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Circularly symmetric filters using McClellan transformation

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keywords: *two-dimensional, one-dimensional, three-dimensional, Fourier transform*

ABSTRACT

Two-dimensional filters with circular symmetry are realized by exploiting their circular symmetry. They can be realized in an efficient manner from the corresponding 1-D filter, without the 2-D coefficients being computed. Only the 1-D filter coefficients need to be stored to realize the two-dimensional filter. Transformation technique using Chebyshev polynomials (called the McClellan transformation) is used in the design. The realization of the 2-D filter using transformation is efficient compared to two-dimensional convolution.

INTRODUCTION

Two-dimensional circularly symmetric digital filters are used for smoothing, sharpening, and restoration of images in image processing (Lu and Antoniou, 1992). Only recently they have been applied to seismic-imaging problems (Hale, 1991), and (Soubaras, 1992). These authors have applied it to 3-D depth migration. In 3-D depth migration, wavefields are extrapolated in depth one frequency at a time via a two-dimensional convolution of the data with a circularly symmetric, temporal frequency and velocity dependent filter. The depth extrapolator function in 3-D is a circularly symmetric operator in the wavenumber domain for a given temporal frequency. This property can be utilized in developing an explicit wavefield extrapolation operator using the techniques described in the paper. Hale (1991) has shown that the 3-D depth extrapolator designed using McClellan transformations are efficient and require less storage. The application of the filter to seismic imaging has been the main motivation for studying the design of these filters here.

In this paper, I discuss the elements of the filter design and illustrate its properties using two examples : A lowpass filter and a wavefield extrapolation filter. In migration, we are interested in implementing the 2-D filters in the space domain rather than in the wavenumber domain since the velocity variations could be handled more

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easily than the $k_x - k_y$ domain. Therefore, I show the implementation of the 2-D lowpass filter in both the $k_x - k_y$ and $x - y$ domains. In a companion paper (Biondi and Palacharla, 1993), the application of the filters to 3-D depth migration is dealt with in more detail.

DESIGN TECHNIQUE

The usual method in designing 2-D filters is to design an optimum filter that corresponds to a given desired two-dimensional spectrum. For designing 2-D filters that exhibit some special symmetry properties usually some transformation techniques are used to exploit the symmetry. The most common method used for generating a 2-D circularly symmetric filter is to use a transformation filter to generate the equivalent 2-D filter given the corresponding filter along one direction. This is done using a transformation filter called the McClellan transformation represented by $F(k)$. The method requires one one-dimensional spectral expansion to get the filter coefficients for the 1-D problem and one two-dimensional spectral expansion to generate the transformation filter.

The particular method for realizing these filters that yields structures with low sensitivity and low computational complexity is one proposed by McClellan and Chan (1977). The steps involved in this method are as follows.

A 1-D nonrecursive filter of odd length can be represented by the transfer function

$$H_1(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} h_1(n) z^{-n} \tag{1}$$

and if the impulse response of the filter is symmetrical such that $h_1(-n) = h_1(n)$ for n = 1,2,...,(N-1)/2, the frequency response of the filter can be expressed as

$$H_1(k) = \sum_{n=0}^{(N-1)/2} a_0(n) \cos(nk) \tag{2}$$

where

$$a_0(n) = \begin{cases} h_1(0) & \text{for } n = 0 \\ 2h_1(n) & \text{otherwise} \end{cases}$$

For any integer n, $\cos(nk)$ can be expressed in terms of an nth-order polynomial of $x = \cos(k)$ known as a Chebyshev polynomial and denoted by $C_n(x)$. The recursion formula is given by equation A-1. Equation 2 can be written in terms of the Chebyshev polynomials as

$$\begin{aligned} H_1(k) &= \sum_{n=0}^{(N-1)/2} a_0(n) C_n \cos(nk) \\ &= \sum_{n=0}^{(N-1)/2} a_1(n) \cos(k)^n \end{aligned} \tag{3}$$

where constants $a_1(nk)$ for $n = $ 0,1,...,(N+1)/2 can be obtained from constants $a_0(nk)$. The McClellan transformation is given by

