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Equations of motion for isotropic porous media with multiple solid constituents

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ABSTRACT

The volume averaging technique for obtaining macroscopic equations of motion for materials that are microscopically inhomogeneous is extended to the situation in which multiple solid constituents form a porous matrix while a uniform fluid fills the pores. Previous volume averaging efforts of Pride *et al.* (1992) and others have concentrated on single solid constituent porous media. The analysis for multiple solid constituents is complicated by the presence of internal interfaces between the solid constituents within the averaging volume. These interfaces are characterized by constants that measure the fraction of the interface on which solid touches solid, fluid touches one solid, fluid touches the other solid, or fluid lies on both sides of the interface. These fractions are easily computed if the interface fractions are assumed to be uncorrelated, but real materials may be expected to exhibit some correlation. On the other hand, these interface fractions do not appear in the volume average equations at the macroscopic level. To complete the analysis, it is found that the jacketed and unjacketed tests of Biot and Willis (1957) together with the thought experiments of Berryman and Milton (1991) for solid matrix composed of two constituents are required in order to obtain definite results. Results are found to be in complete agreement with earlier work of Brown and Korringa (1975) concerning the most general possible form of the quasistatic equations for volume deformation and therefore of the equations of motion for wave propagation through such media.

INTRODUCTION

Recent work by Pride *et al.* (1992) using volume averaging to derive the form of the equations of motion for sound traveling through a fluid-saturated porous medium (Biot, 1962) has been restricted by the assumption that the solid part of such a

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solid/fluid composite was microhomogeneous, *i.e.*, composed of only a single solid constituent. Motivated by the results of Berryman (1992b) showing that at least two solid constituents are needed to explain a variety of laboratory data on fluid-saturated rocks, we show in the present work how to generalize the results of Pride *et al.* (1992) to multicomponent solid frames. Some of the earlier approaches to volume averaging include those of Slattery (1967), Whittaker (1969), and Burridge and Keller (1981). The approach of Pride *et al.* (1992) and the closely related approach taken here have much in common with the methods of Slattery and Whittaker. However, those authors were studying fluid flow through a rigid solid matrix, whereas the present approach necessarily includes the effects of solid deformation.

It will also prove important to make connection with the definitions of Brown and Korringa (1975), which are themselves based on the well-known jacketed and unjacketed thought experiments of Biot and Willis (1957). While generally similar ideas have also been presented by Rice (1975) and Rice and Cleary (1976), we will stress the more detailed discussion presented by Brown and Korringa. If the total volume of the porous sample is V and the pore volume contained in that sample is V_ϕ (where the porosity is given by $\phi = V_\phi/V$), then Brown and Korringa (1975) define constants so that

$$-\frac{\delta V}{V} = \frac{\delta p_d}{K^*} + \frac{\delta p_f}{K_s} \tag{1}$$

and

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_d}{K_p} + \frac{\delta p_f}{K_\phi}. \tag{2}$$

The independent variables in these formulas are the changes in differential pressure δp_d and pore-fluid pressure δp_f. The differential pressure is the difference between the external (confining) pressure δp_c and the fluid pressure, so $\delta p_d = \delta p_c - \delta p_f$. The coefficients are written in terms of the jacketed (or frame) bulk modulus K^*, the unjacketed (or solid grain) bulk modulus K_s, and the unjacketed pore bulk modulus K_ϕ. The remaining modulus K_p (which is called the jacketed pore modulus) can be shown to be related to K^*, K_s, and the porosity ϕ by the formula

$$\frac{\phi}{K_p} = \frac{1}{K^*} - \frac{1}{K_s}, \tag{3}$$

assuming only that an energy density for the bulk deformations exists. Measurements of K_p have been made by Zimmerman *et al.* (1986) for some rocks. One other important fact is that, if the porous solid frame is composed of a single constituent (microhomogeneity), then — and only then — $K_\phi = K_s = K_m$, where K_m is the bulk modulus of the single type of mineral grain present.

In the second section of the paper, we briefly review the averaging theorem and the single solid component results. In the third section, we present the new results for volume averaging of multicomponent solid frames and derive the macroscopic equations of motion, which are just Biot's equations of poroelasticity with coefficients that must be found through an homogenization procedure.

