preserving multiple supression (Thorson, 1984; Ji, 1994a; Lumley et al., 1994).

Solving equation (81) for t_0, we find that the asymptotic inverse and adjoint operators have the elliptic summation path

$$\theta(s;t,h) = \sqrt{t^2 - s^2\,h^2}\;. \tag{82}$$

The weighting functions of the asymptotic pseudo-unitary velocity transform are found using formulas (34) and (35) to have the form

$$w^{(+)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{-1/4} = \frac{1}{\sqrt{\pi}}\,\frac{\sqrt{s\,h}\,\sqrt{t/t_0}}{\sqrt{t}}\;. \tag{83}$$

$$w^{(-)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{3/4} = \frac{1}{\sqrt{\pi}}\,\frac{\sqrt{s\,h}\,\sqrt{t_0/t}}{\sqrt{t}}\;. \tag{84}$$

The factor $\sqrt{s\,h}$ for pseudo-unitary velocity transform weighting has been discovered empirically by Claerbout (**?**).

Offset Continuation and DMO

Offset continuation is the operator that transforms seismic reflection data from one offset to another (Bolondi et al., 1982; Salvador and Savelli, 1982). If the data are continued from half-offset h_1 to a larger offset h_2, the summation path of the post-NMO integral offset continuation has the following form (Biondi and Chemingui, 1994; Fomel, 1995b; Stovas and Fomel, 1996):

$$\theta(x;t,y) = \frac{t}{h_2}\sqrt{\frac{U+V}{2}}\;, \tag{85}$$

where $U = h_1^2 + h_2^2 - (x-y)^2$, $V = \sqrt{U^2 - 4\,h_1^2\,h_2^2}$, and x and y are the midpoint coordinates before and after the continuation. The summation path of the reverse continuation is found from inverting (85) to be

$$\widehat{\theta}(y;z,x) = z\,h_2\,\sqrt{\frac{2}{U+V}} = \frac{z}{h_1}\,\sqrt{\frac{U-V}{2}}\;. \tag{86}$$

The Jacobian of the time coordinate transformation in this case is simply

$$\left|\frac{\partial\widehat{\theta}}{\partial z}\right| = \frac{t}{z}\;. \tag{87}$$

Differentiating summation paths (85) and (86), we can define the product of the weighting functions according to formula (9), as follows:

$$w\,\widehat{w} = \frac{1}{2\,\pi}\,\sqrt{\left|F\,\widehat{F}\right|\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|} = \frac{t}{2\,\pi}\,\frac{\left(h_2^2-h_1^2\right)^2-(x-y)^4}{V^3}\;. \tag{88}$$

The weighting functions of the amplitude-preserving offset continuation have the form[2]

$$w(x;t,y) = \sqrt{\frac{z}{2\,\pi}}\,\frac{h_2^2-h_1^2-(x-y)^2}{V^{3/2}}\;, \tag{89}$$

$$\widehat{w}(y;z,x) = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{h_2^2-h_1^2+(x-y)^2}{V^{3/2}}\;. \tag{90}$$

It easy to verify that they satisfy relationship (88); therefore, they appear to be asymptotically inverse to each other.

The weighting functions of the asymptotic pseudo-unitary offset continuation are defined from formulas (34) and (35), as follows:

$$w^{(+)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{-1/4} = \sqrt{\frac{z}{2\,\pi}}\,\frac{\left(\left(h_2^2-h_1^2\right)^2-(x-y)^4\right)^{1/2}}{V^{3/2}}\;, \tag{91}$$

$$w^{(-)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{3/4} = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{\left(\left(h_2^2-h_1^2\right)^2-(x-y)^4\right)^{1/2}}{V^{3/2}}\;. \tag{92}$$

The most important case of offset continuation is the continuation to zero offset. This type of continuation is known as *dip moveout (DMO)*. Setting the initial offset h_1 equal to zero in the general offset continuation formulas, we deduce that the inverse and forward DMO operators have the summation paths

$$\theta(x;t,y) = \frac{t}{h_2}\,\sqrt{h_2^2-(x-y)^2}\;, \tag{93}$$

$$\widehat{\theta}(y;z,x) = \frac{z\,h_2}{\sqrt{h_2^2-(x-y)^2}}\;. \tag{94}$$

[2]The derivation of formulas (89) and (90) is beyond the scope of this paper. I plan to include this derivation in one of the next SEP reports as a continuation of the "Amplitude preserving offset continuation in theory" series (Fomel, 1995a,b).

