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Partial differential equation for migration to zero-offset

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ABSTRACT

Starting from prestack migration in offset-midpoint coordinates I obtain a partial differential equation (PDE) for migration to zero-offset in constant velocity. The PDE allows migration to zero-offset (MZO) to be performed via an algorithm similar to phase-shift migration. Since the velocity is isolated on depth intervals, this equation could theoretically be used for variable velocity migration to zero-offset. I show synthetic examples of depth variable velocity MZO which match perfectly the zero-offset section.

INTRODUCTION

Migration to zero-offset (MZO) transforms prestack data to zero-offset data. In constant velocity, MZO is equivalent to the processing sequence comprised of dip moveout correction (DMO), normal moveout correction (NMO) and stacking. The output of MZO processing is the input to the zero-offset migration (ZOM) algorithm.

Another definition of MZO given by Deregowski and Rocca (1981) is based on the decomposition of MZO into prestack migration followed by zero-offset modeling. Following this definition an impulse in a prestack constant-offset section is transformed into a depth model that represents all the possible reflector locations (an ellipse in constant velocity). By modeling the reflector back to zero-offset one obtains a curve that represents the impulse response of the MZO operator. The impulse response curve can be applied to constant-offset data as a Kirchhoff integral operator.

Hale (1983) tries to derive the ray-theoretical DMO from wave-equation principles by decomposing the prestack migration equation (double-square root in ω, k_y, k_h) in four major processing steps:

1. DMO,
2. NMO,

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3. Stacking,
4. Zero-offset migration.

I define MZO as the sequence of steps 1,2 and 3 therefore I need to isolate only two steps:

1. Migration to zero-offset (MZO).
2. Zero-offset migration (ZOM).

Examining the MZO processing sequence, I observe that it is also the solution to a partial differential equation which can be used further to generalize MZO to a depth variable velocity.

I apply the variable velocity MZO to three synthetic models and obtain very good coincidence between the initial zero-offset section and the output of the zero-offset migration. The drawback of the method is that summing over the offset is performed as part of the MZO. Further work is needed to isolate the stacking sequence.

MZO from prestack migration

The constant velocity **prestack migration** in offset-midpoint coordinates can be formulated as:

$$p(t=0,k_y,h=0,z) = \int d\omega \int dk_h\; e^{ik_z(\omega,k_y,k_h)z} p(\omega,k_y,k_h,z=0) \tag{1}$$

where $p(\omega,k_y,k_h,z=0)$ is the 3-D Fourier transform of the field $p(t,y,h,z=0)$ recorded at the surface, using Claerbout's (1985) sign convention:

$$p(\omega,k_y,k_h,z=0) = \int dt\; e^{i\omega t} \int dy e^{-ik_y y} \int dh e^{-ik_h h} p(t,y,h,z=0).$$

The constant $k_z(\omega,k_y,k_h)$ is defined in the dispersion relation as

$$k_z(\omega,k_y,k_h) \equiv -\frac{\omega}{v}\left\{\left[1-\frac{v^2}{4\omega^2}(k_y+k_h)^2\right]^{\frac{1}{2}} + \left[1-\frac{v^2}{4\omega^2}(k_y-k_h)^2\right]^{\frac{1}{2}}\right\}. \tag{2}$$

The constant velocity **zero-offset migration** in offset-midpoint coordinates can be formulated as:

$$p(t=0,k_y,z) = \int d\omega_0\; e^{ik_z(\omega_0,k_y)z} p(\omega_0,k_y,z=0) \tag{3}$$

where $p(\omega_0,k_y,z=0)$ is the 2-D Fourier transform of the field $p(t,y,z=0)$. The constant $k_z(\omega_0,k_y)$ is defined in the dispersion relation as

$$k_z(\omega_0,k_y) \equiv -\frac{2\omega_0}{v}\left[1-\frac{v^2k_y^2}{4\omega_0^2}\right]^{\frac{1}{2}}. \tag{4}$$

The goal of the ensuing derivation is to convert equation (1) into a form similar to equation (3). Using a change of variables and integrating over the variable k_h, equation (1) will be transformed into the form:

$$p(t=0,k_y,h=0,z) = \int d\omega_0 \; e^{ik_z(\omega_0,k_y)z} p_0(\omega_0,k_y,z=0),$$

where $p_0(\omega_0,k_y,z=0)$ represents the zero-offset data field. Following Hale's (1983) derivation, a new variable ω_0 is introduced in order to isolate the zero-offset migration operator. The variable ω is expressed as

$$\omega \equiv \omega_0 \left[1 + \frac{v^2 k_h^2}{4\omega_0^2 - v^2 k_y^2}\right]^{\frac{1}{2}}, \tag{5}$$

where ω_0 is considered variable and k_y, k_h are constant. Substituting ω in equation (2), the downward continuation operator $k_z(\omega,k_y,k_h)$ is transformed into

$$k_z \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{\frac{1}{2}} \tag{6}$$

which now has the same form as the operator in equation (4). In order to isolate the zero-offset migration operator, the variable ω_0 is substituted for ω in equation (1) and the order of integration is changed between ω_0 and k_h. The integration boundaries have to be observed carefully as they are modified after each change of variables and integration order. However, for the sake of simplicity, I will ignore in the following demonstration the integration limits, which are discussed in Appendix A.

