Chapter 1

Estimation of elastic constants in homogeneous transversely isotropic media

The elastic constants that control P- and SV-wave propagation in a transversely isotropic medium can be estimated by using P- and SV-wave traveltimes from either cross-well or VSP geometries. This chapter explains the procedure, which consists of two steps. First, elliptical velocity models are used to fit the traveltimes near one axis. The result is four elliptical parameters that represent direct and normal moveout velocities near the chosen axis for P- and SV-waves. Second, the elliptical parameters are used to solve a system of four equations and four unknown elastic constants. The system of equations is solved analytically, yielding simple expressions for the elastic constants as a function of direct and normal moveout velocities. For SH-waves, the estimation of the corresponding elastic constants is easier because the phase velocity is already elliptical. The procedure for homogeneous media introduced in this chapter is generalized to heterogeneous media as explained in chapter 5, by using a tomographic technique described in chapters 3 and 4.

1.1 Introduction

The estimation of elastic constants in anisotropic media from lab measurements differs from the estimation of elastic constants from field measurements in two basic ways. First, in the lab the medium can be assumed to be homogeneous and therefore the propagation of the energy can be safely modeled using straight ray paths. In the field, this assumption is often inadequate and ray bending needs to be properly considered. Second, in the lab it is possible to measure the velocities along all axes of symmetry, which simplifies the calculation of the corresponding elastic constants. In the field, however, the measurements usually have limited aperture, and therefore velocities in one direction have to be estimated by fitting a model to velocities measured in different directions. These two difficulties explain why, when using field data, in situ estimation of elastic constants has been rare, unless a simple model is assumed to describe the anisotropy.

Elliptical anisotropy is the simplest model of anisotropy, since it introduces only one more degree of freedom, and wave and ray relationships are similar to the isotropic case. Elliptical anisotropy is not only simple but also useful when the aperture of the experiment is such that the slowness surface is not properly sampled and is difficult to estimate more than two ray velocities, one based on the arrivals along a certain axis (direct velocities), and the other based on the curvature around that axis (normal moveout velocities). However, since most rocks do not exhibit elliptical anisotropy, the convenient properties of this type of anisotropy haven't found wide applicability, except in a few cases when the departures from isotropy are small or when the measurements are taken around an axis of symmetry.

A model commonly used to describe velocity anisotropy in the earth is the transversely isotropic (TI) model. Even though a TI model hardly ever reduces to an elliptically anisotropic one (Thomsen, 1986), a TI model appears elliptically anisotropic near the axes of symmetry. This property of transverse isotropy is used by Muir (1990a) and Dellinger et al. (1993) to approximate P- and SV-wave slowness surfaces and impulse responses of general TI media with ellipses fitted near *both* horizontal and vertical axes. In a later paper, Muir (1990b) suggests the transformation of the elliptical parameters into elastic constants by using well known expressions that relate them (Levin, 1979; Levin, 1980). However, Muir implicitly assumes in this paper that traveltimes near both axes are available, which doesn't often happen.

In this chapter I show how to obtain the elastic constants that control P- and SV-wave propagation in TI media from limited aperture traveltimes, either from VSP or from cross-well geometries. I start by fitting the traveltimes for P- and SV-waves with elliptical time-distance relations near a single axis (either vertical or horizontal). The result is four velocities: two based on the time-of-arrival and distance along an axis of symmetry (the direct velocities) and two based on the differential traveltime and differential distance as the direction is perturbed (the normal moveout velocities). I use these four elliptical parameters to solve analytically a system of four equations and four unknown elastic constants. Since the procedure is based on fitting the data with elliptical velocity models, it is exact when estimating elastic constants from SH-wave traveltimes.

The data aperture is constrained in two different ways. First, it should be large enough to ensure that there is sufficient curvature to estimate the normal moveout velocities. Second, it should be narrow enough to ensure that the elliptical fit remains accurate for the given wave type. I show in this chapter that there is an intermediate range of ray angles that satisfy these two requirements in the estimation of elastic constants from either VSP or cross-well geometries.

The calculations presented here are valid for homogeneous media. When the model is heterogeneous, it can be described as a superposition of homogeneous regions, and the elliptical parameters needed at each region can be estimated tomographically, as chapter 4 explains. Once the elliptical parameters at each cell are estimated, the procedure developed here for homogeneous media can be applied at each cell to obtain 2-D maps of elastic constants. Chapter 5 presents examples of such a procedure.

The equations I use in this chapter to transform elastic constants into elliptical parameters (forward mapping) are not new. They are the same as the ones summarized by Muir (1990b), which can also be found in Levin (1979) and Levin (1980). What is new is the simultaneous solution of these equations near each axis to obtain elastic constants as a function of elliptical parameters (inverse mapping).

I start by rederiving the basic equations for forward mapping from the expression of P- and SV-wave phase velocities in TI media. The calculations are done near both the horizontal and the vertical axes. Then, using these expressions, I solve the inverse mapping analytically. The final section illustrates the use of the technique when estimating the elastic constants of a homogeneous medium from impulse responses sampled near either the vertical axis or the horizontal axis, to simulate VSP and cross-well configurations, respectively.

1.2 Forward mapping

By doing forward modeling we create data from a given set of model parameters. When the data form a different set of model parameters, I prefer to call the process forward mapping to emphasize that the transformation is done between spaces that cannot be measured directly. In this section I explain how to do the mapping from the elastic constants of a TI medium to different sets of phase velocities.