The weighting functions of the amplitude-preserving inverse and forward DMO are

$$w(x;t,y) = \sqrt{\frac{z}{2\,\pi}}\,\frac{1}{h_2}\,, \tag{95}$$

$$\widehat{w}(y;z,x) = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{h_2\,\left(h_2^2+(x-y)^2\right)}{\left(h_2^2-(x-y)^2\right)^2}\,, \tag{96}$$

and the weighting functions of the asymptotic pseudo-unitary DMO are

$$w^{(+)} = \sqrt{\frac{z}{2\,\pi}}\,\frac{\sqrt{h_2^2+(x-y)^2}}{h_2^2-(x-y)^2}\,, \tag{97}$$

$$w^{(-)} = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{\sqrt{h_2^2+(x-y)^2}}{h_2^2-(x-y)^2}\,. \tag{98}$$

Formulas similar to (95) and (96) have been published by Fomel (1995) and Stovas and Fomel (1996). Formula (96) differs from the similar result of Black et al. (1993) by a simple time multiplication factor. This difference corresponds to the difference in definition of the amplitude preservation criterion. Formula (96) agrees asymptotically with the frequency-domain Born DMO operators (Bleistein, 1990; Bleistein and Cohen, 1995). Likewise, the stacking operator with the weighting function (95) corresponds to Ronen's inverse DMO (Ronen, 1987), as I discussed in an earlier report (Fomel, 1995b). Its adjoint, which has the weighting function

$$\widetilde{w}(x;t,y) = \frac{t/\sqrt{z}}{2\,\pi}\,\frac{1}{h_2}\,, \tag{99}$$

corresponds to Hale's DMO (Hale, 1984).

NUMERIC TEST

For a simple numeric test I choose the stacking DMO operator. The problem is formulated as an iterative least-square inversion of inverse DMO. The input data set is a synthetic three-dimensional common-azimuth common-offset gather containing a reflection response from a point diffractor (Figure 4) The data cube has 64 by 64 traces with the midpoint spacing 20 m. The half-offset is 500 m. Figure 1 compares convergence of the conjugate-gradient inversion with two different types of a DMO operator: adjoint (Hale's) DMO with the weighting function defined by formula (99) and asymptotic pseudounitary DMO with formula (98). Both operators include antialiasing with the method described by Fomel and Biondi (1995). The impulse responses of inverse DMO are plotted in Figure 2. The impulse responses of DMO are plotted in Figure 3. The asymptotic pseudounitary DMO has a noticeably higher amplitudes than the adjoint DMO. We can see that the convergence of the pseudonutary operator is better at the first 5 iterations, though the difference is negligible after 7-th iteration (Figure 1.) The zero-offset data cube obtained after the inversion with 10 conjugate-gradient iterations is shown in Figure 5. Despite some boundary artifacts, caused by the data truncation in the in-line direction, the main kinematic and dynamic features of the solution are correct, and the model of the input data (Figure 6) is accurate. The residual error after 10 iterations is shown in Figure 7. It possesses less than 1% energy of the original signal. To reduce data aliasing artifacts in DMO/inversion, it is desirable to use data with more than one offset (Ronen, 1987; Ronen et al., 1991) and/or add some model constraints in the inversion (Ronen et al., 1995).

