Stanford Exploration Project, Report 77, November 17, 1997, pages 405–414

The split wave equation operator for anisotropic media

Martin Karrenbach[1]

keywords: *modeling, wave equation, anisotropy, elastic*

ABSTRACT

When applying the elastic anisotropic wave equation operator for spatially varying media, several first order derivatives have to be calculated in the process. The wave equation can formally be split into two components such that derivatives are taken with respect to medium parameters and with respect to the propagating wave field. Splitting the wave equation operator allows us to adapt derivative operators to the physical quantities to be differentiated. In particular the adaption can be guided by special properties of the observable quantity. Practical problems can arise since in general anisotropic media the derivated quantities have to be interpolated back to collocation points.

INTRODUCTION

In seismic modeling adaptive techniques are well known. Usually the adaption of a quantity is in the direction of a physical dimension like time or space. In general, adaption is guided by criteria to improve numerical accuracy and stability or to increase the speed of the calculation. In his Ph.D. thesis Cunha-Filho (1992) discussed the notion of treating the propagating elastic wave field differently to the underlying medium. His aim was to improve accuracy of reflected and transmitted amplitudes across an interface. He derived his technique in combination with an averaging scheme across the interface and made use of an unstaggered grid.

When examining the accuracy and stability of finite difference operators, the staggered grid finite difference equation can be separated into components that interpolate to the points where derivatives are evaluated. For anisotropic media Igel et. al (1992) showed an error analysis of FD operators and their spatial distribution. Usual implementations have implicit interpolations of medium and wave field properties built into the wave equation operator. Splitting the wave equation operator and explicitly controlling the interpolation gives a well determined and more accurate wave propagation description.

[1]**email:** not available

FORMAL SPLITTING OF THE WAVE OPERATOR

The constitutive equation links basic wave propagation observables with the medium parameters. Combining a force balance equation with a constitutive equation results in a wave equation. In this exposition I use the elastodynamic wave equation. The constitutive equation and force balance without exterior forces in a perfectly elastic medium are

$$\epsilon = \nabla^t u \tag{1}$$
$$\sigma = \underset{\sim}{\mathbf{C}}\, \epsilon \tag{2}$$
$$\rho\, \ddot{u} = \nabla \sigma \tag{3}$$

where u is the elastic displacement, ϵ the strain, σ the stress and $\underset{\sim}{\mathbf{C}}$ the stiffness tensor. The resulting wave equation involves calculating tensor products and derivatives. This wave equation operator L can be defined as

$$Lu = \nabla(\underset{\sim}{\mathbf{C}} \nabla^t u) \tag{4}$$

∇ is a partial derivative operator matrix acting on its argument. $\underset{\sim}{\mathbf{C}}$ is the stiffness tensor and u is the displacement field. Using the chain rule for covariant derivatives and applying it to the tensor product, we can rewrite equation (4) to

$$L\ u\ =\ (\overset{\downarrow}{\nabla}\overset{\downarrow}{\underset{\sim}{\mathbf{C}}})\ (\nabla^t u)\ +\ (\overset{\downarrow}{\nabla}\ \underset{\sim}{\mathbf{C}})\ (\overset{\downarrow}{\nabla^t} u)\ . \tag{5}$$

The $\downarrow$ pairs indicate the entity on which the matrix of derivative operator is acting upon. In the first term of equation (5) derivatives of the elastic constants are computed; in the second term only derivatives of the strain tensor are computed. The stiffness components in the second term act as constant coefficients for the derivate matrix elements. We can rewrite operator equation (5) in terms of its tensor elements by using (1) and get

$$l_i = \sum_{jkl} [\ (\ \frac{\partial_i}{\partial_j} + \frac{\partial_j}{\partial_i})\ \mathbf{c_{ijkl}}\ ]\ \epsilon_{\mathbf{kl}}\ +\ \sum_{jkl}\ \mathbf{c_{ijkl}}\ [\ (\ \frac{\partial_i}{\partial_j} + \ \frac{\partial_j}{\partial_i}\ )\ \epsilon_{\mathbf{kl}}\ ] \tag{6}$$

where [...] indicates the scope of the partial derivatives and where i,j,k,l = 1,2,3. Equation (6) is written in its most general form and is valid for anisotropic media. The derivative operation in equation (6) appears in both terms. It acts, however, on two different entities. In the first term the derivative of medium properties is taken and in the second term the derivative of the wave field is computed. In the end summation of both terms produces the complete derivative without taking a derivative of the medium and wave field product, as equation (4) would suggest. For a medium with constant properties the first term vanishes, leaving the second term in equation (6), whose Fourier transform represents the Christoffel equation. From an economical point of view equation (6) doesn't seem attractive, but the opportunity lies in computing the solution to equation (4) more accurately and more realistically for certain types of media. In particular the properties of derivatives can be adjusted to match the properties of the observables.