1.2.1 From elastic constants to phase velocities

The phase velocity expression for P- and SV-waves in TI media is (Auld, 1990)

$$\begin{aligned} 2W_{P,SV}(\theta) \;=\; & (W_{33}+W_{44})\cos^2\theta + (W_{11}+W_{44})\sin^2\theta \qquad (1.1a) \\ & \pm\sqrt{[(W_{33}-W_{44})\cos^2\theta - (W_{11}-W_{44})\sin^2\theta]^2 + 4(W_{13}+W_{44})^2\sin^2\theta\cos^2\theta}, \end{aligned}$$

where $W_{P,SV}(\theta)$ is the phase velocity squared, and θ is the phase angle from the vertical. W_{ij} is the $(ij)^{th}$ elastic modulus divided by density, with units of velocity squared; I refer to the quantity W_{ij} as an *elastic constant* in the remainder of this chapter. The plus sign (+) in front of the square root corresponds to P-waves and the minus sign (−) to SV-waves. For SH-waves, the expression for the phase velocity is (Auld, 1990)

$$W_{SH}(\theta) \;=\; W_{44}\cos^2\theta + W_{66}\sin^2\theta. \qquad (1.1b)$$

Near the vertical and horizontal axes ($\theta \approx 0$ and $\theta \approx 90$) equation (1.1a) is elliptical. The velocities that describe the corresponding ellipses are called elliptical velocities. In the next two sections, I rederive equations contained in previous papers (Levin, 1979; Levin, 1980; Muir, 1990b) that are needed to calculate these elliptical velocities from the elastic constants. The expressions that result are used later in the chapter to solve the inverse mapping when the data have a narrow aperture.

1.2.2 From elastic constants to phase velocities near the vertical axis

Expanding equation (1.1a) around $\theta = 0$ and neglecting terms in $\sin^4\theta$, results in

$$\begin{aligned} 2W_{P,SV}(\theta) &= (W_{33}+W_{44})\cos^2\theta + (W_{11}+W_{44})\sin^2\theta \qquad (1.2)\\ &\pm\left((W_{33}-W_{44})\cos^2\theta - (W_{11}-W_{44})\sin^2\theta + \frac{2(W_{13}+W_{44})^2}{W_{33}-W_{44}}\sin^2\theta\right). \end{aligned}$$

Choosing the positive root yields the P-wave phase velocity near the vertical axis, as follows:

$$W_P(\theta) = W_{P,z}\ c^2 + W_{P,x\mathrm{NMO}}\ s^2, \qquad (1.3)$$

where $c = \cos\theta$, $s = \sin\theta$,

$$W_{P,z} = W_{33}, \qquad (1.4)$$

and

$$W_{P,x\mathrm{NMO}} = W_{44} + \frac{(W_{13}+W_{44})^2}{W_{33}-W_{44}}. \qquad (1.5)$$

$W_{P,z}$ is the vertical P-wave phase velocity squared and $W_{P,x\mathrm{NMO}}$ is the horizontal normal moveout (NMO) phase velocity squared.

Choosing the negative root in equation (1.3) yields SV-wave phase velocities near the vertical axis, as follows:

$$W_{SV}(\theta) = W_{SV,z}\ c^2 + W_{SV,x\mathrm{NMO}}\ s^2, \qquad (1.6)$$

where

$$W_{SV,z} = W_{44}, \qquad (1.7)$$

and

$$W_{SV,x\mathrm{NMO}} = W_{11} - \frac{(W_{13}+W_{44})^2}{W_{33}-W_{44}}. \qquad (1.8)$$

The previous expressions for the NMO velocities agree with the results of Thomsen (1986) and Vernik and Nur (1992).

The expression for SH-wave phase velocities near the vertical axis is

$$W_{SH}(\theta) = W_{SH,z}\ c^2 + W_{SH,x\mathrm{NMO}}\ s^2, \qquad (1.9)$$

where

$$W_{SH,z} = W_{44}, \qquad (1.10)$$

and

$$W_{SH,x\mathrm{NMO}} = W_{SH,x} = W_{66}. \qquad (1.11)$$

1.2.3 From elastic constants to phase velocities near the horizontal axis

P- and SV-wave phase velocities near the horizontal axis can be obtained by interchanging W_{11} and W_{33} and c^2 and s^2 wherever they occur in equations (1.3) through (1.8). Thus, for P-waves the result is

$$W_P(\theta) \;=\; W_{P,x}\; s^2 + W_{P,z\mathrm{NMO}}\; c^2, \tag{1.12}$$

where

$$W_{P,x} \;=\; W_{11}, \tag{1.13}$$

and

$$W_{P,z\mathrm{NMO}} \;=\; W_{44} + \frac{(W_{13} + W_{44})^2}{W_{11} - W_{44}}. \tag{1.14}$$

For SV-waves, the expression for the phase velocity near the horizontal axis is

$$W_{SV}(\theta) \;=\; W_{SV,x}\; s^2 + W_{SV,z\mathrm{NMO}}\; c^2, \tag{1.15}$$

where

$$W_{SV,x} \;=\; W_{44}, \tag{1.16}$$

and

$$W_{SV,z\mathrm{NMO}} \;=\; W_{33} - \frac{(W_{13} + W_{44})^2}{W_{11} - W_{44}}. \tag{1.17}$$

Near the horizontal axis the SH-wave phase velocities are

$$W_{SH}(\theta) \;=\; W_{SH,x}\; s^2 + W_{SH,z\mathrm{NMO}}\; c^2, \tag{1.18}$$

where

$$W_{SH,x} \;=\; W_{66}, \tag{1.19}$$

and

$$W_{SH,z\mathrm{NMO}} \;=\; W_{SH,z} \;=\; W_{44}. \tag{1.20}$$

In the rest of the chapter I refer to the elliptical parameters $W_{P,x}$, $W_{P,z}$, $W_{P,x\mathrm{NMO}}$, $W_{P,z\mathrm{NMO}}$, $W_{SV,x}$, $W_{SV,z}$, $W_{SV,x\mathrm{NMO}}$, $W_{SV,z\mathrm{NMO}}$, $W_{SH,x}$, $W_{SH,z}$, $W_{SH,x\mathrm{NMO}}$, and $W_{SH,z\mathrm{NMO}}$ as W_*, direct, or NMO phase velocity squared for P-, SV-, and SH-waves.