REVIEW OF AVERAGING THEOREM AND PREVIOUS RESULTS

The issue addressed by Pride *et al.* (1992) concerns the method by which one arrives at the equations of motion for sound traveling through a solid/fluid mixture when it is assumed that the solid is porous, but contains only a single type of mineral. The fluid is homogeneous and completely fills the pores.

The averaging theorem

The averaging theorem used is due to Slattery (1967) and is based on the idea that volume averages of derivatives are closely related to derivatives of volume averages, but care must be taken to account properly for behavior of the averaged quantities at points or surfaces where abrupt changes occur. In particular, when the quantity to be averaged exists on one side of an interface and does not exist on the other side, an interior interface term will contribute to the volume average of the derivative, but not to the derivative of the volume average.

Suppose that Q is a quantity to be averaged; Q can be a scalar, vector, or tensor. For convenience of the discussion, we will assume that the averaging volume is a finite sphere centered at position $\mathbf{x}$, although other choices are also possible. We label this volume $\Omega(\mathbf{x})$ and the surface of this volume is $\partial\Omega$. The exterior surface has two parts $\partial\Omega = \partial E_0 + \partial E_Q$, with ∂E_0 being the part where the quantity of interest Q vanishes identically and ∂E_Q being the part where $Q \neq 0$. In addition to the exterior surface, there is also an interior surface where Q changes abruptly to zero and we label this surface ∂I_Q, for interior. This interior surface is the bounding surface for the region we will label Ω_Q, *i.e.*, the region wherein the quantity to be averaged Q is nonzero. With these definitions, it is straightforward to show that

$$\int_\Omega \nabla Q \, d^3x = \int_{\Omega_Q} \nabla Q \, d^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, dS + \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS, \tag{4}$$

where dS is the infinitesimal of the surface volume element, and $\hat{\mathbf{n}}_Q$ is the unit outward normal vector from the region containing nonzero Q. The main point of (4) is just that $\partial E_Q + \partial I_Q$ is the entire bounding surface of Q in the volume Ω. As an example of the meaning of this result, consider Q to be a vector quantity, take the trace of (4), and the result is just a statement of the well-known divergence theorem for vectors.

The second result is that

$$\nabla \int_\Omega Q \, d^3x = \nabla \int_{\Omega_Q} Q \, d^3x = \int_{E_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{5}$$

The result (5) follows from the fact that the volumes $\Omega(\mathbf{x})$ and $\Omega(\mathbf{x} + \delta\mathbf{x})$ contain virtually the same internal surfaces and so these do not contribute to the gradient.

Combining these results finally gives

$$\int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, dS = \nabla \int_\Omega Q \, d^3x = \int_\Omega \nabla Q \, d^3x - \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{6}$$

Then, dividing by the volume $V = \int_\Omega d^3x$ contained in Ω gives the averaging theorem:

$$\nabla \langle Q \rangle = \langle \nabla Q \rangle - \frac{1}{V} \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{7}$$

One further definition is required to understand the notation to be used for the single solid analysis. The average $\langle Q \rangle$ is an average over the whole volume of Ω, while we will also want to consider the partial average $\bar{Q}$, related to the full volume average by

$$\langle Q \rangle = \bar{v}_Q \bar{Q}, \tag{8}$$

where $\bar{v}_Q$ is the volume fraction of Ω in which Q is nonzero.

Note that, although we generally neglect to show this dependence, all the average quantities are in fact functions of the particular choice of averaging volume $\Omega(\mathbf{x})$. In principle, $\Omega(\mathbf{x})$ can be as large as the sample being studied, or as small as desired. The legitimacy of the averaging theorem does not depend on the size of the averaging volume. However, some intermediate choice will generally be made for $\Omega(\mathbf{x})$. Too small of an averaging volume implies rapid fluctuations in the quantities of interest (like the fluid and solid dilatations), while a very large averaging volume implies all the coefficients in the equations are universal constants and therefore prevents us from studying the effect of local inhomogeneities. Also, if the averaging volume is too large, then oscillatory changes in particle displacement must average to zero over the averaging volume, which is clearly an undesirable result when studying wave propagation. Pride *et al.* (1992) provide further discussion of criteria for choosing the size of the averaging volume.