For simplicity I define the variables

$$\begin{aligned} v_h &= \tfrac{vk_h}{2} \\ v_y &= \tfrac{vk_y}{2}. \end{aligned}$$

By substituting the variable ω with the expression in ω_0 in equation (1) and changing the integration order between ω_0 and k_h the prestack migration equation becomes

$$\begin{aligned} p(t=0,k_y,h=0,z) &= \int dk_h \int d\omega \; e^{ik_z(\omega,k_y,k_h)z} p(\omega,k_y,k_h,z=0) \\ &= \int d\omega_0 \; e^{ik_z(\omega_0,k_y)z} \int dk_h \left[\frac{d\omega}{d\omega_0}\right] p^*(\omega_0,k_y,k_h,z=0) \\ &= \int d\omega_0 \; e^{ik_z(\omega_0,k_y)z} p_0(\omega_0,k_y). \end{aligned} \tag{7}$$

The new field $p^*(\omega_0,k_y,k_h,z=0)$ represents a remapping (interpolation) from ω to ω_0 of the field $p(\omega,k_y,k_h,z=0)$. The meaning of this remapping will be discussed in the next section. The field $p_0(\omega_0,k_y)$ defined as

$$p(\omega_0,k_y) = \int dk_h \left[\frac{d\omega}{d\omega_0}\right] p^*(\omega_0,k_y,k_h,z=0) \tag{8}$$

with the Jacobian

$$\begin{aligned} J &= \left[\frac{d\omega}{d\omega_0}\right] \\ &= (1+\frac{v_h^2}{\omega_0^2 - v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2 v_y^2}{(\omega_0^2 - v_y^2)^2}\right] \end{aligned} \tag{9}$$

represents the zero-offset field.

The last equation in (7) represents zero-offset downward continuation and imaging as introduced by Gazdag (1978) or Stolt (1978). Equation (8) represents a way of obtaining the zero-offset section from constant-offset sections.

So far the operations needed to obtain the zero-offset stacked section from the constant-offset field are:

1. Fourier transform the constant-offset field $p(t, y, h) \to p(\omega, k_y, k_h)$.
2. Remap (interpolate) the ω axis into ω_0.
3. Multiply by the Jacobian.
4. Integrate over k_h.
5. Inverse Fourier transform $p_0(\omega_0, k_y) \to p_0(t_0, y)$.

Remapping in ω_0

As the variable ω is replaced with ω_0 in the integral

$$\int d\omega p(\omega, k_y, k_h) = \int d\omega_0 \left[\frac{d\omega}{d\omega_0}\right] p^*(\omega_0, k_y, k_h),$$

each value of ω_0 in the integral needs to be associated with the appropriate value of the field $p(\omega, k_y, k_h)$ which is $p(\omega_0\sqrt{1+\frac{v_h^2}{\omega_0^2-v_y^2}}, k_y, k_h)$. The new field $p^*(\omega_0, k_y, k_h)$ represents a remapping of the original field $p(\omega, k_y, k_h)$. Each value in the new field $p^*(\omega_0, k_y, k_h)$ with coordinates (ω_0, k_y, k_h) corresponds to the value in the field $p(\omega, k_y, k_h)$ with coordinates $(\omega = \omega_0\sqrt{1+\frac{v_h^2}{\omega_0^2-v_y^2}}, k_y, k_h)$.

The $\omega \to \omega_0$ remapping can be understood easier if one takes the particular case $k_y = 0$. The change of variable in ω (5) becomes

$$\omega^2 = \omega_0^2 + v_h^2,$$

the Jacobian

$$\left[\frac{d\omega}{d\omega_0}\right] = \frac{\mid \frac{2\omega_0}{v} \mid}{\sqrt{\frac{4\omega_0^2}{v^2} + k_h^2}},$$

and the zero-offset migrated field becomes

$$p_0(\omega_0, k_y = 0) = \int dk_h \frac{\mid \frac{2\omega_0}{v} \mid}{\sqrt{\frac{4\omega_0^2}{v^2} + k_h^2}} p(\omega = sign(\omega_0)\frac{v}{2}\sqrt{\frac{4\omega_0^2}{v^2} + k_h^2}, k_y = 0, k_h). \qquad (10)$$

Compare equation (10) with the Gazdag (1978) and Stolt (1978) zero-offset migration equations:

$$\begin{aligned} p(z, k_y) &= \int d\omega\; e^{-isign(\omega)\sqrt{\frac{4\omega^2}{v^2} - k_y^2}z} p(\omega, k_y) \\ &= \int d\omega_0\; e^{-i\omega_0 z} \frac{v}{2} \frac{\mid \omega_0 \mid}{\sqrt{\omega_0^2 + k_y^2}} p(\omega = sign(\omega_0)\frac{v}{2}\sqrt{\omega_0^2 + k_y^2}, k_y). \end{aligned} \qquad (11)$$