1.3 Inverse mapping

The preceding equations that relate phase velocities and elastic constants suggest several approaches to solve the inverse mapping that depend on the recording geometry, the recording aperture, and the wave types available. From SH-wave phase velocities near either axis, it is always possible to estimate W_{44} and W_{66} because the NMO velocities are equal to corresponding direct velocities. However, from P- and SV-wave phase velocities near either axis, the estimation of the corresponding elastic constants is not straightforward. This section explains what to do in such a case.

1.3.1 Using P- and SV-wave full aperture phase velocities

To estimate W_{11}, W_{33}, W_{44}, and W_{13} directly from equation (1.1a), we need at least four phase velocities at four different angles between 0 and 90 degrees. W_{ij} is the solution of a system of nonlinear equations where the independent term is formed by these phase velocities. Along the axes, the system of equations is almost diagonal, and the estimation of three elastic constants (W_{33}, W_{11}, and W_{44}) is straightforward:

$$W_{33} = W_{P,z}, \tag{1.21a}$$

$$W_{11} = W_{P,x}, \tag{1.21b}$$

$$W_{44} = W_{SV,z} = W_{SV,x}. \tag{1.21c}$$

The elastic constants are estimated directly from phase velocities along the axes. W_{13} can be estimated from the previous elastic constants and one phase velocity at an oblique angle, typically 45 degrees.

This approach, though simple in theory, is not applicable in many practical situations because wide aperture data are required (i.e., the angles of the observations must include 0, 90 degrees, and one intermediate measurement) to simplify the system of equations. This is not the case for most single-geometry data sets (either surface, or cross-well or VSP) where no rays travel along at least one axis and therefore phase velocities along both axes cannot be estimated without having to assume a velocity symmetry (e.g., isotropic, elliptical, or TI).

1.3.2 Using P- and SV-wave narrow aperture phase velocities

1.3.3 Elastic constants from phase velocities near the vertical

When the phase angles are close to zero, it is possible to estimate elastic constants from the corresponding phase velocities by using equations (1.4), (1.5), (1.7), and (1.8), a system of four equations and four unknowns. The independent term (that I will explain how to obtain in section 2.4, "Obtaining the phase velocities") is formed by $W_{P,z}$, $W_{P,x\mathrm{NMO}}$, $W_{SV,z}$, and $W_{SV,x\mathrm{NMO}}$. The solution of this system of equations is

$$W_{33} = W_{P,z}, \tag{1.22a}$$

$$W_{44} = W_{SV,z}, \tag{1.22b}$$

$$W_{13} = \sqrt{(W_{P,x\mathrm{NMO}} - W_{SV,z})(W_{P,z} - W_{SV,z})} - W_{SV,z}, \tag{1.22c}$$

$$W_{11} = W_{SV,x\mathrm{NMO}} + W_{P,x\mathrm{NMO}} - W_{SV,z}. \tag{1.22d}$$

In (1.22d), W_{11} is a linear combination of elliptical parameters independent of the horizontal P-wave phase velocity, unlike the W_{11} estimated from equation (1.21c). Equation (1.22d) says, roughly, that summing P- and SV-wave NMO velocities (squared) is the same as summing the elastic constants that control the horizontal wave propagation. The examples in the final section of the chapter show the range of angles near the vertical for which these equations are valid.

Since this approximation simultaneously uses the elliptical parameters of two ellipses fitted near the vertical, I call it *vertical*, double elliptic approximation, analogous to Muir's double elliptic approximation that uses horizontal and vertical ellipses to approximate the slowness surface and impulse response for all angles (Muir, 1990a; Dellinger et al., 1993). It is important to recognize the differences between these two approximations. On the one hand, the *vertical*, double elliptic approximation is used to estimate elastic constants from phase velocities near the vertical axis. Slowness surfaces and impulse responses can be calculated from these elastic constants using (1.1a) and the exact relationships between phase and group velocities. On the other hand, Muir's double elliptic approximation is used to estimate directly slowness surfaces and impulse responses from data near both axes without having to know the elastic constants.

Fitting P- and SV-wave phase velocities with ellipses near the vertical is not the same as using the vertical, double elliptic approximation. However, the fitting is a necessary intermediate step in the estimation of elastic constants using equations (2.22). When the elliptical fitting is done near the horizontal axis, the result is the *horizontal*, double elliptic approximation, as follows.

1.3.4 Elastic constants from phase velocities near the horizontal

When the phase angle is close to 90 degrees, the expressions for the elastic constants as a function of P- and SV-wave phase velocities are obtained by solving the system of equations (1.13), (1.14), (1.16), and (1.17), with the following result:

$$W_{11} = W_{P,x}, \tag{1.23a}$$

$$W_{44} = W_{SV,x}, \tag{1.23b}$$

$$W_{13} = \sqrt{(W_{P,z\mathrm{NMO}} - W_{SV,x})(W_{P,x} - W_{SV,x})} - W_{SV,x}, \tag{1.23c}$$

$$W_{33} = W_{SV,z\mathrm{NMO}} + W_{P,z\mathrm{NMO}} - W_{SV,x}. \tag{1.23d}$$

This set of equations forms the *horizontal*, double elliptic approximation. It uses elliptical P- and SV-wave phase velocities near the horizontal to approximate the elastic constants. Notice that the estimation of W_{33} is independent of the P-wave phase velocity along the vertical axis. Equations (2.23) can be obtained from (2.22) by interchanging W_{11} and W_{33}, and x and z.

