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Huygen's waveforms and half-order derivatives

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ABSTRACT

This short tutorial paper shows why in two dimensions, Huygen's secondary source, otherwise known as the Hankel tail, is a fractional order derivative. In three-dimensions the tail reduces to a simple derivative.

INTRODUCTION

First we will review the calculus of step functions and power functions and their Fourier transforms leading up to the half-order differential operator. With this background we can answer the question, what is the difference between a line and the sequence of short dashes that make up the line? Likewise, what is the difference between a plane and a dense array of tiny patches that make up the plane? Is each tiny patch the same as a point source? These questions will be answered from the viewpoint of wave propagation where it is conceptually and computationally convenient to make plane waves by superposing an array of spherical waves. We will find that a plane wave with an impulsive waveform can be made from a superposition of circles (or hyperbolas) or spheres (or hyperboloids of revolution) but that in three dimensions the hyperbolid carries not an impulsive waveform, but a time derivative, d/dt, and in two dimensions the hyperbola carries the half-order derivative waveform. These waveforms are generally clear on synthetic data but on field data they are usually obscured by waveforms from other sources.

FRACTIONAL ORDER OPERATORS

Causal integration is represented in the time domain by convolution with a step function. In the frequency domain this amounts to multiplication by $1/(-i\omega)$. Integrating twice amounts to convolution by a ramp function, $t\,\mathrm{step}(t)$, which in the Fourier domain is multiplication by $1/(-i\omega)^2$. Integrating a third time is convolution

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with $t^2\,\mathrm{step}(t)$ which in the Fourier domain is multiplication by $1/(-i\omega)^3$. In general

$$t^{n-1}\ \mathrm{step}(t) \quad = \quad \mathrm{FT}\ \left(\frac{1}{(-i\omega)^n}\right) \tag{1}$$

Proof of the validity of equation (1) for integer values of n is by repeated indefinite integration which also indicates the need of an $n!$ scaling factor. Proof of the validity of equation (1) for fractional values of n would take us far afield mathematically. (Fractional values of n, however, are exactly what we need to interpret Huygen's secondary wave sources in 2-D.) The factorial function of n in the scaling factor becomes a gamma function. The poles suggest that a more thorough mathematical study of convergence is warranted, but this is not the place for it.

A common application is when $n = 1/2$ and (ignoring the scale factor) equation (1) becomes

$$\frac{1}{\sqrt{t}}\ \mathrm{step}(t) \quad = \quad \mathrm{FT}\ \left(\frac{1}{\sqrt{-i\omega}}\right) \tag{2}$$

It is well known that

$$\frac{d}{dt} \quad = \quad \mathrm{FT}\ (-i\omega) \tag{3}$$

A product in the frequency domain corresponds to a convolution in the time domain. A time derivative is like convolution with a doublet $(1,-1)/\Delta t$. Thus, from equation (2) and equation (3) we obtain

$$\frac{d}{dt}\,\frac{1}{\sqrt{t}}\ \mathrm{step}(t) \quad = \quad \mathrm{FT}\ \left(\sqrt{-i\omega}\right) \tag{4}$$

HANKEL TAIL

The waveform in equation (4) often arises in practice (as the 2-D Huygen's wavelet). Because of the discontinuities on the left side of equation (4), it is not easy to visualize. Thinking again of the time derivative as a convolution with the doublet $(1,-1)/\Delta t$, we imagine the 2-D Huygen's wavelet as a positive impulse followed by negative signal decaying as $-t^{-3/2}$. This decaying signal is sometimes called the "Hankel tail." In the frequency domain $-i\omega = |\omega|e^{-i90^\circ}$ has a 90 degree phase angle and $\sqrt{-i\omega} = |\omega|^{1/2}e^{-i45^\circ}$ has a 45 degree phase angle.

In practice, it is easiest to represent and to apply the 2-D Huygen's wavelet in the frequency domain. Subroutine `halfdif()` is provided for that purpose. Instead of using $\sqrt{-i\omega}$ which has a discontinuity at the Nyquist frequency and a noncausal time function, I use the square root of a causal representation of a finite difference, i.e. $\sqrt{1-Z}$, (see any reference on Z-transforms) which is well behaved at the Nyquist frequency and has the advantage that the modeling operator is causal (vanishes when $t < t_0$). (Fourier transform is done using subroutine `ftu()` from PVI.) Passing an impulse function into subroutine `halfdif()` gives the response seen in Figure 1.

```
# Half order causal derivative.
#
subroutine halfdif(  conj, n, x,    y)
integer n2, i,       conj, n
real    omega,                x(n), y(n)
complex cz,                              cv(4096)
n2=1; while(n2<n) n2=2*n2;         if( n2 > 4096) call erexit('halfdif memory')
do i= 1, n2 {             cv(i) = 0.}
do i= 1, n
        if( conj == 0)  { cv(i) = x(i)}
        else            { cv(i) = y(i)}
call ftu( +1., n2, cv)
                        do i= 1, n2 {
                                omega = (i-1.) * 2.*3.14159265 / n2
                                cz = csqrt( 1. - cexp( cmplx( 0., omega)))
                                if( conj != 0)   cz = conjg( cz)
                                cv(i) = cv(i) * cz
                                }
call ftu( -1., n2, cv)
do i= 1, n
        if( conj == 0)  { y(i) = cv(i)}
        else            { x(i) = cv(i)}
return; end
```