Notice that the Stolt migration formula is comprised of an inverse Fourier transform and a remapping (interpolation). Equation (10) inside the integral in k_h has the same form as the remapping in the Stolt formula, save for a constant coefficient $\frac{v}{2}$. This similitude suggests the equivalent formulation:

$$\begin{aligned} p_0(t_0, k_y = 0) &= \int dk_h \int d\omega\; e^{-isign(\omega)\frac{v}{2}\sqrt{\frac{4\omega^2}{v^2} - k_h^2}t_0} p(\omega, k_y = 0, k_h) \\ &= \int dk_h \int d\omega\; e^{-isign(\omega)\sqrt{\omega^2 - v_h^2}t_0} p(\omega, k_y = 0, k_h). \end{aligned} \qquad (12)$$

The validity of the new equation is verified by replacing in equation (12) the exponential expression

$$sign(\omega)\sqrt{\omega^2 - v_h^2}$$

by a new variable ω_0 to get back the inverse Fourier transform of the initial expression.

The integration in k_h represents the inverse Fourier transform at zero offset ($h = 0$). However because equation (12) actually performs summation along hyperbolas in the spatial coordinate h, replacing the integral in k_h by the inverse Fourier transform will not return the constant-offset sections.

Applying the same technique to the case $k_y \neq 0$ I obtain

$$p_0(t_0, k_y) = \int dk_h \int d\omega\; e^{-isign(\omega)\left[\frac{1}{2}(\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2})\right]^{\frac{1}{2}} t_0} p(\omega, k_y, k_h) \quad (13)$$

which can be verified by substituting the exponential expression in (ω, v_y, v_h) with a new variable ω_0 defined as

$$\omega_0^2 = \frac{1}{2}\left[\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}\right]. \qquad (14)$$

PDE

The function inside the integral in equation (13) satisfies the partial differential equation

$$\frac{2}{v^2}\frac{\partial^2 p}{\partial t_0^2} = \frac{1}{v^2}p_{tt} + p_{yy} - p_{hh} + \sqrt{(\frac{1}{v^2}p_{tt} + p_{yy} - p_{hh})^2 - \frac{4}{v^2}p_{tt}p_{yy}}. \tag{15}$$

Notice that for flat layers ($p_{yy} = 0$) the equation is reduced to

$$\frac{1}{v^2}\frac{\partial^2 p}{\partial t_0^2} = \frac{1}{v^2}p_{tt} - p_{hh}$$

which is solved by doing NMO as time migration in time and offset coordinates. For the zero-offset case ($p_{hh} = 0$), the equation becomes an identity. To verify the partial differential equation, I coded equation (13) for a constant velocity case and for several depth varying velocity functions. Equation (13) can be used to extrapolate the initial wavefield in t_0 using a v_{rms} velocity for each zero-offset time value. The results are shown in the following synthetic cases.

Figure **??** shows the velocity and the travel-time map used to model the constant-offset sections. Since for this first case the velocity was constant the information in this figure is trivial, however I kept it just to be consistent with the layout of the other figures. The number of offsets used is 32, with a maximum half-offset $h = 930$ meters. In all cases the structural model was a diffractor at a depth of 1250 m. Figure **??** shows a comparison between the first zero-offset panel obtained from modeling and the output of the MZO algorithm applied to the 32 constant-offset sections. The kinematics of the initial zero-offset section in Figure **??**a coincide with the output of the MZO algorithm in Figure **??**b. The artifacts in Figure **??**b are due to the Fourier domain implementation of the algorithm. The next figures use a depth variable velocity to model the constant-offset sections and the corresponding v_{rms} to migrate the data to zero-offset.

Figure **??**a shows a root-mean-square velocity with a slow increase, corresponding to an interval velocity with several linearly increasing velocity layers. The travel-time map model in Figure **??**b can be used to identify the boundaries of sharper velocity changes. Comparing Figures **??**a and **??**b we notice the excellent kinematic match between the zero-offset model and the output of the migration to zero-offset algorithm using the depth variable velocity.

Figure **??** shows a velocity model with an increasing jump, while in Figure **??**, the velocity has a decreasing jump. Again in both cases, comparing Figures **??**a with **??**b and Figures **??**a with **??**b, the kinematics of the initial zero-offset section and the output of the migration to zero-offset are extremely similar even though the velocities used show quite large variations.