The estimation of W_{33} from near-horizontal phase velocities [equation (1.23d)] and W_{11} from near-vertical phase velocities [equation (1.22d)] is in both cases the sum of NMO velocities minus W_{44}. I show in appendix **??** that when estimating velocities tomographically, NMO velocities correspond to the smallest singular values of the problem. The largest singular values correspond to velocities estimated from rays that travel along the axes. Therefore, as expected, estimating W_{33} from cross-well traveltimes alone is a harder problem than estimating W_{11} from the same data. The opposite is true when estimating W_{33} and W_{11} from VSP measurements.

1.3.5 Using only P-wave phase velocities near the axes

When the medium is TI and no information is available about SV-wave phase velocities, it is still possible to obtain four elastic constants from only P-wave phase velocities near both axes. This is done by solving the system of equations (1.4), (1.5), (1.13), and (1.14), which yields

$$W_{11} = W_{P,x}, \tag{1.24a}$$

$$W_{33} = W_{P,z}, \tag{1.24b}$$

$$W_{44} = \frac{W_{P,x\mathrm{NMO}} W_{P,z} - W_{P,z\mathrm{NMO}} W_{P,x}}{W_{P,x\mathrm{NMO}} + W_{P,z} - W_{P,z\mathrm{NMO}} - W_{P,x}}, \tag{1.24c}$$

$$W_{13} = \sqrt{(W_{P,z\mathrm{NMO}} - W_{44})(W_{P,x} - W_{44})} - W_{44}. \tag{1.24d}$$

This set of equations forms the P-wave, double elliptic approximation of the elastic constants in a TI medium.

In this approximation, as well as in the previous ones, the assumption of transverse isotropy is crucial. When the medium is isotropic ($W_{P,x} = W_{P,z} = W_{P,x\mathrm{NMO}} = W_{P,z\mathrm{NMO}}$) there is no way to calculate W_{44} (the shear modulus) from P-wave phase velocities alone because equation (1.24c) is indeterminate. When the medium is weakly anisotropic the estimation of W_{44} using this approximation may still be unreliable because both the numerator and the denominator in (1.24c) are close to zero. We will see from the examples later in the chapter that even when the medium is moderately anisotropic this approximation breaks down quickly for phase angles not close to the axes, unlike the previous vertical and horizontal double elliptic approximations [equations (2.22) and (2.23), respectively] that have a wider range of validity.

If only SV-wave phase velocities near the axes are available, it is not possible to obtain the corresponding elastic constants, because the system of equations (1.7), (1.8), (1.16), and (1.17) is underdetermined.

1.4 Obtaining the phase velocities

Equations (1.3) , (1.6), (1.12), and (1.15) show that the phase velocities of P- and SV-waves are elliptical near the axes of symmetry. Those of *SH*-waves are also elliptical [as equation (2.1b) shows]. When the phase velocity has an elliptical shape, the corresponding impulse response is also elliptical (Levin, 1978; Byun, 1982). Therefore, the group slowness expression that corresponds to these equations has the general form

$$S^2(\phi) \;=\; S_z^2 \cos^2\phi + S_x^2 \sin^2\phi, \tag{1.25}$$

where ϕ is the ray angle measured from the vertical, and S_* (the ray slowness) is

$$S_*^2 \;=\; \frac{1}{W_*}. \tag{1.26}$$

To estimate S_*, I use the expression for the traveltime of a ray that travels a distance $l = \sqrt{\Delta x^2 + \Delta z^2}$ between two points:

$$t^2 \;=\; \Delta x^2 S_x^2 + \Delta z^2 S_z^2. \tag{1.27}$$

This equation, which has the same form as the isotropic moveout equation, is obtained after multiplying equation (1.25) by l^2. Velocities estimated from the curvature around one axis of symmetry using this traveltime equation are called NMO velocities, and velocities estimated from arrival times along the same axis are called direct velocities. Hence the different names chosen for the phase velocities W_*.

1.5 Using traveltimes to estimate elastic constants

The procedure for estimating the elastic constants of a homogeneous TI medium from traveltime measurements near one axis of symmetry is the following:

1. Fit the traveltimes with elliptically anisotropic models, one model for each wave type. This gives direct and NMO group slownesses.

2. From the direct and NMO group slownesses, find the corresponding direct and NMO phase velocities, using equation (1.26).

3. From the estimated phase velocities, find the elastic constants, using the equations that correspond to the given recording geometry [either (2.22) or (2.23)]. For *SH*-waves, the estimated phase velocities squared are the same as the corresponding elastic constants.

This procedure can be generalized to heterogeneous media by describing the model as a superposition of homogeneous blocks. Chapters 3 and 4 explain the basic ingredients of this approach which are anisotropic ray tracing and tomographic estimation of elliptical velocities.

1.6 Examples

This section contains examples that illustrate how four elastic constants of a homogeneous TI medium can be estimated when the data are (a) P- and SV-wave traveltimes from VSP geometries with different apertures, (b) P- and SV-wave traveltimes from cross-well geometries with different apertures, and (c) P-wave traveltimes from cross-well and VSP geometries with different apertures.

The impulse response is given, and it is the same for all examples. The elastic constants that describe the medium are: $\sqrt{W_{11}} = 7400$, $\sqrt{W_{33}} = 6295$, $\sqrt{W_{13}} = 5575$, and $\sqrt{W_{44}} = 2160$, all with units of (ft/s) (Byun and Corrigan, 1990). I calculate the ellipse that fits the impulse response near the chosen axis by using a ray along the axis and a ray close to the axis. Once I have found the four needed elliptical parameters I use equation (2.22), (2.23), or (2.24) to convert those elliptical parameters into elastic constants.