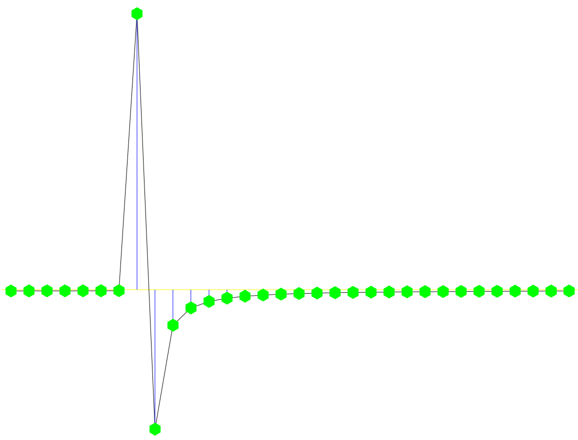

Figure 1: Impulse response (delayed) of finite difference operator of half order. Twice applying this filter is equivalent to once applying $(1, -1)$. jon4-hankel [ER]

HUYGEN'S SECONDARY SOURCE

Huygen's secondary source is the concept that a plane wave can be regarded as broken into many point sources. Each point makes a circular wave in (x, z)-space or equivalently a hyperbola in (x, t)-space. The hyperbolas shown in Figure 2 are all tangent to the horizontal line. Observe that the line density on the plot is great near the line but weaker away from it. We begin analysis from the assumption that each hyperbola carries a simple impulse function of time. We will add these hyperbolas together and discover that the sum is a plane wave, but the waveform is not an impulse function of time, but another function with a tail in the time domain and an $\omega^{-1/2}$ amplitude spectrum. From this we conclude that if the original waveform on the hyperbola had a $\omega^{1/2}$ amplitude spectrum (instead of being an impulse with a constant spectrum) then we would have an improved representation of the plane wave as a sum of points.

HUYGEN'S SECONDARY SOURCE IN 2-D

First we calculate the density of curves as a function of time. The density goes to infinity as the hyperbola separation Δx tends to zero in Figure 2. Then the hyperbolas sum to a plane wave which carries the waveform that we want to know.

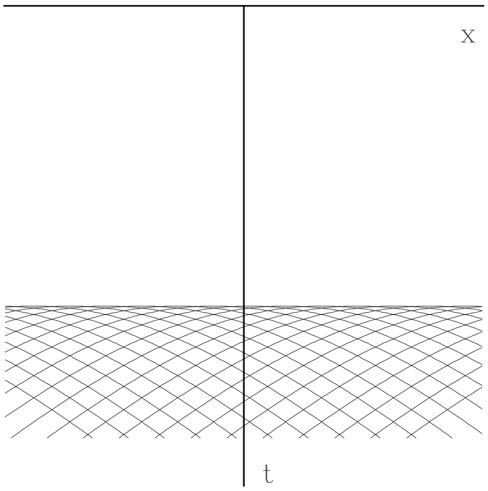

Figure 2: Many hyperbolas tangent to a line. jon4-hyplay [ER]

When a hyperbolic event carries a wavelet, the wavelet covers an area in the (t, x)-plane. Define this area as that between the two hyperbolas shown in Figure 3. These two curves are

$$t_1 = v^{-1} \sqrt{z^2 + x^2} \tag{5}$$
$$t_2 = v^{-1} \sqrt{z^2 + x^2} + \Delta t \tag{6}$$

In the limit $\Delta t \to 0$, the waveform is an impulse function whose temporal spectrum is constant. In that limit, the many waveforms in Figure 2 superpose giving a great concentration at the apex. Since the many hyperbolas are identical with one another, the time dependence of their sum is (within a scale factor) the same time dependence of the shaded area of the single hyperbola in Figure 3.