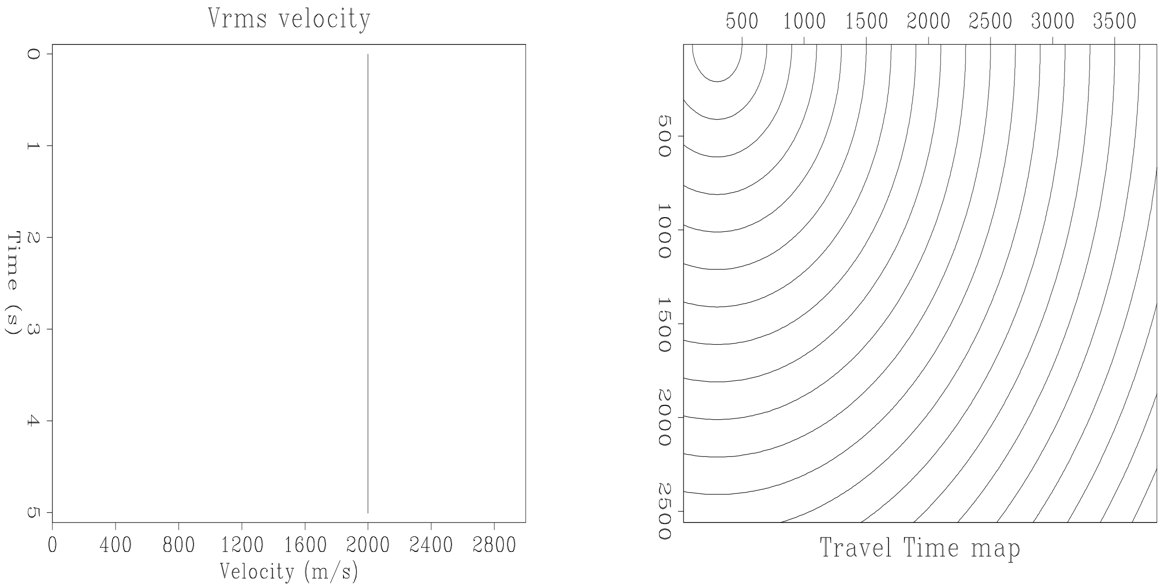

Figure 1: The constant velocity medium case.
a. V_{rms} velocity used.
b. Zero-offset travel-time map used for modeling. mihai1-Fig1 [ER]

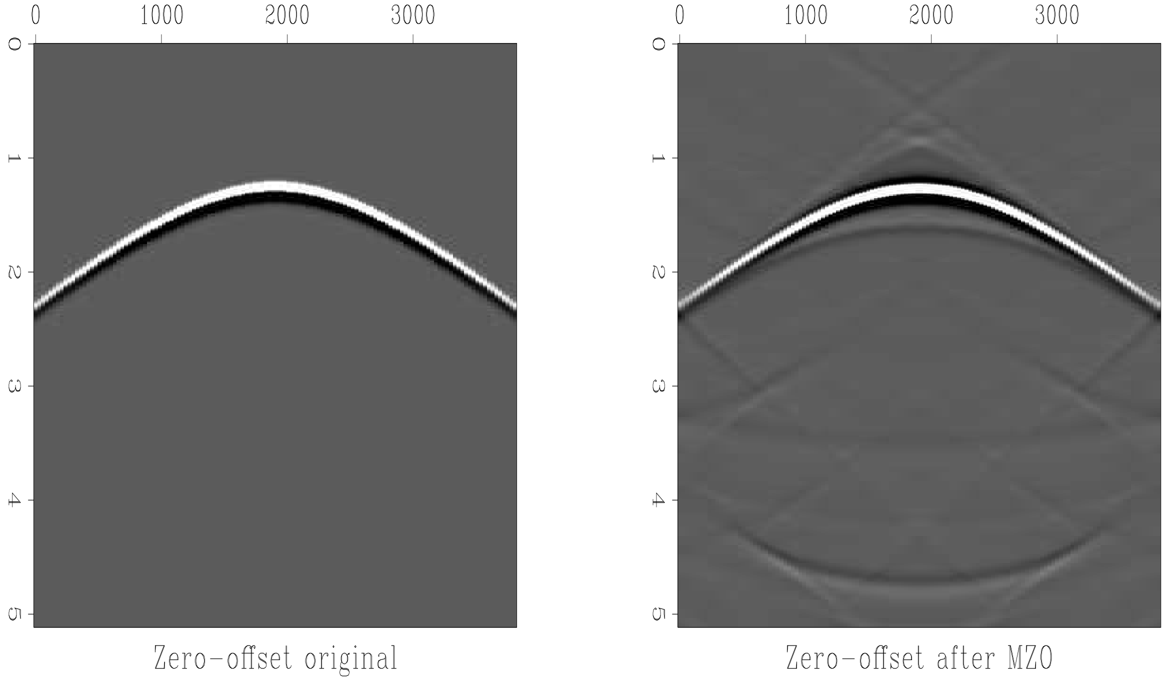

Figure 2: Comparison between zero-offset input and constant velocity MZO output.
a. First panel in the input cube: zero-offset.
b. MZO output after stacking. $\boxed{\text{mihai1-Fig2}}$ [CR]

CONCLUSIONS

Starting from the prestack migration equation in common midpoint and offset coordinates, I find a partial differential equation that can be used to transform prestack data to zero-offset data in a constant velocity medium. The PDE can be extended to depth variable velocity via an algorithm similar to Gazdag's phase-shift downward continuation. The synthetic results show a very good match between the actual zero-offset section and the output of the depth variable velocity MZO.