1.6.1 Estimating W_{ij} from VSP P- and SV-wave traveltimes

Figure 1.1 shows how the vertical double elliptic approximation works for different angles around the vertical. The parameters of the ellipses that approximate the impulse response have been calculated at the angles shown by the straight lines. The left column compares the given impulse responses for P- and SV-waves (continuous lines) with the elliptical approximations around the vertical (dashed lines). With the four elliptical parameters obtained for each aperture, I calculate the elastic constants of the medium by using equation (2.22). From the estimated elastic constants, I then calculate the corresponding impulse responses for both P- and SV-waves. The result is shown in the central column (dashed lines) together with the given impulse responses. In most cases (except with the 40-degree aperture) the agreement is excellent. With the 40-degree aperture the horizontal P-wave group velocity has been overestimated and the shear wave triplication is larger than expected. The right column shows the absolute value of the percentage error made in the estimation of the elastic constants. For small angles (≤ 10 degrees), the error is negligible. For angles between 10 and 30 degrees, the error is smaller in W_{11} (A) than in W_{13} (F). For large angles, the error in W_{11} (A) is the largest, almost 30 percent. Notice that up to 30 degrees, even though the error in the estimation of the elastic constants is not zero but a few percent, the differences between given and estimated impulse responses are hard to see.

In Figure 1.1, the elliptical approximations to the impulse responses are calculated using two ray angles at a time, one zero and one nonzero. Figure 1.2 shows what happens when all these angles (or different VSP offsets) are used simultaneously to calculate the elliptical approximations and the elastic constants. The horizontal P-wave group velocity is now slightly overestimated, and the shear velocities are retrieved well. The errors in the estimated elastic constants are six percent in W_{11} (A) and four percent in W_{13} (F). The errors made when using only large angles have been compensated by also using small angles.

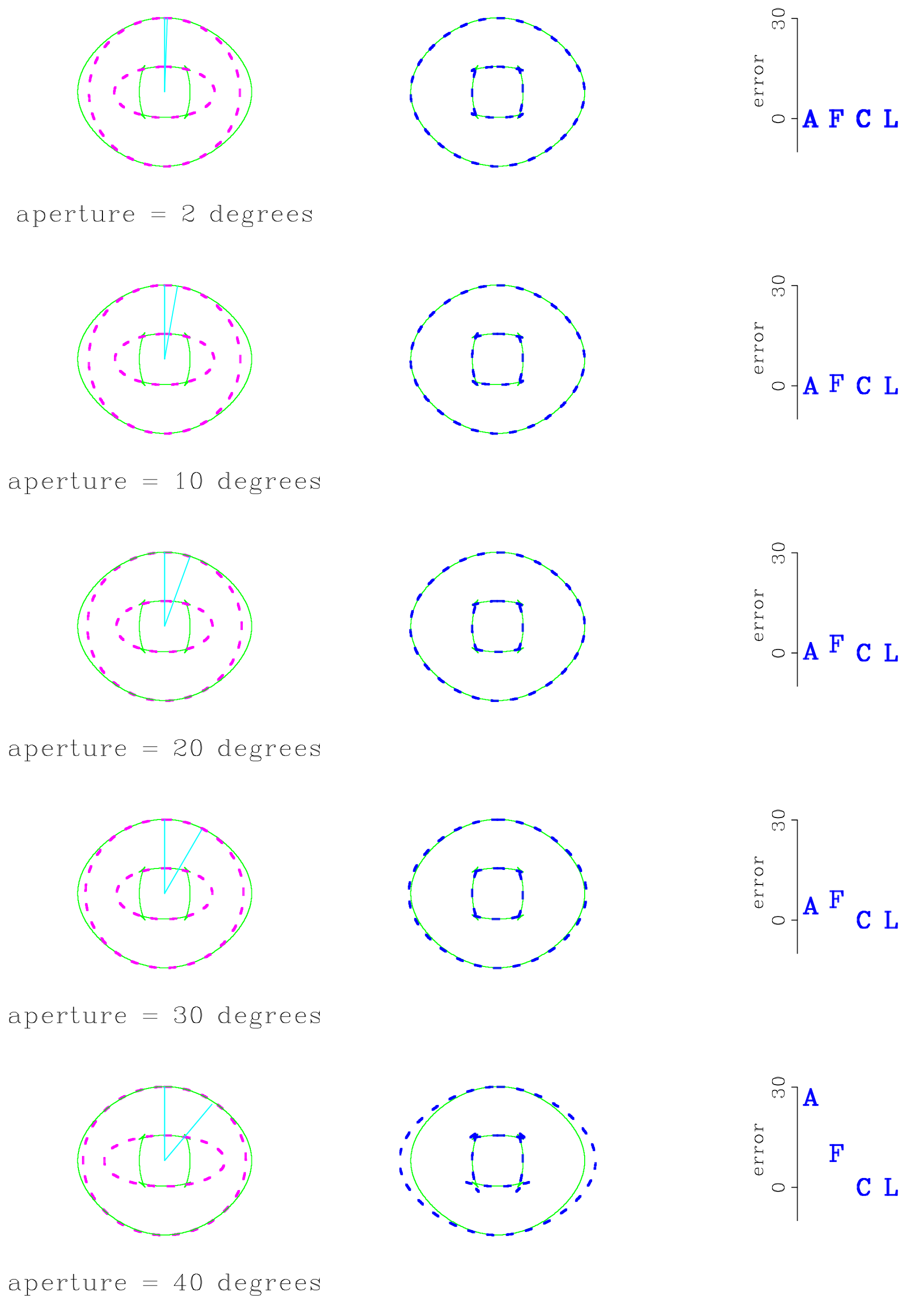

Figure 1.1: Left: impulse response for P- and SV-waves (continuous lines) compared with their elliptical approximations around the vertical (dashed lines). Center: given impulse responses (continuous) compared with the ones calculated from the estimated elastic constants (dashed). Right: absolute value of the error made in the estimation of the elastic constants when using the vertical, double elliptic approximation. The elastic constants are A = W_{11}, F = W_{13}, C = W_{33}, and L = W_{44}. The density is assumed to be unity. vsp [NR]