$$cos(k) = Acos(k_x) + Bcos(k_y) + Ccos(k_x)cos(k_y) + D \tag{4}$$

$$= F(k_x, k_y)$$

where A, B, C, and D are constants. In the original McClellan transformation A=B=C=0.5 and D=-0.5. The McClellan filter is an approximation to the circular filter $\cos k$ where $k = \sqrt{k_x^2 + k_y^2}$ in the (k_x, k_y) domain. Hale(1991) suggests a modified McClellan transformation which is a better approximation than the original one, it is given by A-3. The contours of constant amplitude and phase for the original and the modified McClellan transformation filters are shown in Figure **??**. The modified filter has better accuracy at higher wavenumbers as can be seen from the contours. Applying the McClellan transformation to the frequency response of a 1-D nonrecursive filter given in Equation 3 yields

$$H_2(k_x, k_y) = \sum_{n=0}^{(N-1)/2} \sum_{n=0}^{(N-1)/2} a_2(n)(\cos(k_x))^n(\cos(k_y))^n \tag{5}$$

and since $(\cos k_x)^n$ can be expressed as linear combinations of $\cos k_x, \cos 2k_x, ..., \cos nk_x$, Equation 5 can be written as

$$H_2(k_x, k_y) = \sum_{n=0}^{(N-1)/2} \sum_{n=0}^{(N-1)/2} a_3(n)\cos(nk_x)\cos(nk_y) \tag{6}$$

This frequency response can be realized using the chebyshev filter structure shown in Figure **??**. The upper part of the structure does the Chebyshev recursion and the lower part implements the convolution with the one-dimensional filter.

EXAMPLES

Lowpass Filter

Circularly symmetric lowpass filters can be designed by assuming an idealized frequency response of the form

$$H(k_x, k_y) = \begin{cases} 1 & \text{for } \sqrt{k_x^2 + k_y^2} \le k_c \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where k_c is the cutoff frequency in the one-dimensional case. The first step is to design the low-pass filter for the one-dimensional case. This can be done using any of the linear filtering techniques which determine the filter in some least squares sense

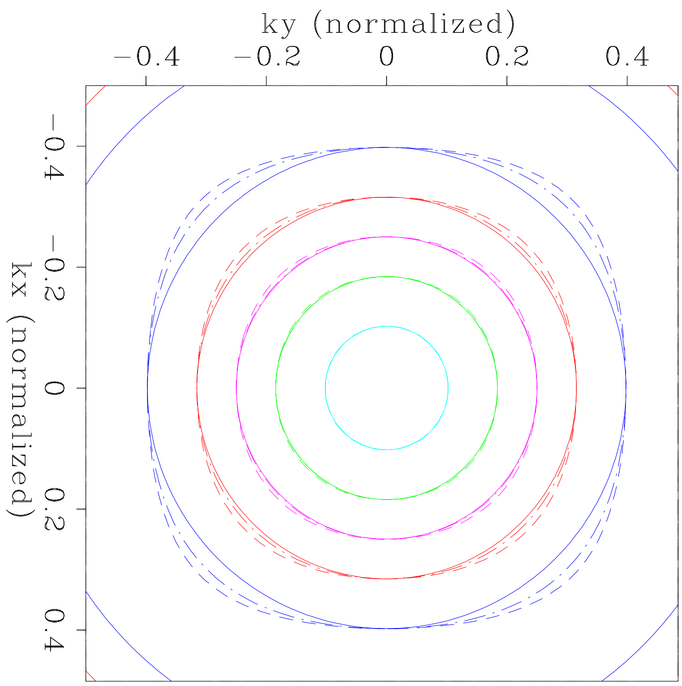


Figure 1: Contours of constant amplitude and phase for the original McClellan transformation filter(large dash), the modified McClellan transformation filter (dot dash) and the ideal circularly symmetric filter (continuous). gopal1-mco3 [ER]