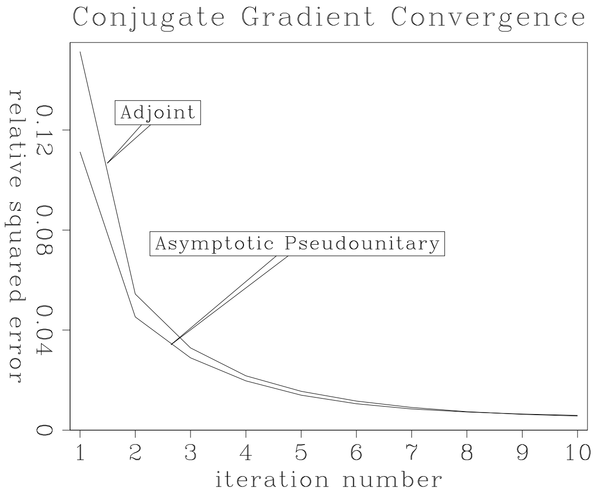

Figure 1: Comparison of convergence of the iterative inversion with different DMO operators. The relative squared residual error is plotted against the number of iteration. sergey3-adjcon [ER]

CONCLUSIONS

The mathematical theory of stacking operators leads to the fundamental concept of asymptotic inversion. When the integral continuation operators are constructed by the asymptotic Green-function solution of the partial differential equation, they often appear to be asymptotically inverse to the reverse continuation.

The concept of the adjoint operator is fundamental for the practical least-square inversion. From a practical point of view, every linear operator, including the operators of stacking type, can be represented with a matrix, and the adjoint operator corresponds to the matrix transposition.

This paper fills the gap between the concept of asymptotically inverse operators and the concept of adjoint operators by introducing the notion of asymptotic pseudo-unitary stacking operators. To what extent this notion is useful for practical least-square inversion largely depends on the particular form of the inverted operator. Practical applications may require specialized numeric tests.