ADAPTION OF THE DERIVATIVE OPERATOR

Two basic observables in the wave equation are the wave field (eg. stress, displacement) and the medium properties (stiffnesses, velocities). If the frequency content of the wave field and the material properties is similar, the same type of derivative operators can be used for both. Then (4) should be used rather than (6). The necessary condition is that model and wave field roughness are of the same scale. For example, if the medium properties vary smoothly in space then a fairly long derivative operator can be adequate to approximate numerically the analytical derivative. However, if the medium is varying very rapidly, the same derivative operator may not be adequate. As an extreme example, realistic modeling wave propagation in a random medium is affected by the length of the finite difference operator. A long operator implicitly averages medium properties. To describe local effects a much shorter operator would be necessary, but for the same prescribed spacing, that operator gives a less accurate result. A layered model is an extreme case. The spatial variability in the horizontal direction is zero, the spatial variability in depth occurs in discontinuous steps. Applying equation (4) with conventional finite difference operators to such a model would implicitly assume a continuous and band limited medium. Equation (6) allows to "separate" the medium from the wave field. We can adapt the derivative operators to the properties of the observables they are operating on.

Practical Issues

The splitting of model and wave field does not address the problem of interpolation of quantities to the desired grid points. That has to be handled separately. In practical implementations one problem can arise when calculating derivatives of the quantities. The derivatives have to be interpolated onto collocation points. This can result in a potential loss in accuracy and diminish improvements achieved by the operator adaption. Examining the partitioned constitutive equation, one can see how a commonly used staggered grid is chosen.

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ \begin{pmatrix} \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} & \begin{pmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36} \end{pmatrix} \\ \begin{pmatrix} c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \end{pmatrix} & \begin{pmatrix} c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \\ \begin{pmatrix} \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \end{pmatrix} \tag{7}$$

Stiffness values in the first diagonal quadrant and the first part of the σ and ϵ vectors are chosen to lie on the primary grid. The second diagonal quadrant and the second part of σ and ϵ are then necessarily chosen to lie on the secondary grid. That scheme works fine as long as the derivative computation evaluates the quantities at the correct collocation points. In anisotropic elastic media up to orthorhombic symmetry, the previous notion works fine, since the off-diagonal elements of the stiffness matrix are zero and thus do not have to be evaluated. In more anisotropic systems (beyond orthorhombic) those elements are non zero, such that even in conventional schemes interpolation to collocation points has to be included for staggered grid methods. When splitting the wave operator, one has to take care of proper collocation interpolation. Thus the interpolation issue can complicate the primary goal of adapting derivative operators to the quantities.

SYNTHETIC EXAMPLES

Equivalence of modeling equations

First I show that equation (5) and (4) are numerically equivalent and produce identical results, as predicted analytically. Figure 1 illustrates a wave field propagating in the model that shows a step discontinuity but is otherwise homogeneous. The elastic wave propagates across a discontinuity in v_p and v_s. Figure 2 shows a comparison of two modeling responses. One result was calculated using equation (5) and one result was obtained using equation (4). For comparison I extract a single trace from the recorded wave fields, as shown in Figure 1. There is no noticeable difference between the two curves for each of the two components. For both components the resulting responses overlie each other perfectly and indicate that both modeling equations are also numerically equivalent.

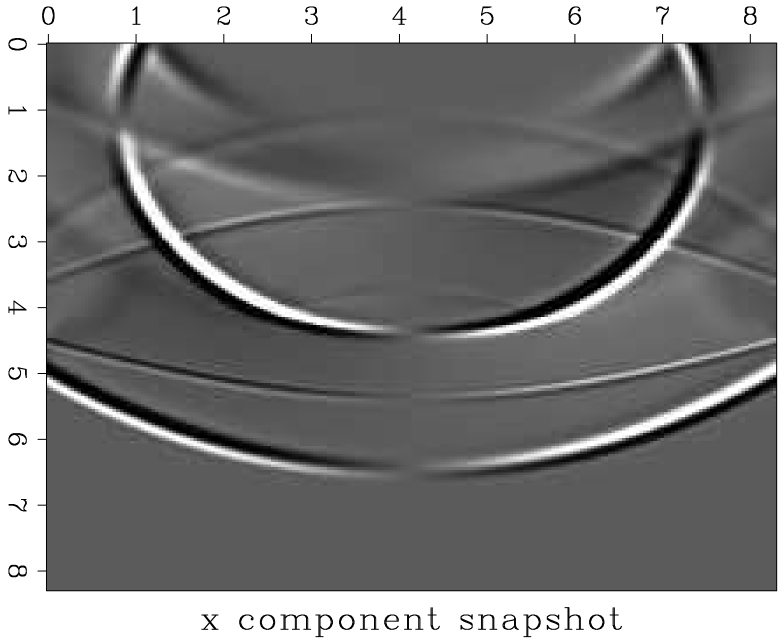

Figure 1: The x component of an elastic wave field in a homogeneous medium with a step discontinuity has propagated ca. 240 timesteps. martin4-equiwava [ER]