Dynamical equations

Following Pride *et al.* (1992), we assume that the dynamical equations for constituents of the solid/fluid mixture composing the porous medium can be linearized to

$$\rho_\xi \frac{\partial^2 \mathbf{u}_\xi}{\partial t^2} = \nabla \cdot \tau_\xi + \mathbf{f}_\xi, \tag{9}$$

where subscripts $\xi = f, s$ refer to fluid or solid mineral, respectively, and the other symbols are density ρ, displacement $\mathbf{u}$, stress tensor τ, and body force $\mathbf{f}$, with t being the independent variable of time. Assuming Hooke's law for the isotropic solid, we have

$$\tau_s = K_m \nabla \cdot \mathbf{u}_s \mathbf{I} + G_m \left(\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T - \tfrac{2}{3} \nabla \cdot \mathbf{u}_s \mathbf{I} \right), \tag{10}$$

where K_m and G_m are, respectively, the bulk and shear moduli of the constituent mineral. The identity tensor is symbolized by $\mathbf{I}$. Similarly, the fluid is assumed to be a linearly viscous Newtonian fluid obeying

$$\tau_f = \left(-p_f + \kappa_f \nabla \cdot \dot{\mathbf{u}}_f \right) \mathbf{I} + \mu_f \left(\nabla \dot{\mathbf{u}}_f + \nabla \dot{\mathbf{u}}_f^T - \tfrac{2}{3} \nabla \cdot \dot{\mathbf{u}}_f \mathbf{I} \right), \tag{11}$$

where κ_f and μ_f are, respectively, the coefficients of bulk and shear viscosity. Dots over displacement indicate a single time derivative. The increment of fluid pressure associated with conservative work is related to the fluid dilatation by the bulk modulus K_f through

$$-p_f = K_f \nabla \cdot \mathbf{u}_f. \tag{12}$$

Performing the bulk averages on the microscopic stress/strain relations and using the averaging theorem gives the general constitutive relations for the solid and fluid stress tensors

$$\begin{aligned}(1-\phi)\bar{\tau}_s = {} & (1-\phi)K_m \nabla \cdot \bar{\mathbf{u}}_s \mathbf{I} - K_m \delta\phi \mathbf{I} \\ & +(1-\phi)G_m \left(\nabla \bar{\mathbf{u}}_s + \nabla \bar{\mathbf{u}}_s^T - \tfrac{2}{3}\nabla \cdot \bar{\mathbf{u}}_s \mathbf{I}\right) - G_m \mathbf{D}\end{aligned} \tag{13}$$

and

$$\phi\bar{\tau}_f = \phi K_f^* \nabla \cdot \bar{\mathbf{u}}_f \mathbf{I} + K_f^* \delta\phi \mathbf{I} + \phi\mu_f \frac{\partial}{\partial t}\left(\nabla \bar{\mathbf{u}}_f + \nabla \bar{\mathbf{u}}_f^T - \tfrac{2}{3}\nabla \cdot \bar{\mathbf{u}}_f \mathbf{I}\right) + \mu_f \frac{\partial}{\partial t}\mathbf{D}, \tag{14}$$

where

$$\delta\phi = \frac{1}{V}\int_{\partial I} \mathbf{n} \cdot \mathbf{u}\, dS, \tag{15}$$

$$\mathbf{D} = \frac{1}{V}\int_{\partial E} \left(\mathbf{nu} + \mathbf{un} - \tfrac{2}{3}\mathbf{n} \cdot \mathbf{uI}\right) dS, \tag{16}$$

and

$$K_f^* = K_f \left(1 + \frac{\kappa_f}{K_f}\frac{\partial}{\partial t}\right). \tag{17}$$

Quasistatic constitutive relations for isotropic materials

Taking the trace of equations (13) and (14) gives the following results. The constitutive relations for dilatations and porosity are

$$-\frac{\delta p_s}{K_m} = \nabla \cdot \bar{\mathbf{u}}_s - \frac{\delta\phi}{1-\phi} \tag{18}$$

and

$$-\frac{\delta p_f}{K_f} = \nabla \cdot \mathbf{u}_f + \frac{\delta\phi}{\phi}, \tag{19}$$

where the partial averages $\bar{\mathbf{u}}_s$ and $\bar{\mathbf{u}}_f$ are related to the full volume averages by $\langle \mathbf{u}_s \rangle = (1-\phi)\bar{\mathbf{u}}_s$ and $\langle \mathbf{u}_f \rangle = \phi\bar{\mathbf{u}}_f$. One assumption implicit in (18) and (19) is that ϕ changes much more slowly in space than the displacement variables $\bar{\mathbf{u}}_s$ and $\bar{\mathbf{u}}_f$. This assumption allows us to remove the factors involving the porosity from the divergence terms, but we will show in a later publication that this assumption is not crucial to our analysis.