The obvious drawback is the fact that the MZO defined in this paper does an implicit summation over offset. I need to understand better this partial differential equation, so there will be more work on this subject in the next SEP report.

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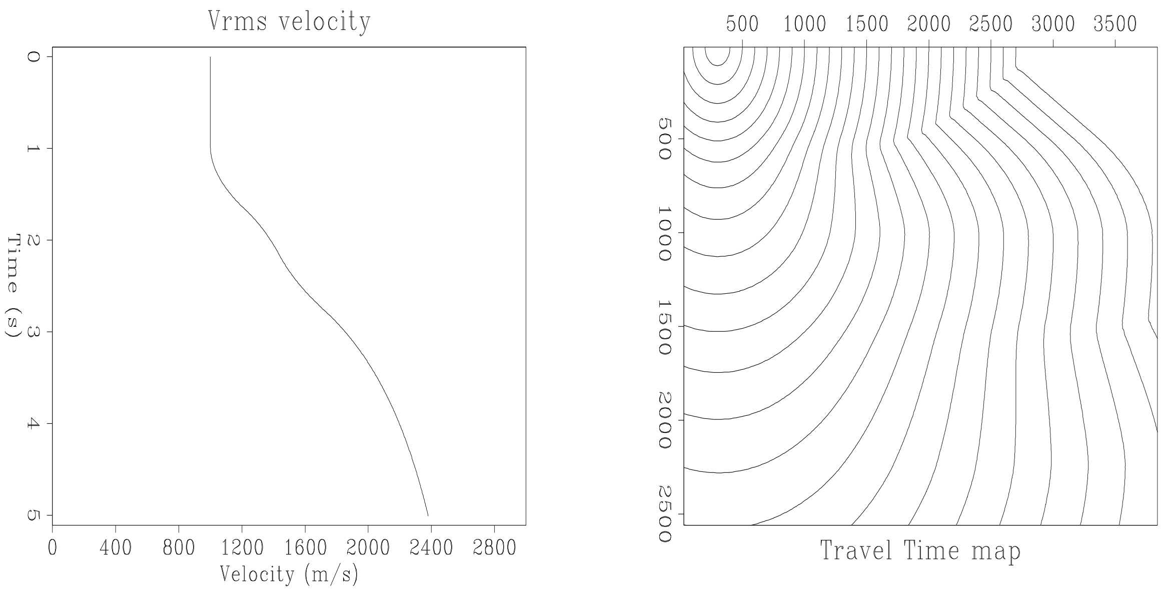

Figure 3: The depth variable velocity case.
a. V_{rms} velocity used.
b. Zero-offset travel-time map used for modeling. mihai1-Fig3 [ER]