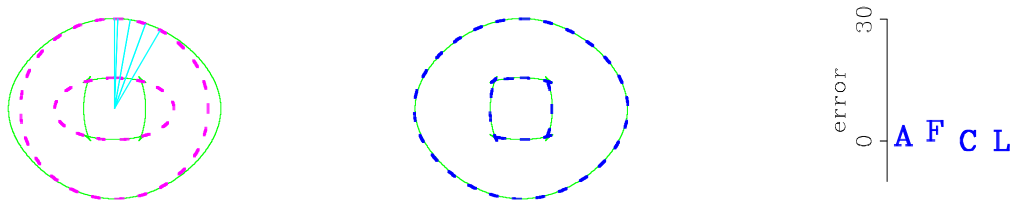

Figure 1.2: Left: impulse response for P- and SV-waves (continuous lines) compared with their elliptical approximations around the vertical (dashed lines). All ray angles shown are used simultaneously to calculate the elliptical approximations. Center: given impulse responses (continuous) compared with the ones calculated from the estimated elastic constants (dashed). Right: absolute value of the error made in the estimation of the elastic constants when using the vertical, double elliptic approximation. The elastic constants are A = W_{11}, F = W_{13}, C = W_{33}, and L = W_{44}. vsp-4angles [NR]

1.6.2 Estimating W_{ij} from cross-well P- and SV-wave traveltimes

Figure 1.3 shows how the horizontal, double elliptic approximation works for different ray angles measured from the horizontal. When comparing this case with the VSP case (Figure 1.1), we see that the approximation works better for ray angles around the horizontal than around the vertical, even when the angles are large. The estimated impulse responses agree well with the given ones, and the errors in the elastic constants are negligible for angles less than 30 degrees.

The reason the horizontal, double elliptic approximation works better than the vertical, double elliptic approximation is that the elliptical approximation for P-wave impulse response is more adequate at larger angles around the horizontal than around the vertical, where the group velocity is lower. For P-waves, the elliptic approximations around the vertical are almost circular, which explains why in practice the NMO velocity is a good estimate for the actual vertical velocity of the medium.

Figure 1.4 shows that when small and large ray angles are used simultaneously to calculate the elliptical approximations around the horizontal, the result is roughly an average of the results shown in Figure 1.3. The agreement between given and estimated impulse responses is excellent and the error in the estimation of the elastic constants is $\approx$ two percent for W_{13} (F) and $\approx$ one percent for W_{33} (C).

1.6.3 Estimating W_{ij} from cross-well and VSP P-wave traveltimes

For TI media, the elastic constants that control the P- and SV-wave propagation can be estimated from P-wave measurements near the axes, as equation (2.24) shows. Figure 1.5 is an example. The left column shows the given P-wave impulse response (continuous line) and the approximating ellipses (dashed lines) around the horizontal