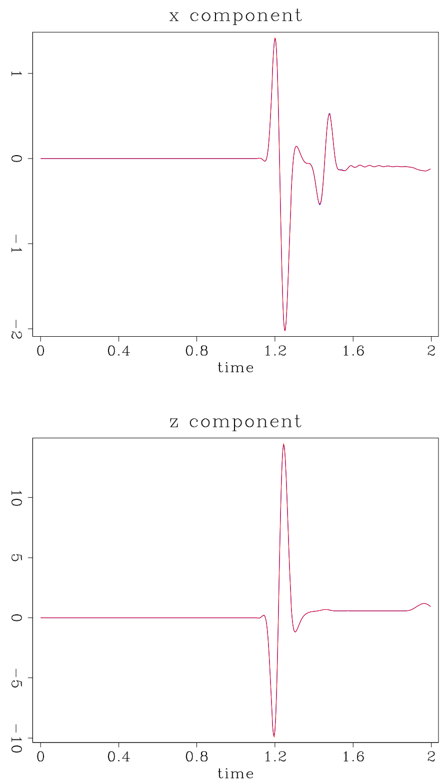

Figure 2: Trace 90 is taken from two wave fields. The results from the two different modeling equations are overlain and confirm that they are not only analytically but also numerically equivalent. The modeling was carried out on a staggered grid. The source is located on top of the step and the receivers are located below. martin4-equivala [ER]

A long and short operator combination

Figure 3 is using the same model as Figures 1 and 2. However, the difference lies in the types of operators used for calculating the wave field. For derivating the field quantities a 16 point operator was used, while for derivating the medium parameters a 6 point wide operator was used. The 16 point operator smoothes the the propagating wave field more than the 6 point operator. The short operator is more local and smoothes the medium parameters less. It does not remove as many high spatial frequencies in the derivation process, as the longer 16 point operator does. The step discontinuity really has a very high frequency content and we prefer an accurate computation of the derivative of that discontinuity.

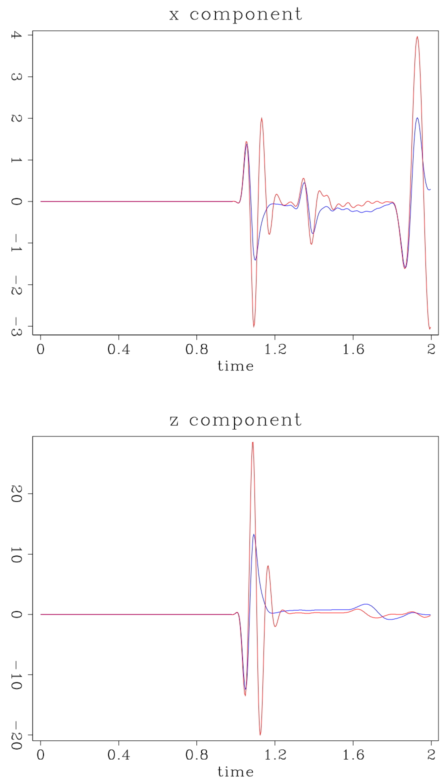

Figure 3: Trace 90 is taken from two wave fields. The results from the two different modeling equations are overlain. The smoother response is generated by the composite modeling equation with a fixed 16 point operator. The rougher response is produced by the split modeling equation , which leaves higher frequencies in the wave field, if the model itself contains those high frequencies. The modeling was carried out on a staggered grid. Again the source is located on top of the step and the receivers are located below. On the CD you are able to interactively experiment with different operator lengths yourself. martin4-long+short [ER]

CONCLUSIONS

The wave operator can formally and numerically be split into parts that involve taking derivatives of either model parameters or propagating field quantities. For identical derivative operators this is the same as the conventional method which takes nested derivatives of those quantities. However, the main advantage lies in the possibility to treat the two parts differently. I have given the splitting for completely anisotropic media. By assuming different a priori properties of the medium, the operators can be adjusted to match the medium. A possible topic of future research is the use of wavelet transform for derivative calculations for medium and field variables.

REFERENCES

Filho, C. A. C., 1992, Elastic modeling in discontinuous media: Testing the dynamic response: SEP–**73**, 317–330.

Igel, H., Riollet, B., and Mora, P., 1992, Accuracy of staggered 3-d finite-difference grids for anisotropic wave propagation: Annual International SEG meeting 1992, Soc. Expl. Geophys., Expanded Abstracts, 1244–1246.