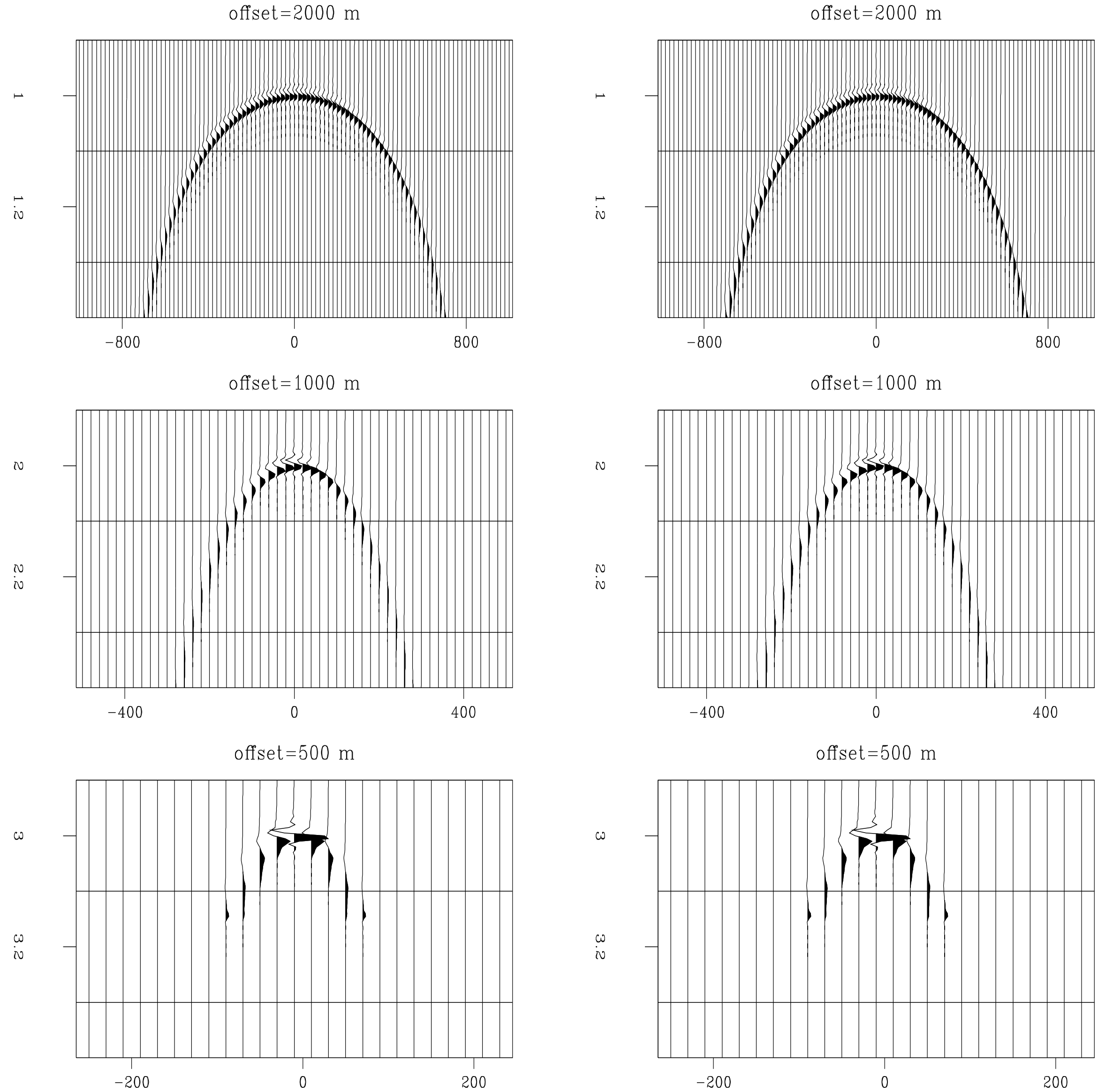

Figure 2: Impulse responses of inverse DMO operators. Left column: adjoint DMO. Right column: asymptotic pseudo-unitary DMO. sergey3-adjinv [ER]