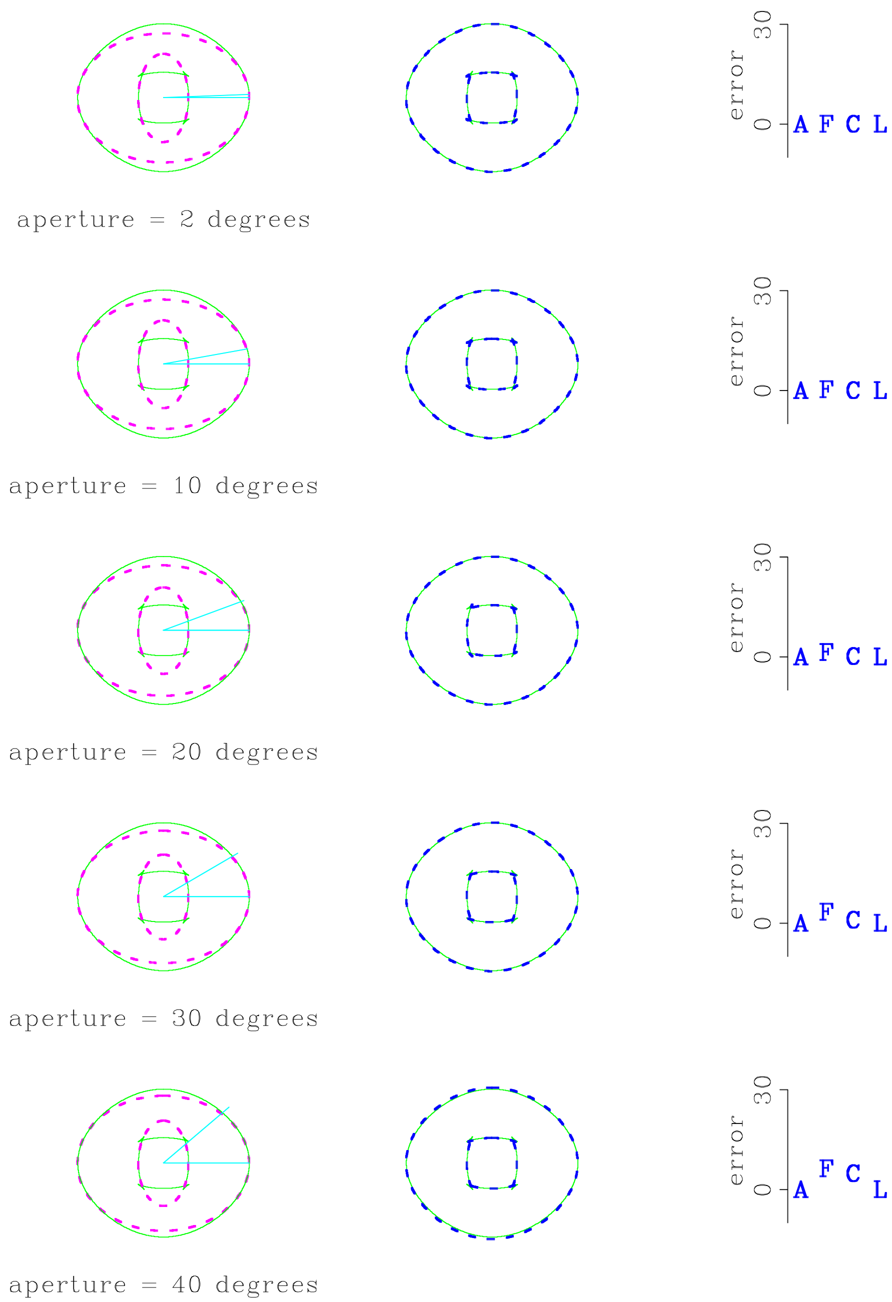

Figure 1.3: Left: impulse response for P- and SV-waves (continuous lines) compared with their elliptical approximations around the horizontal (dashed lines). Center: given impulse responses (continuous) compared with the ones calculated from the estimated elastic constants (dashed). Right: absolute value of the error made in the estimation of the elastic constants when using the horizontal, double elliptic approximation. The elastic constants are A = W_{11}, F = W_{13}, C = W_{33}, and L = W_{44}. $\boxed{\text{xw}}$ [NR]

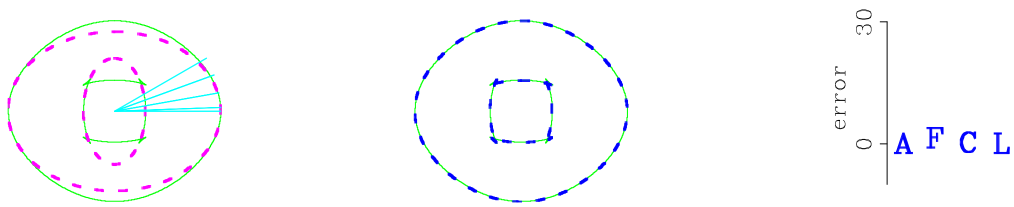

Figure 1.4: Left: impulse response for P- and SV-waves (continuous lines) compared with their elliptical approximations around the horizontal (dashed lines). All ray angles shown are used simultaneously to calculate the elliptical approximations. Center: given impulse responses (continuous) compared with the ones calculated from the estimated elastic constants (dashed). Right: absolute value of the error made in the estimation of the elastic constants when using the horizontal, double elliptic approximation. The elastic constants are A = W_{11}, F = W_{13}, C = W_{33}, and L = W_{44}. xw-4angles [NR]

and around the vertical for two different apertures. The central column shows the given and approximate impulse responses for P- and SV-waves, the approximate impulse responses calculated from the estimated elastic constants. The right column shows the absolute value of the error made in the estimation of the elastic constants. Unlike the previous approximations, the P-wave, double elliptic approximation is valid only at very small angles ($\approx$ two degrees). At greater angles it yields highly inaccurate results, particularly in the estimation of W_{44} (L), which controls the shear wave propagation along the axes. Note that for an aperture of 10 degrees, the shear wave impulse response has been considerably underestimated.

This approximation may be hard to use in practical situations because it works only for very small angles near the axes.

1.7 Constraints on elliptical velocities in layered media

I have shown that traveltimes from both compressional and shear waves are necessary for an accurate estimation of the elastic constants from measurements near the axes of symmetry. However, in cases when only one set of traveltimes is available, we may still obtain useful information about the medium from the elliptical velocities that best fit the given traveltimes, if we know in advance possible causes of the anisotropy of the medium.

In particular, when the anisotropy is caused by fine layering of isotropic materials, as Backus (1962) shows, $W_{66} \geq W_{44}$, which, combined with the energy constraints on W_{ij}, leads to

$$W_{11} \geq W_{44}. \tag{1.28}$$

For this type of anisotropy, an additional inequality is provided by Berryman (1979) and Helbig (1979):

$$W_{13} < W_{13E}, \tag{1.29}$$

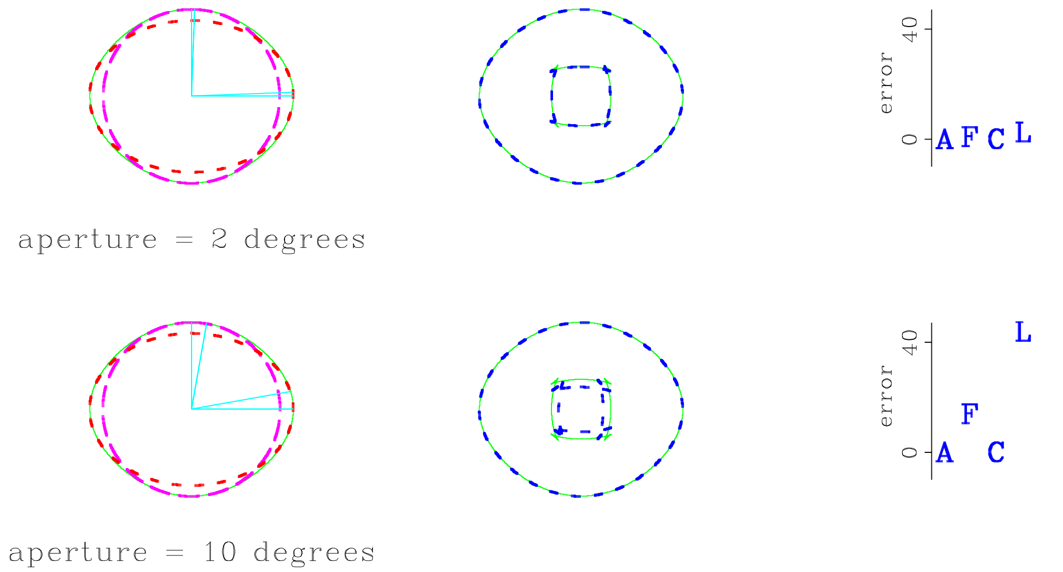

Figure 1.5: P-wave impulse response used to estimate the elastic constants from measurements around the axes. Left: given P-wave impulse response (continuous lines) and elliptical approximations around the axes (dashed lines). Center: given impulse response (continuous) and estimated ones (dashed) for P- and SV-waves. Right: absolute value of the error made in the estimation of the elastic constants when using the P-wave double elliptic approximation. xw-vsp [NR]