The separation between the two curves, Δx is

$$\Delta x = \sqrt{t^2 v^2 - z^2} - \sqrt{(t - \Delta t)^2 v^2 - z^2} \tag{7}$$

The area in Figure 3 is a length, integrated over time. Between the two hyperbolas of separation Δt, we take a constant signal of strength $1/\Delta t$ which adds up to an

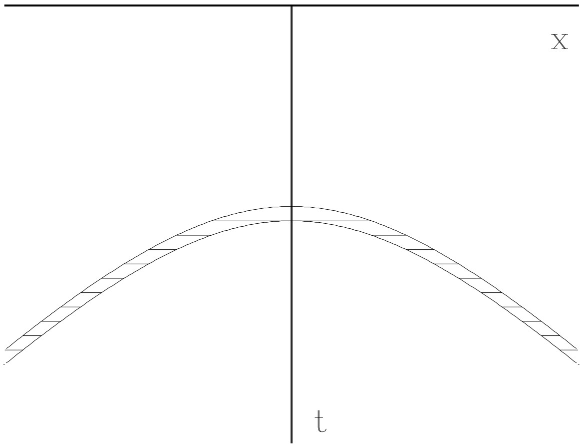

Figure 3: Detailed view of one of the many hyperbolas in figure 2. Two hyperbolas are separated by Δt. An impulsive signal is defined as one that is zero everywhere but between the two hyperbolas where its amplitude is $1/\Delta t$. jon4-hyparea [ER]

amplitude at time t of $A_{\rm hyp}(t)$.

$$A_{\rm hyp}(t) \quad = \quad \frac{\Delta x}{\Delta t} \quad = \quad \frac{d}{dt}\sqrt{t^2v^2 - z^2} \quad = \quad \frac{tv^2}{\sqrt{t^2v^2 - z^2}} \qquad \text{for } t > z/v \tag{8}$$

To avoid much clutter that belongs in a mathematics book rather than in a seismology book, we concentrate our attention on the discontinuity itself. Mathematically, this amounts to ignoring all but the highest frequencies. Seismologically, we say that after a signal like equation (8) passes through the filters and gain control processes typical of data collection, the numerator t does not warrant comment, but the power of t at the pole itself is important.

By making this high frequency approximation, we are setting aside some calculations that we should be able to do for a constant velocity medium but which would not apply to a medium with velocity as a function of depth. Divergence considerations, for example, suggest that the amplitude of each hyperbola should not be a constant, but should drop off proportional to $t^{-1/2}$. Another example is that of parabolas instead of hyperbolas, i.e. $x(t) = \sqrt{t - t_0}$. (When velocity changes with depth, we no longer have hyperbolas, but the tops of the traveltime curve could be approximated by a parabola.) Then the amplitude is $A_{\rm par}(t) = 1/\sqrt{t - t_0}$. Notice that the divisor in equation (8) for the hyperbola $\sqrt{t^2v^2 - z^2} = \sqrt{tv - z}\sqrt{tv + z}$ has the same half power discontinuity at $z = tv$ as the parabola. Thus the amplitude spectrum at high frequencies for either the parabola or the hyperbola will drop off as the inverse half power of frequency according to equation (2). To get a plane wave with the constant spectrum of an impulse instead of this inverse-square-root spectrum, we must use hyperbolas (or parabolas), not carrying an impulse function, but instead carrying a square-root spectrum (to cancel the *inverse*-square-root spectrum that arises from superposing the hyperbolas). Equation (4) gives us a causal waveform with the desired spectrum. Thus the decomposition of an impulsive 2-D plane wave into hyperbolas requires the hyperbolas to carry the "half-order differential" waveform given in equation (4).

THREE DIMENSIONS

Interestingly, the Huygen's wavelet is different in three dimensions than in two dimensions. In three dimensions, there is no Hankel tail. In three dimensions we have the definition $x^2+y^2=r^2$ and from $t^2v^2=x^2+y^2+z^2$ we find on a plane of constant z, that the equation for a circle expanding with time is $r=\sqrt{t^2v^2-z^2}$. Between time t and $t+\Delta t$ is a ring with an area $2\pi r\Delta r$. Taking the signal amplitude in the ring to be $1/\Delta t$, analogous to equation (8) the amplitude at time t is

$$A(t) \quad = \quad \mathrm{step}(t-z/v)\ 2\pi r\ \frac{\Delta r}{\Delta t} \tag{9}$$

$$A(t) \quad = \quad \mathrm{step}(t-z/v)\ 2\pi\sqrt{t^2v^2-z^2}\ \frac{d}{dt}\ \sqrt{t^2v^2-z^2} \tag{10}$$

$$A(t) \quad = \quad \mathrm{step}(t-z/v)\ 2\pi v^2\ t \tag{11}$$

As before, in seismology we are interested in the high frequency behavior so the scaling t in equation (11) is not nearly so important as is the step function. By equation (1) the step function causes the spectrum to decay as ω^{-1}. Our original erroneous assumption that Huygen's hyperbola of revolution should carry a positive impulse leads to the contradiction that an impulsive plane wave decomposed into Huygen's sources and added together again does not preserve the constant spectrum of the original impulsive waveform. We can get the missing ω back into the spectrum by having the hyperbola of revolution carry a time-derivative filter instead of an impulse. Thus in three dimensions, a plane wave can be regarded, approximately, as the superposition of many hyperboloidal responses, each carrying a d/dt waveform.