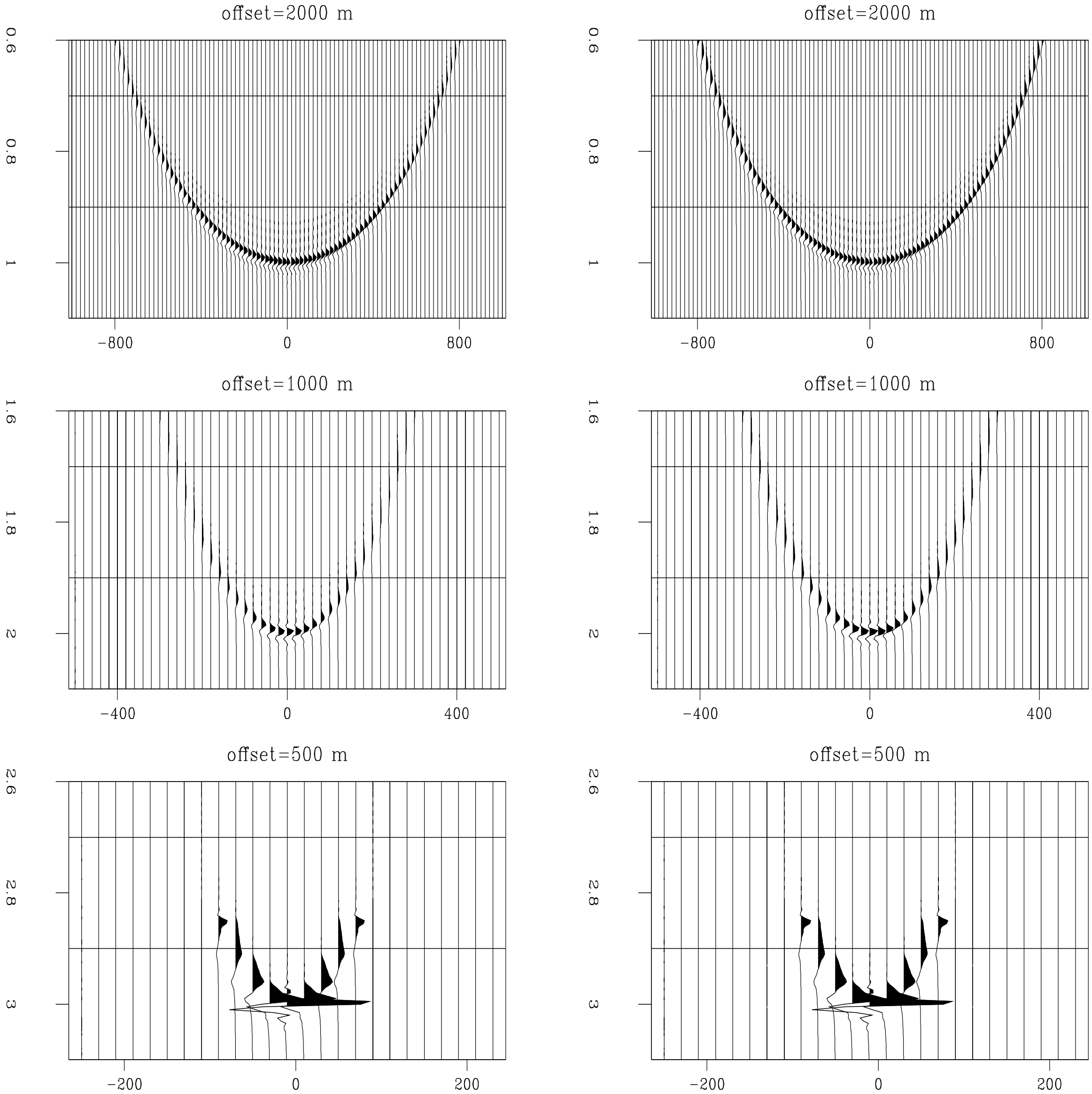

Figure 3: Impulse responses of DMO operators. Left column: adjoint DMO. Right column: asymptotic pseudo-unitary DMO. sergey3-adjimp [ER]

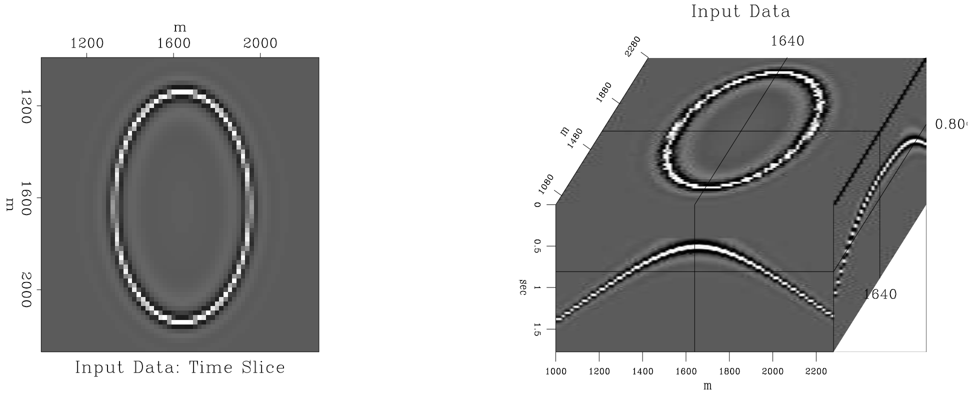

Figure 4: Input data for the numeric test: a synthetic common-azimuth common-offset gather recording reflection from a point diffractor (an appropriate NMO correction has been applied). sergey3-adjdat [ER]

