0.1 Splitting the wave operator

When one applies the elastic wave equation operator for spatially varying media, several first order derivatives have to be calculated in the process. The wave equation can formally be split into two components so that derivatives are taken with respect to medium parameters and with respect to the propagating wave field. Splitting the wave equation operator allows one to adapt derivative operators to the physical quantities to be differentiated. In particular the adaption can be guided by special properties of the quantity. Practical problems can arise since the derivated quantities in general complex media have to be interpolated back to collocation points.

In seismic modeling adaptive techniques are well known. Usually the adaption of a quantity is in the direction of a physical dimension such as time or space. In general, adaption is guided by criteria to improve numerical accuracy and stability or to increase the speed of the calculation. Cunha (?) discussed the notion of treating the propagating isotropic elastic wave field differently from the underlying medium. For anisotropic media, Igel et. al (?) showed an interpolation error analysis of FD operators and their spatial distribution in the case of a staggered grid. In this section I combine both notions for anisotropic media.

0.1.1 Formal splitting of the wave operator

The general wave equation (??) involves calculating tensor products and derivatives, where the wave operator $\mathbf{L}$, introduced in a previous section, is defined as

$$\mathbf{Lu} = \underset{\sim}{\mathbf{a}} \nabla (\underset{\sim\sim}{\mathbf{b}} \underset{\sim}{\epsilon}) \; . \tag{0.1}$$

∇ is a partial derivative operator matrix (??) acting on its argument, $\underset{\sim}{\mathbf{b}}$ is a medium property tensor and $\mathbf{u}$ is the displacement field. Using the chain rule for covariant derivatives and applying it to the tensor product, one can rewrite equation (0.1) to

$$l_i = a_j \{ \sum_{jkl} [\, \frac{1}{2} (\, \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}) \; \mathbf{b_{ijkl}} \;] \; \epsilon_{\mathbf{kl}} \; + \; \sum_{jkl} \; \mathbf{b_{ijkl}} \; [\, \frac{1}{2} (\, \frac{\partial}{\partial x_i} + \; \frac{\partial}{\partial x_j} \;) \; \epsilon_{\mathbf{kl}} \;] \} \; , \tag{0.2}$$

where [...] indicates the scope of the partial derivatives and where i,j,k,l = 1,2,3 is the range of tensor components. Equation (0.2) is written in terms of its tensor elements and is valid for arbitrary anisotropic media. In the first term of equation (0.2) derivatives of the elastic constants are computed; in the second term only derivatives of the strain tensor are computed. The stiffness components in the second term act as constant coefficients for the derivated matrix elements. The derivative operation in equation (0.2) appears in both terms. It acts, however, on two different entities. In the first term the derivative of medium properties is taken and in the second term the derivative of the wave field is computed. In the end, summation of both terms produces the complete derivative without taking a derivative of the medium and wave field product, as equation (0.1) would suggest. For a medium with constant properties the first term vanishes, leaving the second term in equation (0.2), whose Fourier transform represents the Christoffel equation (?; ?). From an economical point of view of equation (0.2), some expensive computations can be saved if the medium derivatives are calculated once and are then reused through the course of computation, but the real opportunity lies in computing the solution to equation (0.1) more accurately and more realistically for certain types of media. In particular, the derivative properties can be adjusted to match the properties the quantities.

0.1.2 Adaption of the derivative operator

Two basic observables in the wave equation are the wave field (e.g. stress, displacement) and the medium properties (stiffnesses, velocities). If the frequency content of the wave field and the material properties are similar, the same type of derivative operators can be used for both. Then (0.1) should be used rather than (0.2). The necessary condition is that model and wave field roughness are of the same scale. For example, if the medium properties vary smoothly in space then a fairly long derivative operator can be adequate to approximate the analytical derivative numerically. However, if the medium is varying rapidly, the same derivative operator may not be adequate. As an extreme example, realistic modeling of wave propagation in a random medium is affected by the length of the finite difference operator. A long operator implicitly averages medium properties. To describe local effects, a much shorter operator would be necessary, but for the same prescribed spacing, that operator might give a less accurate derivative. A layered model is an extreme case. The spatial variability in the horizontal direction is zero, while the variability in depth occurs in discontinuous steps. Applying equation (0.1) along with conventional finite difference operators to such a model would implicitly assume a continuous and band-limited medium. Equation (0.2) allows us to "separate" the medium from the wave field. One can adapt the derivative operators to the properties of the observables they are operating on.