It is important at this point to understand the interpretations of all the symbols appearing in the last two equations. First, the variable δp_f is just the change in the average fluid pressure throughout the fluid phase. The change in average solid pressure δp_s is related to the macroscopic confining pressure change δp_c by the averaging relation $\delta p_c = (1-\phi)\delta p_s + \phi\delta p_f$. Thus, δp_s is just the average change in solid pressure experienced by the solid. Since δp_c and δp_f may be viewed as the pressures we can control, δp_s is the (solid volume) weighted average of the confining pressure after subtracting that part of the confining pressure supported by the fluid pressure. The change in porosity is given by $\delta\phi$. The porosity change occurs naturally in these expressions because

$$\delta\phi = \frac{1}{V}\int_{\partial I_f} \hat{\mathbf{n}}_f \cdot \mathbf{u}_f \, dS = -\frac{1}{V}\int_{\partial I_s} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS. \tag{20}$$

This result is demonstrated in the next section.

The divergence of the average solid displacement $\nabla \cdot \bar{\mathbf{u}}_s$ is properly interpreted as the dilatation of the porous solid frame (not the dilitation of the solid alone). This interpretation is not obvious, but it follows from the fact that the term arises from the external surface integral [*c.f.*, Eq. (5)]

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \frac{1}{V}\int_{E_s} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS, \tag{21}$$

which is exactly the surface integral needed to define the overall behavior of the porous solid frame. Thus, in terms of the definitions of Brown and Korringa (1975),

$$\nabla \cdot \bar{\mathbf{u}}_s = \frac{\delta V}{V} = e = -\frac{\delta p_d}{K^*} - \frac{\delta p_f}{K_m}. \tag{22}$$

This interpretation is the same one reached by Pride *et al.* (1992) using a combination of the standard thought experiments (jacketed and unjacketed) of Biot and Willis (1957). To check that this is so, we can easily show that

$$\frac{\delta\phi}{1-\phi} = -\left[\frac{1-\phi}{K^*} - \frac{1}{K_m}\right](\delta p_s - \delta p_f), \tag{23}$$

using either approach when a single constituent is present so that the Brown and Korringa unjacketed constants satisfy $K_m = K_\phi$. Note that (23) can also be written as

$$\delta\phi = -\left[\frac{1-\phi}{K^*} - \frac{1}{K_m}\right]\delta p_d, \tag{24}$$

emphasizing that porosity is constant if differential pressure is constant — a general result for microhomogeneous porous frames, but not true otherwise. Thus, the left hand side of (18) is just the solid dilatation $\delta V_s/V_s$, while the two terms on the right hand side are $\delta V/V + \delta(1-\phi)/(1-\phi)$.

Similarly, it is important to understand that the expression $\nabla \cdot \bar{\mathbf{u}}_f$ is not just a fluid dilatation, but also includes the effects of fluid motion in and out of the volume. In fact, this is already apparent from (19) since the strict fluid dilatation satisfies

$$-\frac{\delta V_f}{V_f} = \frac{\delta p_f}{K_f}, \tag{25}$$

yet (19) contains an additional term related to changes in porosity. The correct physical interpretation of $\nabla \cdot \bar{\mathbf{u}}_f$ is provided by its relation to the increment of fluid content

$$\zeta = \phi\left(\nabla \cdot \bar{\mathbf{u}}_s - \nabla \cdot \bar{\mathbf{u}}_f\right), \tag{26}$$

where ζ is defined as

$$\zeta \equiv \frac{\delta V_\phi - \delta V_f}{V} = \phi(\frac{\delta V}{V} - \frac{\delta V_f}{V_f}) + \delta\phi \tag{27}$$

and has the interpretation (Biot, 1973; Berryman and Thigpen, 1985) of the relative change in fluid mass per unit volume of initial fluid mass. Note that (25) and (27) are in agreement with (26) if the averaging equation (19) is also satisfied.