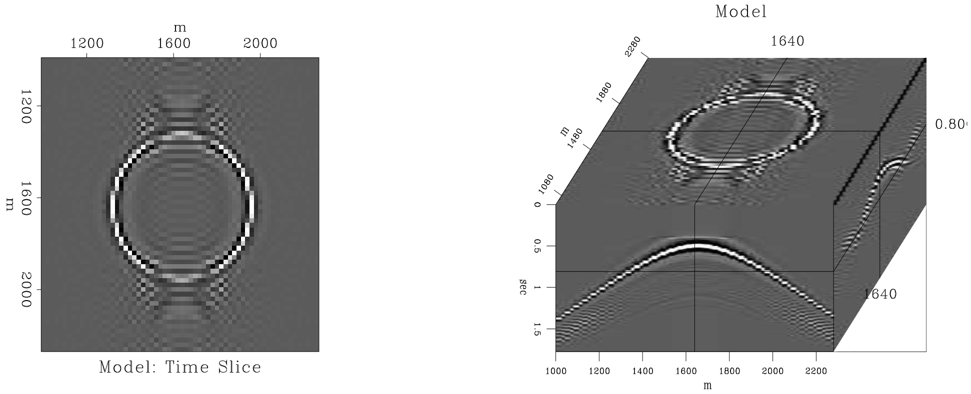

Figure 5: Zero-offset diffraction from a point diffractor obtained as the result of 10 iterations of iterative least-square DMO/inversion with the asymptotic pseudounitary operator. sergey3-adjmod [ER]

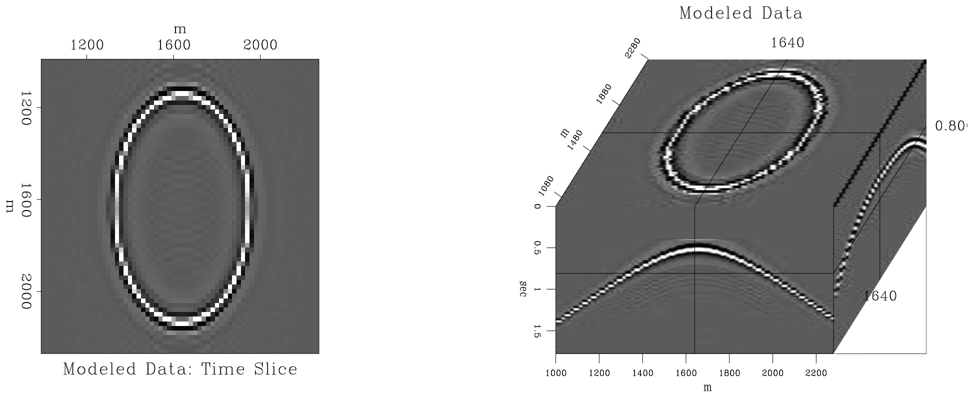

Figure 6: Modeled common-offset common-azimuth data after 10 iterations of iterative least-square DMO/inversion with the asymptotic pseudounitary operator. sergey3-adjrdt [ER]

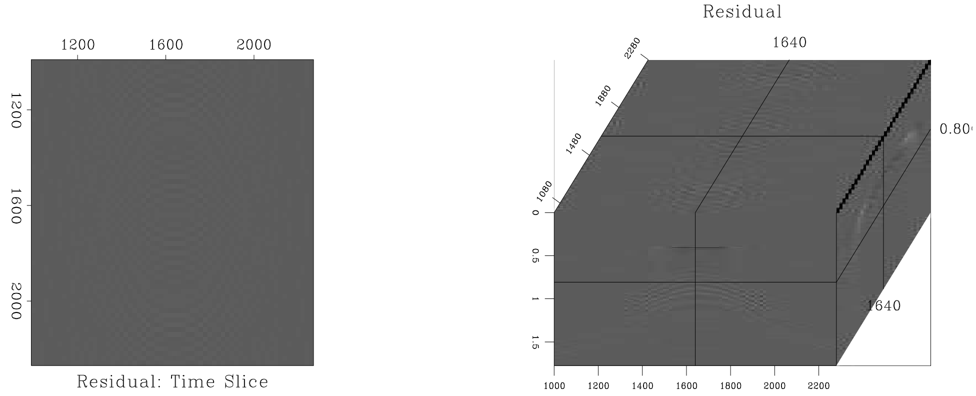

Figure 7: Residual error after 10 iterations of iterative least-square DMO/inversion, plotted at the same scale as the data. sergey3-adjres [ER]

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