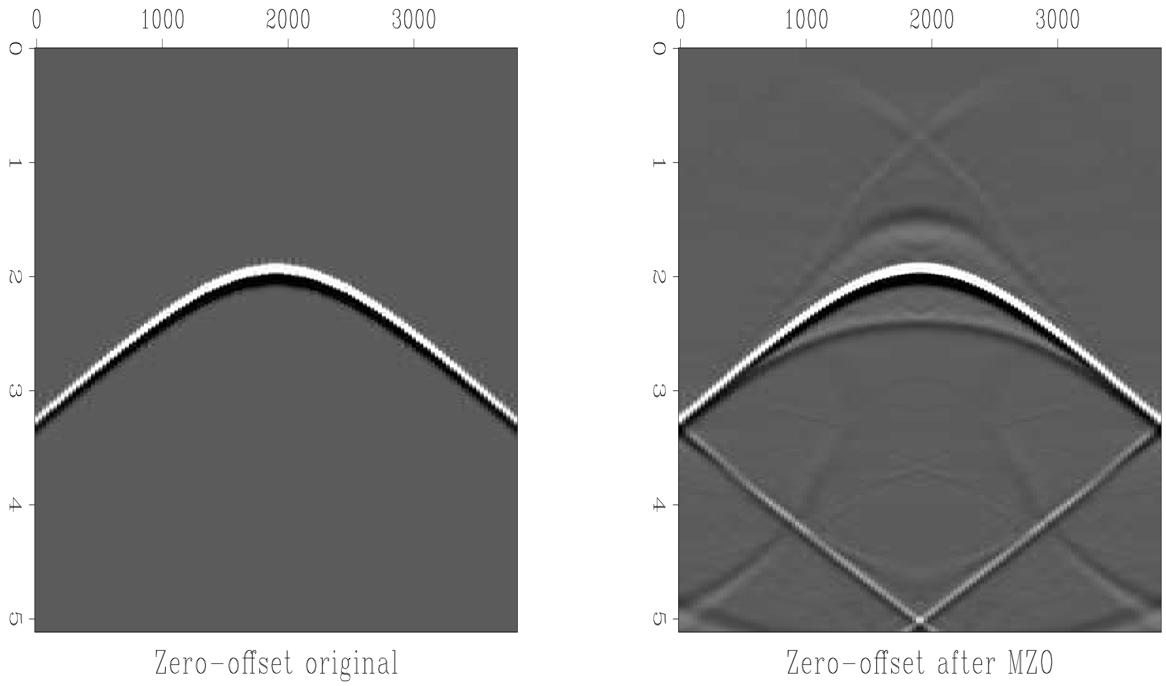

Figure 4: Comparison between zero-offset input and v(z) velocity MZO output.
a. First panel in the input cube: zero-offset.
b. MZO output after stacking. mihai1-Fig4 [CR]

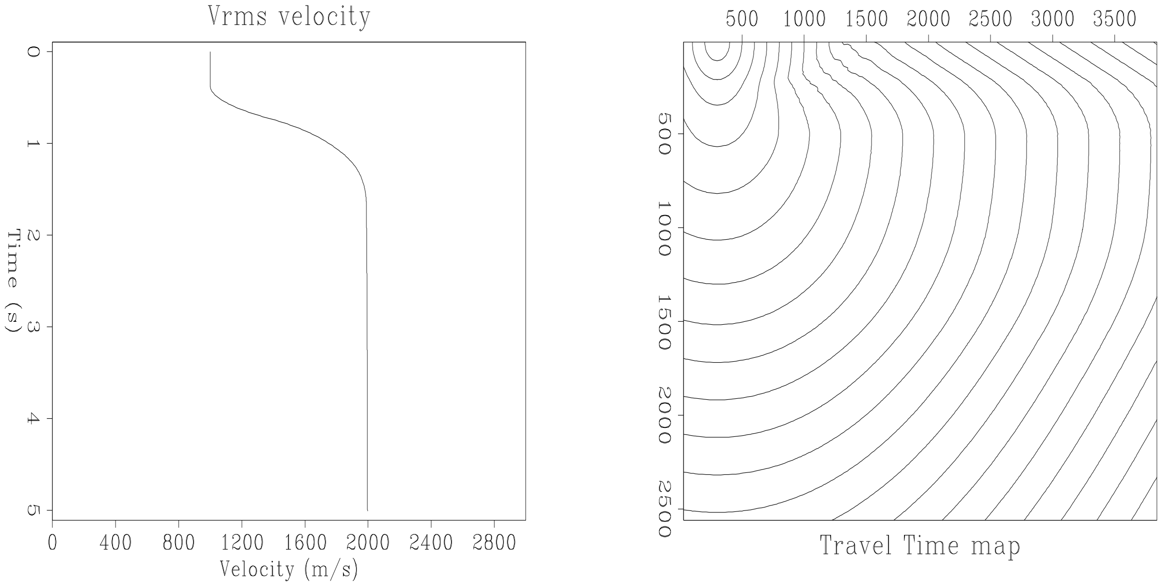

Figure 5: The depth variable velocity case.
a. V_{rms} velocity used.
b. Zero-offset travel-time map used for modeling. mihai1-Fig5 [ER]