The equations (18), (19), and (22) are sufficient to arrive at the standard form of the equations relating e and ζ to the macroscopic pressures δp_c and δp_f for a single constituent porous medium given by

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K^* & 1/K_m - 1/K^* \\ 1/K_m - 1/K^* & 1/K^* + \phi/K_f - (1+\phi)/K_m \end{pmatrix} \begin{pmatrix} -\delta p_c \\ -\delta p_f \end{pmatrix}. \tag{28}$$

These equations are completely consistent with the results of Pride *et al.* (1992) as can be demonstrated by substituting the definitions given above into their formulas (48) and (49), and then doing a straightforward (though somewhat tedious) 2×2 matrix inversion. The relationship between these matrix elements and the coefficients H, C, and M in Biot's equations are given by Berryman (1992a). They are: $H = K^* + \alpha C + 4G/3$, $C = \alpha M$, and $1/M = (\alpha - \phi)/K_m + \phi/K_f$, where the Biot-Willis parameter is defined by $\alpha = 1 - K^*/K_m$.

EQUATIONS OF MOTION WITH TWO SOLID CONSTITUENTS

When two solid constituents are present, the averaging theorem gives some significantly different results that we will present and discuss here. For simplicity, we will limit the discussion to averages of divergence of displacement and of the displacement itself. We make one assumption implicitly here, that the averaging volume Ω is large enough so that statistical differences between the bulk porosity in the volume and the outcrop of porosity at the surface of the averaging volume are negligible.

We restrict discussion to a problem studied previously by Berryman and Milton (1991): two porous constituents in fully welded contact. Welded contact between porous constituents implies that no cracks/fractures can open up between these constitutents due to applied temperature or stress. Welded contact is somewhat easier to analyze than nonwelded or partially welded contact. Our intention is to treat these more general situations in a later publication, but the main ideas will be presented here.

As a means of simplifying the algebra in the following analysis, we introduce in the second subsection two new quantities that we call $\tilde{\mathbf{u}}_A$ and $\tilde{\mathbf{u}}_B$, the divergences of which are just the dilatations of the corresponding solid constituents. This step helps to avoid introducing various terms that would ultimately cancel in the final formulas.

Results for all solids

If there are two solids present, for our present purposes of averaging we may lump them together and act as if they are both generic "solids," using subscript "s" to refer to the regions occupied. The averaging theorem (or in this case just the normal divergence theorem) states that

$$\langle \nabla \cdot \mathbf{u}_s \rangle = \frac{1}{V}\int_{\Omega(\mathbf{x})} \nabla \cdot \mathbf{u}_s \, d^3x = \frac{1}{V}\int_{\partial E} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS + \frac{1}{V}\int_{\partial I} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS, \tag{29}$$

where Ω is the averaging volume and $V = \int_\Omega d^3x$, with ∂E being the external boundary and ∂I being the internal or pore boundary. The divergence of the average (found by taking the Frechet derivative with respect to the averaging volume) is

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \frac{1}{V}\int_{\partial E} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS. \tag{30}$$

The internal surface integral is easily interpreted as the negative of the change in porosity $\delta\phi$, since the displacement integrated along the bounding surface produces a volume change that is then normalized by the total volume V in the averaging volume $\Omega(\mathbf{x})$, so

$$\frac{1}{V}\int_{\partial I} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS = -\delta\phi. \tag{31}$$

The left hand side of (29) is also easily interpreted as the total change in solid volume $\delta[V(1-\phi)]$ divided by the averaging volume V. Comparing these expressions shows that

$$\nabla \cdot \langle \mathbf{u}_s \rangle = (1-\phi)\frac{\delta V}{V}. \tag{32}$$

Note that there has been no change in the averaging volume Ω, but there has been movement of solid in or out of volume and/or a change in state of compression of the solid. In this regard, our notation is trying to reflect the fact that experimentally we normally start with a fixed volume of material V and then measure changes δV in that volume.

Accounting for the volume occupied by the solid initially, we have $\langle \mathbf{u}_s \rangle \equiv (1-\phi)\bar{\mathbf{u}}_s$, so

$$\nabla \cdot \bar{\mathbf{u}}_s = \frac{\delta V}{V} + \frac{\bar{\mathbf{u}}_s \cdot \nabla \phi}{1-\phi}. \tag{33}$$

We normally neglect the second term on the right hand side of (33), since we assume that the scales of variation of the displacement field are much smaller than those for the porosity, so that $|\bar{\mathbf{u}}_s \cdot \nabla\phi| << |\nabla \cdot \bar{\mathbf{u}}_s|$. However, a completely general analysis must account for the presence of this term.