0.1.3 Practical issues

In practical implementations, problems arise when one calculates derivatives of quantities on a staggered grid. A staggered grid (cf. Figure 0.1) can be chosen such that the outcome of a derivative calculation falls onto the grid point of the quantity, and is multiplied together in the next step of the algorithm. In such a scheme no quantity needs to be interpolated, and thus there is no loss in accuracy. Examining the partitioned constitutive equation, one can see how a commonly used staggered grid is chosen:

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ \begin{pmatrix} \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} & \begin{pmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36} \end{pmatrix} \\ \begin{pmatrix} c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \end{pmatrix} & \begin{pmatrix} c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \\ \begin{pmatrix} \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \end{pmatrix} \tag{0.3}$$

The 4 quadrants of the stiffness matrix c are abbreviated by $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and their locations on the grid are noted in Figure 0.1. Stiffness values in the first diagonal A_{11} quadrant and the first portion of the $\sigma_{1..3}$ and $\epsilon_{1..3}$ vectors are chosen to lie on the primary grid. The second diagonal quadrant and the second part of $\sigma_{4..6}$ and $\epsilon_{4..6}$ are then necessarily chosen to lie on the secondary grid. The primary cube center coincides with the secondary cube edge. That scheme causes the derivative computation to evaluate quantities at the correct collocation points. In anisotropic elastic media up to orthorhombic symmetry, the previous notion works well, since the off-diagonal elements A_{12}, A_{21} of the stiffness matrix are zero and thus do not have to be evaluated. In anisotropic systems beyond orthorhombic those elements are non zero, such that even in non-split schemes interpolation to collocation points has to be included for staggered grid methods. When splitting the wave operator, one has to take care of proper collocation interpolation when using a staggered grid. Thus the interpolation issue can complicate the primary goal of adapting derivative operators to the quantities. Splitting of the wave operator also affects the numerical stability

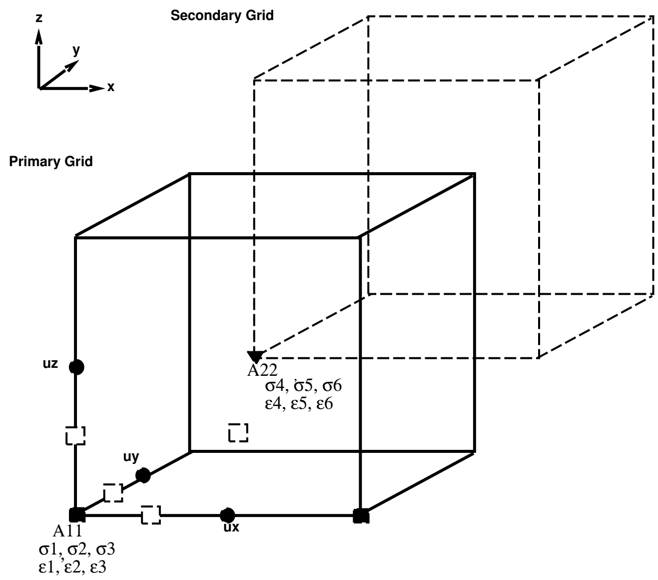

Figure 0.1: The primary grid and secondary grid as usually defined for staggered grid applications for elastic wave modeling. The primary cube center coincides with the secondary cube edge. The positions of stress, strain and stiffness components are indicated (black triangles), together with the evaluation points of the split wave operator scheme (empty boxes). stagsplit [NR]

and numerical dispersion of the algorithm. One can apply Neumann-style analysis to obtain stability conditions and dispersion curves. Each of the two terms of the split wave operator each influences stability such that the overlap of those two is the region where the algorithm is stable and non dispersive.