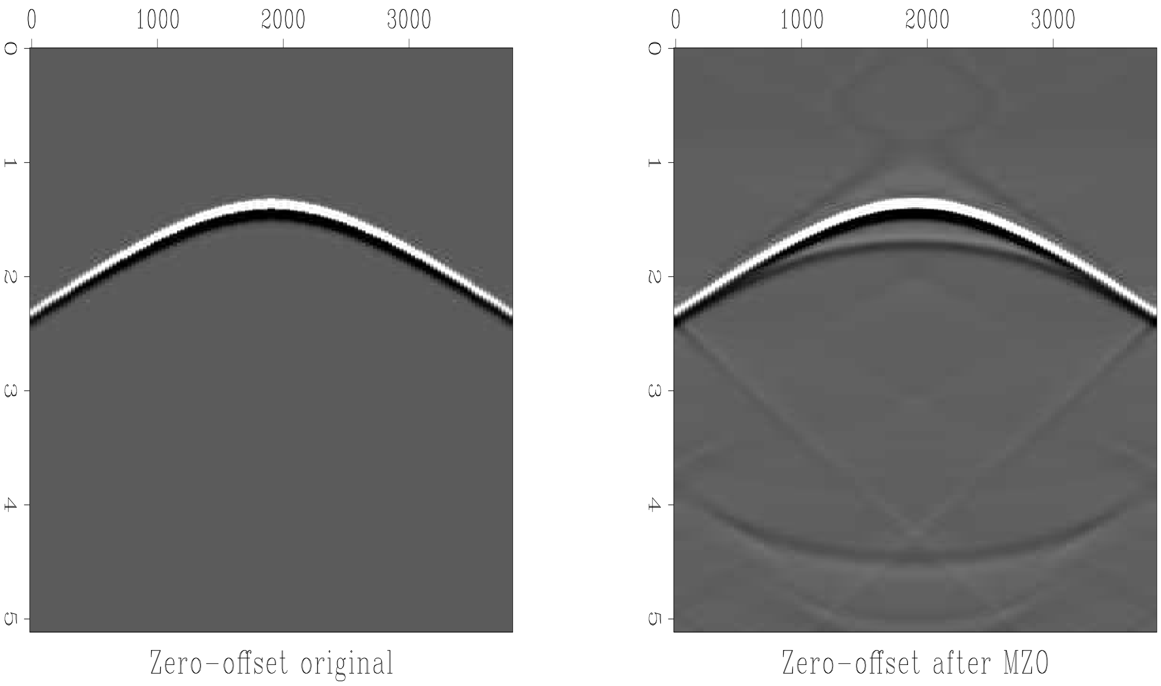

Figure 6: Comparison between zero-offset input and v(z) velocity MZO output.
a. First panel in the input cube: zero-offset.
b. MZO output after stacking. mihai1-Fig6 [CR]

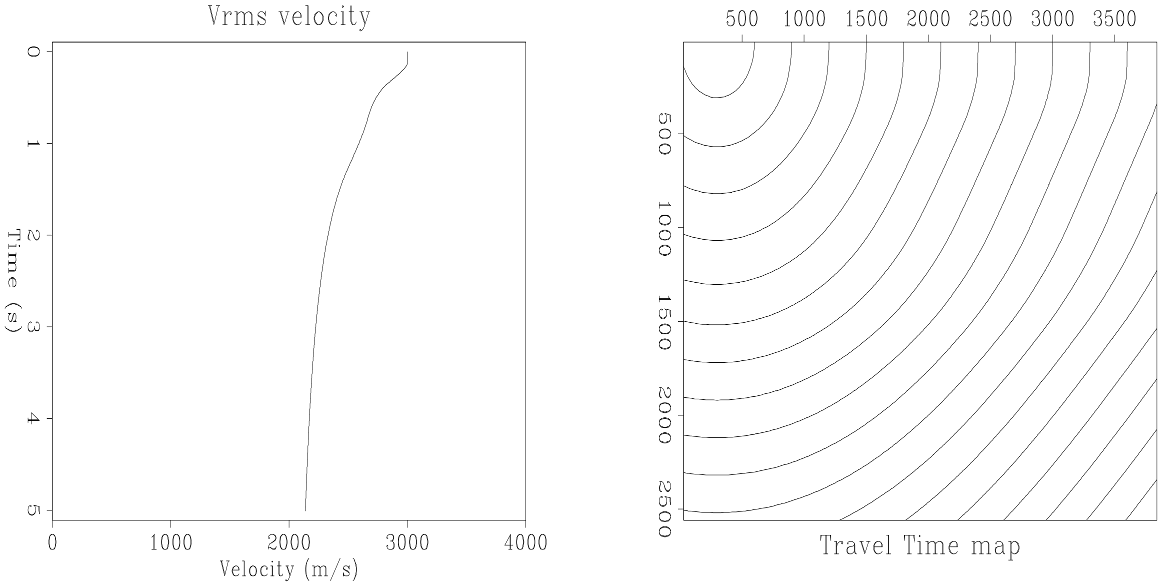

Figure 7: The depth variable velocity case.
a. V_{rms} velocity used.
b. Zero-offset travel-time map used for modeling. $\boxed{\text{mihai1-Fig7}}$ [ER]