Results for constituents A and B

Now, when we want to distinguish the properties of the solids A and B, we can break up the averaging volume into two pieces such that

$$\langle \nabla \cdot \mathbf{u}_s \rangle = \langle \nabla \cdot \mathbf{u}_A \rangle + \langle \nabla \cdot \mathbf{u}_B \rangle \,, \tag{34}$$

which follows from the fact that material A and B occupy disjoint parts of the averaging volume. The averaging theorem for the divergence of $\mathbf{u}_A$ alone then states that

$$\langle \nabla \cdot \mathbf{u}_A \rangle = \nabla \cdot \langle \mathbf{u}_A \rangle + \frac{1}{V}\int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS + \frac{1}{V}\int_{\partial I_B} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS, \tag{35}$$

where we have explicitly noted that the interior interface has two parts: one boundary ∂I_f with the fluid (or pore space) and one ∂I_B with the other solid (B).

Since

$$\nabla \cdot \langle \mathbf{u}_A \rangle = \frac{1}{V}\int_{\partial E_A} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS, \tag{36}$$

and a similar expression for $\nabla \cdot \mathbf{u}_B$, we find easily from the identity

$$\frac{1}{V}\int_{\partial E_s} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS = \frac{1}{V}\int_{\partial E_A} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS + \frac{1}{V}\int_{\partial E_B} \hat{\mathbf{n}}_B \cdot \mathbf{u}_B \, dS \tag{37}$$

that

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \nabla \cdot \langle \mathbf{u}_A \rangle + \nabla \cdot \langle \mathbf{u}_B \rangle \,. \tag{38}$$

In order to determine the physical significance of $\nabla \cdot \langle \mathbf{u}_A \rangle$, we need to repeat the analysis for all solids, taking into account the fact that when there are two or more solids there must also be additional interior interfaces between these various constituents. Of the four terms in (35), each requires some interpretion. First, the left hand side has an interpretation similar to that of the left hand side of (29). Thus, we have the volume average of the dilatation of A material must be

$$\langle \nabla \cdot \mathbf{u}_A \rangle = \frac{\delta[V_A(1-\phi_A)]}{V}, \tag{39}$$

where V_A is the total porous volume of A material and $V_A(1-\phi_A)$ is the total solid volume of A material.

The two integrals on the right hand side of (35) are more difficult to interpret because they involve the contact region of two porous materials having possibly different porosities. Statistically the A material should have solid material at this interface occupying the fraction $1-\phi_A$ of the total interface area and A-pore the remaining fraction ϕ_A of the total. The B material has corresponding proportions. Now these continuous surfaces may be statistically correlated or uncorrelated. If uncorrelated, we can easily compute the coefficients we will need. But if they are correlated, we must introduce some new constants with the following properties: $\eta_{AB} = \eta_{BA}$ is the fraction of the interface on which solid A touches solid B, η_{Af} is the fraction of the interface on which solid A touches the fluid in B, and similarly η_{Bf} is the fraction on which the solid B touches the fluid in A. Within our general assumption of statistical homogeneity, these constants should obey the relations $\eta_{AB} + \eta_{Af} = 1 - \phi_A$ and $\eta_{BA} + \eta_{Bf} = 1 - \phi_B$. One immediate general result is that the difference $\eta_{Af} - \eta_{Bf} = \phi_B - \phi_A$. The solid/solid contact area should be proportional to η_{AB}, which may be very small or it can be as large as the minimum of the two solid fractions $(1-\phi_A)$, $(1-\phi_B)$. For uncorrelated surfaces, we expect $\eta_{AB} = (1-\phi_A)(1-\phi_B)$, $\eta_{Af} = (1-\phi_A)\phi_B$, and $\eta_{Bf} = (1-\phi_B)\phi_A$. These identities are easily shown to satisfy the statistical sum rules for these coefficients. For correlated interfaces, we may view η_{AB} as a new microstructural parameter that characterizes the internal (to the averaging volume) solid/solid interface.