Equivalence of modeling equations

First I show that equation (0.2) and (0.1) are numerically equivalent and produce the same results, as predicted analytically. I propagate a wave field in a model that shows a step discontinuity but is otherwise homogeneous. The elastic wave propagates across a discontinuity in v_p and v_s. Figure 0.2 shows a single trace comparison of two modeling responses. One result is calculated using equation (0.2) and one result is obtained using equation (0.1). For both components the resulting responses overlie each other and indicate that both modeling equations are also numerically equivalent.

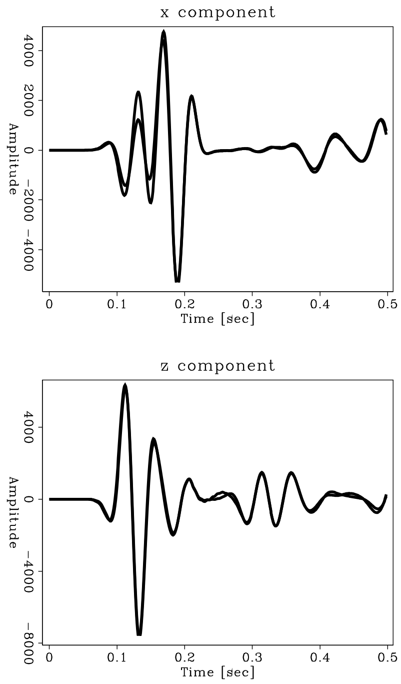

Figure 0.2: Trace 90 is taken from two wave fields. The results from the two different modeling equations are overlaid and confirm that they are not only analytically but also numerically equivalent. The modeling was carried out on a non-staggered grid. The source is located on top of the step and the receivers are located below. equivalns [ER]

A long and short operator combination

Figure **??** is using the same model as in Figure 0.2. However, the difference lies in the types of operators used for calculating the wave field. For derivating the field quantities, a 6-point operator was used, while for derivating the medium parameters, a 16-point wide operator was used. The 16 point operator calculates the derivative of the step function more accurately and is less local than the shorter 6-point operator. The shorter operator is adequate for the wave field, since it is a smoother function than the medium and it is not necessary to use the full 16 point operator. The results of the split operator, the composite 16-point and the composite 6-point operator, are overlaid in Figure **??**. One can see that the results from the composite 16-point and the split operator are hardly distinguishable, but that the result from the composite 6-point operator clearly shows differences and cannot adequately propagate the wave field. The step discontinuity has a very high frequency content and one prefers an accurate computation of the derivative of that discontinuity.

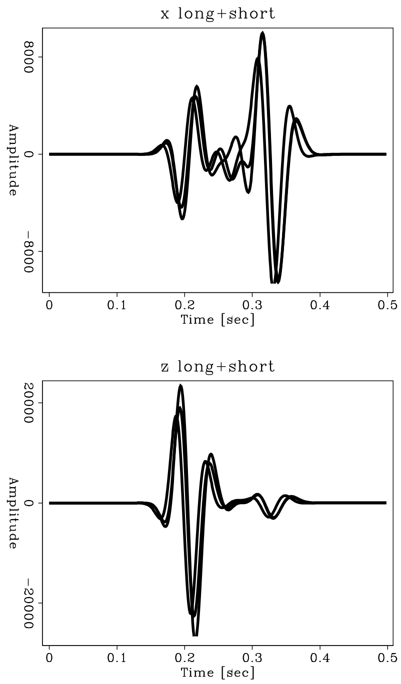

Figure 0.3: A trace is taken from three wave fields, using three different modeling operators. The composite 16-point operator and a split operator, which match well and a composite 6-point operator which shows differences. are over lain. The source is located above a step discontinuity and the receivers are located below. mlong+fshort [ER]

0.1.4 Summary

The wave operator can formally and numerically be split into parts that involve taking derivatives of either model parameters or propagating field quantities. For identical derivative operators this is the same as the conventional method that takes nested derivatives of those quantities. However, the advantage of using the split operators is the possibility to treat the two parts differently. I have given the splitting for completely anisotropic media. If one assumes different a priori properties of the medium, the operators can be adjusted to match the medium. Another advantage is the increase in efficiency that a certain optimum operator combination can achieve given a desired accuracy. So far adaptation to particular media have not been used much; a possible topic of future research is the use of wavelet transform for derivative calculations for medium and field variables and their proper back interpolation into the primary grid.