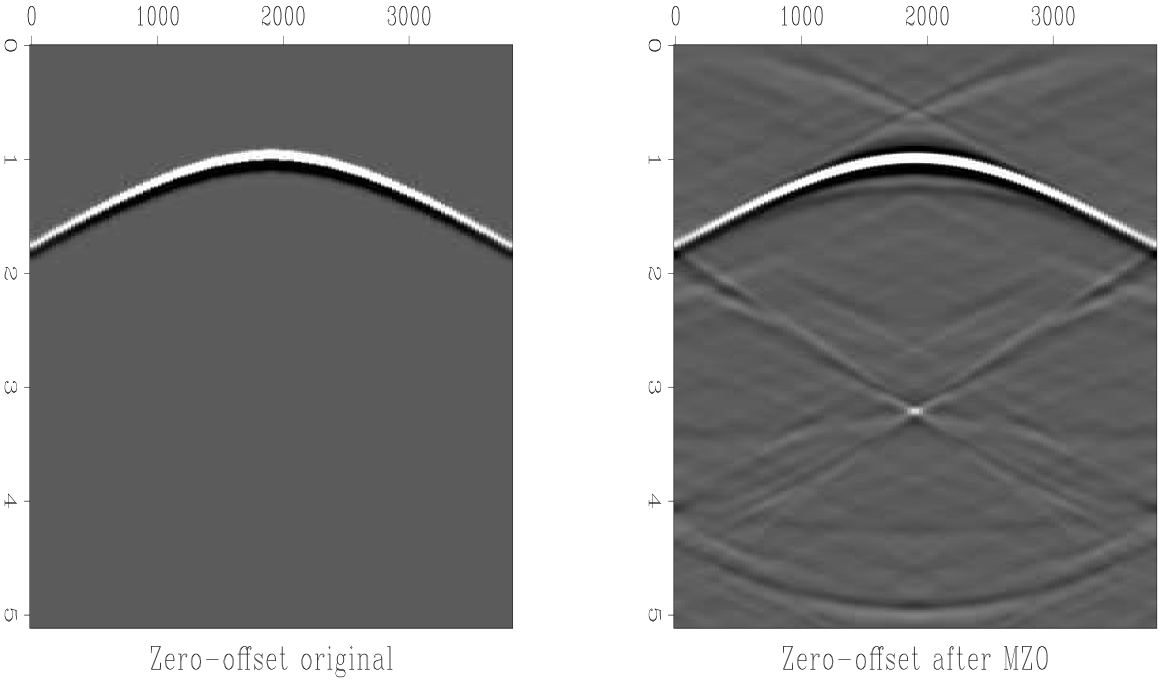

Figure 8: Comparison between zero-offset input and v(z) velocity MZO output.
a. First panel in the input cube: zero-offset.
b. MZO output after stacking. $\boxed{\text{mihai1-Fig8}}$ [CR]

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APPENDIX A

In equation (1) the values of the constant k_z, given by equation (2), have to be real. Imaginary values of k_z do not satisfy the downward continuation ordinary differential equation

$$\frac{\partial^2 P}{\partial z^2} = -k_z^2 P$$

and have to be excluded. Real values of k_z from equation (2) require the conditions:

$$\begin{aligned} \mid \omega \mid \quad &\geq \quad \mid v_y + v_h \mid \\ \mid \omega \mid \quad &\geq \quad \mid v_y - v_h \mid \end{aligned}$$

which can be reduced to the condition

$$\mid \omega \mid \quad \geq \quad \mid v_y \mid + \mid v_h \mid \ . \tag{A-1}$$

After the change of variable from ω to ω_0 in equation (5)

$$\omega \equiv \omega_0 \left[1 + \frac{v^2 k_h^2}{4\omega_0^2 - v^2 k_y^2} \right]^{\frac{1}{2}},$$

we want to express ω_0 function of ω and determine the integration boundaries for ω_0. We start with the expression for ω:

$$\omega^2 = \omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2} \tag{A-2}$$

and after reducing

$$\omega^2 \omega_0^2 - \omega^2 v_y^2 = \omega_0^4 - \omega_0^2 v_y^2 + \omega_0^2 v_h^2$$

and grouping

$$\omega_0^4 - \omega_0^2(\omega^2 + v_y^2 - v_h^2) + \omega^2 v_y^2 = 0$$

we have

$$\omega_0^2 = \frac{1}{2}(\omega^2 + v_y^2 - v_h^2 \pm \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}). \tag{A-3}$$

The discriminant Δ is

$$\Delta = (\omega - v_y - v_h)(\omega - v_y + v_h)(\omega + v_y - v_h)(\omega + v_y + v_h).$$

From the conditions on k_z, Δ is always positive and therefore ω_0^2 is always real within the ω limits.

The integration limits for ω_0 are found by starting with the limits for ω :

$$\omega = \mid v_h \mid + \mid v_y \mid$$

and after we square both sides

$$\omega^2 = v_h^2 + v_y^2 + 2 \mid v_h v_h \mid$$

and replacing ω^2 in the equation for ω_0^2 we have

$$\begin{aligned}
\omega_0^2 &= \frac{1}{2}\left[v_h^2 + v_y^2 + 2 \mid v_h v_y \mid + v_y^2 - v_h^2 + \sqrt{(2v_y^2 + 2 \mid v_h v_y \mid)^2 - 4v_y^2(v_h^2 + v_y^2 + 2 \mid v_h v_y \mid)} \right] \\
&= \frac{1}{2}\left[2v_y^2 + 2 \mid v_h v_y \mid + \sqrt{4v_y^4 + 4v_h^2 v_y^2 + 8v_y^2 \mid v_h v_y \mid - 4v_y^2 v_h^2 - 4v_y^4 - 8v_y^2 \mid v_h v_y \mid} \right] \\
&= v_y^2 + \mid v_h v_y \mid.
\end{aligned} \tag{A-4}$$

The integration in ω_0 is done from $(-\infty, -\sqrt{v_y^2 + \mid v_h v_y \mid})$ and from $(\sqrt{v_y^2 + \mid v_h v_y \mid}, \infty)$. After changing the order of integration from ω_0 to k_h, the integration boundaries for k_h become

$$k_h \in (-\frac{2}{v}\frac{\omega_0^2 - v_y^2}{\mid v_y \mid}, \frac{2}{v}\frac{\omega_0^2 - v_y^2}{\mid v_y \mid}). \tag{A-5}$$