The first integral on the right hand side of (35) is the surface integral of displacement along the fluid boundary. This term has the same significance as the corresponding one for the whole solid; it is the change in porosity associated with A material. The total pore volume associated with A is $V_A\phi_A$, so the change in pore volume must be a change in this quantity. However, the surface integral is strictly over the original boundary of the A material (prior to the displacements $\mathbf{u}_A$), so the correct expression for this change in the absence of other solids is clearly $V_A\delta\phi_A$. But, in the presence of other solids, we must account for the possibility of changes in overall porosity due to changes in volume fraction. Thus, the full contribution of this term is

$$\int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS = -V_A\delta\phi_A + V\eta_{Af}\delta v_A, \tag{40}$$

using one of the constants introduced in the preceding paragraph. When the volume fraction does not change, as in the case when the averaging volume happens to contain only A material, we see that this expression reduces correctly to (31). When we write the corresponding relation for the B phase and then consider that it must be true that

$$\int_{\partial I_f} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, dS = \int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS + \int_{\partial I_f} \hat{\mathbf{n}}_B \cdot \mathbf{u}_B \, dS, \tag{41}$$

then we see that the extra terms proportional to change in volume fraction are exactly what were needed to guarantee that (41) is equivalent to $\delta\phi = v_A\delta\phi_A + v_B\delta\phi_B + (\phi_A - \phi_B)\delta v_A$.

The second integral on the right hand side of (35) is the surface integral over the AB solid/solid interface. This term also has the important characteristic that it must be exactly the negative of the corresponding term for the B material. So however we interpret it, the expression should be easily identified by the fact that interchanging A and B should change the sign of the term. Making the identification that

$$\frac{1}{V}\int_{\partial I_B} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS \equiv \eta_{AB}\delta v_A, \tag{42}$$

where δv_A is the change in volume fraction of porous constituent A, we find that this interpretation is reasonable. If an imaginary continuous surface between the porous constituents is drawn and the corresponding surface integral taken, then the result would be exactly δv_A. Since the AB interface occupies only a fraction η_{AB} of this total interface area, we see that (42) follows.

The remaining term to be interpreted in (35) is proportional to the surface integral of the A component displacement. Combining the previous results, this average must be given by

$$\int_{\partial E_A} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, dS = (1-\phi_A)[\delta V_A - V\delta v_A] = (1-\phi_A)v_A\delta V, \tag{43}$$

with a matching expression for the B phase. That these two integrals must satisfy the sum rule in (37) together with (32) implies that their sum must be equal to $(1-\phi)\delta V$ which is easily seen to be true.

Comparing all these expressions, we finally obtain the result

$$\nabla \cdot \langle \mathbf{u}_A \rangle = v_A(1-\phi_A)\frac{\delta V_A}{V_A} - (1-\phi_A)\delta v_A, \tag{44}$$

which is the desired expression for divergence of the average displacement of A.

Although the divergence of $\langle \mathbf{u}_A \rangle$ is given rigorously by (44), the quantities that actually appear in the quasistatic equations of motion are simply the dilatations of the constituents, so we will define a new quantity $\tilde{\mathbf{u}}_A$ satisfying

$$\nabla \cdot \tilde{\mathbf{u}}_A \equiv \frac{\delta V_A}{V_A}, \tag{45}$$

which is related to $\bar{\mathbf{u}}_A$ by

$$\nabla \cdot \bar{\mathbf{u}}_A = \nabla \cdot \tilde{\mathbf{u}}_A - \frac{\delta v_A}{v_A}, \tag{46}$$

where the change in volume fraction is itself related to $\tilde{\mathbf{u}}_A$ and the corresponding expression of B by $\delta v_A = v_A v_B(\nabla \cdot \tilde{\mathbf{u}}_A - \nabla \cdot \tilde{\mathbf{u}}_B)$. These definitions and interrelations will be important for our analysis of wave propagation issues for multicomponent rocks.

Fully welded contact

For two porous components A and B in welded contact, the volume fractions of the components v_A, v_B satisfy

$$v_A + v_B = 1, \tag{47}$$

while the overall porosity is given by

$$\phi = v_A\phi_A + v_B\phi_B, \tag{48}$$

where ϕ_A, ϕ_B are respectively the porosities of the porous constituents A and B. The fractions of the total volume occupied by the solid components are $\bar{v}_A = v_A(1-\phi_A)$ and $\bar{v}_B(1-\phi_B)$, respectively. The solid components in A and B are individually pure, so that each porous constituent may be thought of as a microhomogeneous (or Gassmann) material.