where

$$W_{13E} = \sqrt{(W_{11} - W_{44})(W_{33} - W_{44})} - W_{44} \tag{1.30}$$

is a particular value of W_{13} that makes expression (1.1a) elliptical for P-waves and circular for SV-waves. By combining equation (1.28) and equation (1.30), results

$$W_{33} \geq W_{44}. \tag{1.31}$$

Then, by using equations (1.28), (1.29), and (1.31), and fact that in general

$$W_{11} \geq W_{33}, \tag{1.32}$$

it is possible to obtain bounds for the normal moveout velocities derived in previous sections. The procedure is simple. One bound is obtained from the equation of the corresponding normal moveout velocity as a function of the elastic constants. The other bound is obtained by using equation (1.29). The results can be summarized as follows:

$$W_{SV,x} < W_{P,x\mathrm{NMO}}, W_{SV,x\mathrm{NMO}} < W_{P,x}, \tag{1.33}$$

$$W_{SV,z} < W_{P,z\mathrm{NMO}}, W_{SV,z\mathrm{NMO}} < W_{P,z}. \tag{1.34}$$

These inequalities are valid when the axis of symmetry is either vertical or horizontal.

If the anisotropy is caused by fine layering of isotropic materials then the relative sizes of the direct and NMO velocities are given by inequalities (1.33) and (1.34).

In other words, fine layering $\Rightarrow$ (1.33) or (1.34). However, the reverse [i.e., (1.33) or (1.34) $\Rightarrow$ fine layering] is not necessarily true but it can aid the interpretation when the causes of anisotropy are suspected or when a complete inversion of all the elastic constants is not possible. For example, from cross-well measurements the estimated $W_{P,z\mathrm{NMO}}$ is smaller than $W_{P,x}$ when the axis of symmetry is vertical (i.e., $W_{P,z} < W_{P,x}$) but this is not necessarily the case when the axis of symmetry is horizontal (i.e., $W_{P,z} > W_{P,x}$). Notice that inequalities (1.33) and (1.34) are strict. When direct and normal moveout velocities are found to be equal, the anisotropy of the medium cannot be caused by fine layering.

When the conditions $W_{SV,x\mathrm{NMO}} = W_{SV,z}$ and $W_{SV,z\mathrm{NMO}} = W_{SV,x}$ are substituted into equations (1.8) and (1.17) respectively, we obtain $W_{13} = W_{13E}$, and therefore P-wave velocities are elliptical and SV-wave velocities are circular. This means that, for shear waves, isotropy around one axis indicates that the velocities are isotropic for all angles.[1] For compressional waves, isotropy around one axis does not always mean isotropy for all angles. When the condition $W_{P,z\mathrm{NMO}} = W_{P,x}$ is substituted into equation (1.14), we get $W_{13} = W_{11} - 2W_{44}$, and when $W_{P,x\mathrm{NMO}} = W_{P,z}$ in equation (1.5), it follows that $W_{13} = W_{33} - 2W_{44}$. None of these values of W_{13} transform (1.1a) into a circle, unless $W_{11} = W_{33}$.

1.8 Conclusions

I have shown how to estimate the elastic constants of homogeneous TI media from P-, SV-, and SH-wave traveltimes near a single axis of symmetry (either from VSP or cross-well geometries). The technique uses the parameters obtained by fitting traveltimes near one axis with elliptical models. For SH-wave traveltimes, the estimation of the corresponding TI elastic constants is simple because SH-wave phase velocities are also elliptical in TI media. For P- and SV-wave traveltimes, four parameters are needed to estimate the phase velocities at all angles from measurements near one axis. These parameters are the direct and NMO phase velocities for P- and SV-waves.

The transformation from elliptical parameters to elastic constants is simple for both VSP and cross-well geometries.

The accuracy in the estimation of the elastic constants from the elliptical parameters depends on the accuracy in the estimation of the NMO velocities. For small ray angles the approximation is excellent when using either VSP or cross-well geometries because the elliptical approximations are adequate in both cases. In practice, however, using only small ray angles may hinder an accurate estimation of NMO velocities. For intermediate ray angles, the estimation of vertical P-wave velocities from cross-well geometries is more accurate than the estimation of horizontal P-wave velocities from VSP. This is because the elliptical approximation for the P-wave impulse response fits wider angles around the horizontal than around the vertical where

[1] Here *isotropy* means that the phase velocity is circular. This use of the term does not take into account the nature of the particle motion, which may not correspond to a pure SV-mode (Dellinger, 1991).

the group velocity is smaller. For large ray angles the approximation doesn't work because elliptical fits are not as accurate.

When compressional and shear velocity logs are also available, they can be used either to add redundancy in the estimation of W_{11} and W_{44} or to check whether the assumption of transverse isotropy is valid, in particular when using cross-well measurements where vertical velocities are not well sampled.

In chapter 5, I show how we can combine this technique and anisotropic traveltime tomography to estimate spatial variations of elastic constants. But to do anisotropic traveltime tomography, we should first be able to trace rays in anisotropic models. The next chapter explains how.