Averaging equations

Volume averaging for the three components (two solid and one fluid) yields

$$-\frac{\delta p_A}{K_A} = \nabla \cdot \tilde{\mathbf{u}}_A - \frac{\delta\phi_A}{(1-\phi_A)}, \tag{49}$$

$$-\frac{\delta p_B}{K_B} = \nabla \cdot \tilde{\mathbf{u}}_B - \frac{\delta\phi_B}{(1-\phi_B)}, \tag{50}$$

and

$$-\frac{\delta p_f}{K_f} = \nabla \cdot \bar{\mathbf{u}}_f + \frac{\delta\phi}{\phi}. \tag{51}$$

These three equations should be compared to (18) and (19). Now, to provide some further insight, notice that the left hand side of (49) is equal to $\delta[V_A(1-\phi_A)]/V_A(1-\phi_A)$, while the right hand side is $\delta V_A/V_A - \delta\phi_A/(1-\phi_A)$.

Other relationships

It is useful to think of equations (49) and (50) as equations for the changes in the the constituent porosities $\delta\phi_A$ and $\delta\phi_B$. To relate these values to the expressions above, we need another pair of equations. First, note that from (48)

$$\delta\phi = v_A\delta\phi_A + v_B\delta\phi_B + \delta v_A(\phi_A - \phi_B), \tag{52}$$

so we need an expression for the change in v_A. For welded contact, we obtain such an expression by noting that by definition

$$v_A + \delta v_A = \frac{V_A(1 + \nabla \cdot \tilde{\mathbf{u}}_A)}{V_A(1 + \nabla \cdot \tilde{\mathbf{u}}_A) + V_B(1 + \nabla \cdot \tilde{\mathbf{u}}_B)}, \tag{53}$$

which upon expansion and neglect of second order terms yields

$$\delta v_A = v_A v_B \left(\nabla \cdot \tilde{\mathbf{u}}_A - \nabla \cdot \tilde{\mathbf{u}}_B \right), \tag{54}$$

while for welded contact $\delta v_B = -\delta v_A$. Note that, if A and B expand or contract at the same rate so $\nabla \cdot \tilde{\mathbf{u}}_A = \nabla \cdot \tilde{\mathbf{u}}_B$, then $\delta v_A = 0$ as expected.

We also want to view the combined solid volume $V_s = V_A(1 - \phi_A) + V_B(1 - \phi_B)$ as a whole in order to recover Biot's macroscopic equations for the inhomogeneous material. Then, it is important to recognize that the solid dilatations must satisfy

$$\nabla \cdot \bar{\mathbf{u}}_s = v_A \nabla \cdot \tilde{\mathbf{u}}_A + v_B \nabla \cdot \tilde{\mathbf{u}}_B, \tag{55}$$

and, similarly, the solid pressures must satisfy

$$(1 - \phi)\delta p_s = \bar{v}_A \delta p_A + \bar{v}_B \delta p_B. \tag{56}$$

Relation (55) may be easily derived by considering the denominator of the right hand side of (53), whereas (56) is just a statement of force conservation across the material boundary.

CONCLUSIONS

A volume averaging technique for obtaining macroscopic equations of motion for materials that are microscopically inhomogeneous has been extended to the situation in which multiple solid constituents form a porous matrix while a uniform fluid fills the pores. The analysis for multiple solid constituents was complicated by the presence of internal interfaces between the solid constituents within the averaging volume. These interfaces are characterized by constants that measure the fraction of the interface on which solid touches solid, fluid touches one solid, fluid touches the other solid, or fluid lies on both sides of the interface. These fractions are easily computed if the interface fractions are assumed to be uncorrelated, but real materials may be expected to exhibit some correlation. Although the presence of these interfaces complicates our analysis, the interface fractions nevertheless do not appear in the volume average equations at the macroscopic level. To complete the derivation, it was found that the jacketed and unjacketed tests of Biot and Willis (1957) together with the thought experiments of Berryman and Milton (1991) for solid matrix composed of two constituents were required in order to obtain definite results. Results were then found to be in complete agreement with earlier work of Brown and Korringa (1975) concerning the most general possible form of the quasistatic equations for volume deformation and therefore of the equations of motion for wave propagation through such media. The results therefore provide independent confirmation of the form of the equations that has been assumed to be correct since the work of Brown and Korringa (1975) and Rice and Cleary (1976).

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