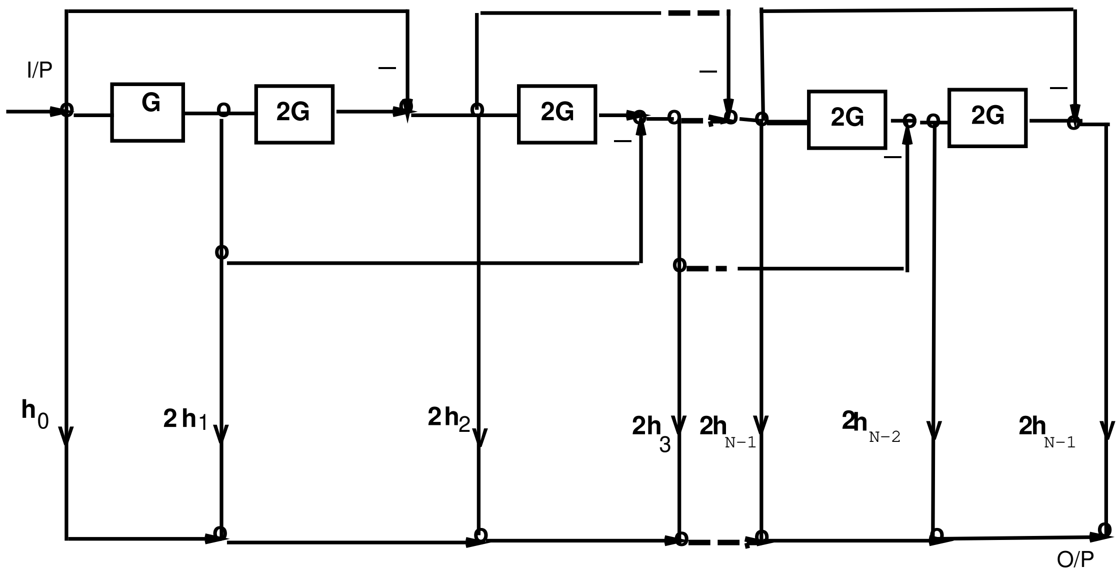


Figure 2: Chebyshev filter structure for the realization of circularly symmetric filters. The lower part does the convolution with the filter h_n and the upper part implements the Chebyshev recursion. gopal1-cheny [NR]

with respect to the desired response. I used a Remez exchange algorithm (Parks and Burrus, 1987) which is a Fourier synthesis algorithm for real symmetric spectral functions with the weighted L_∞ norm. It solves the problem: Given an ideal real symmetrical spectrum $F_0(f)$, $f = [0, \pi]$ find $a(n)$, $n = 0, N$ such that:

$$max[W(f)\left(\sum_{n=0}^{N} a(n)\cos(nf) - F_0(f)\right)]$$

is minimum, where $W(f)$ is weighting function.

Figure **??** shows the desired and the fitted spectrum obtained using a 21 term zero-phase filter. The filter in the space domain is a symmetrical filter as shown in Figure **??**. The filter coefficients correspond to h_n in Figure **??**. The McClellan transformation filter is designed using the 5x5 filter shown in the Appendix. This filter corresponds to the G filter shown in Figure **??**. Once the filter coefficients h_n are computed they are incorporated into the recursive structure shown in Figure **??**. The implementation is done in both space and wavenumber domains. The wavenumber domain response of the filter is shown in Figure **??**. A spike in $x - y$ space was used to test the space-domain implementation. The output of the lowpass filter is a smoothed spike as shown in Figure **??**

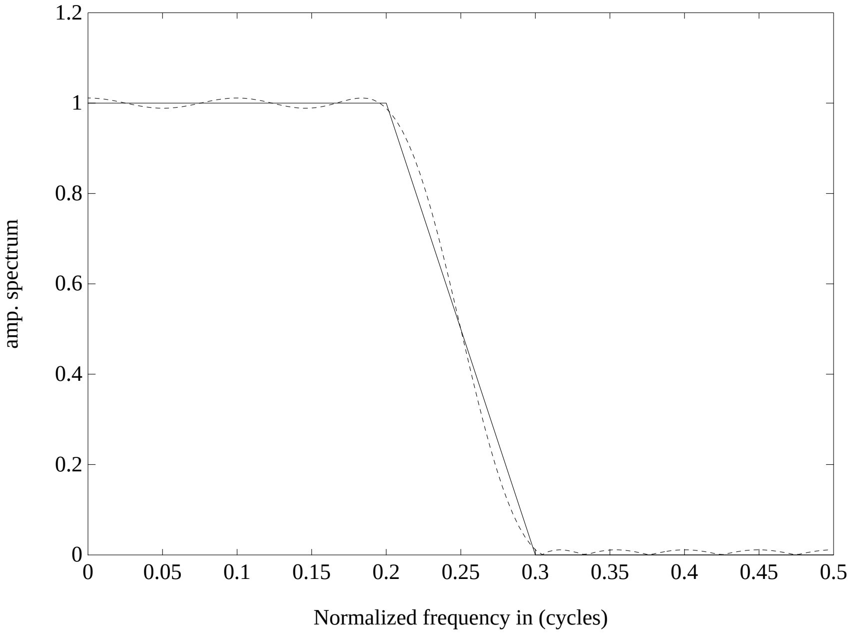


Figure 3: 1-D actual and desired spectra for the lowpass filter. The desired spectrum is given by the continuous curve and the fitted spectrum is given by the dashed curve. gopal1-rez [CR]

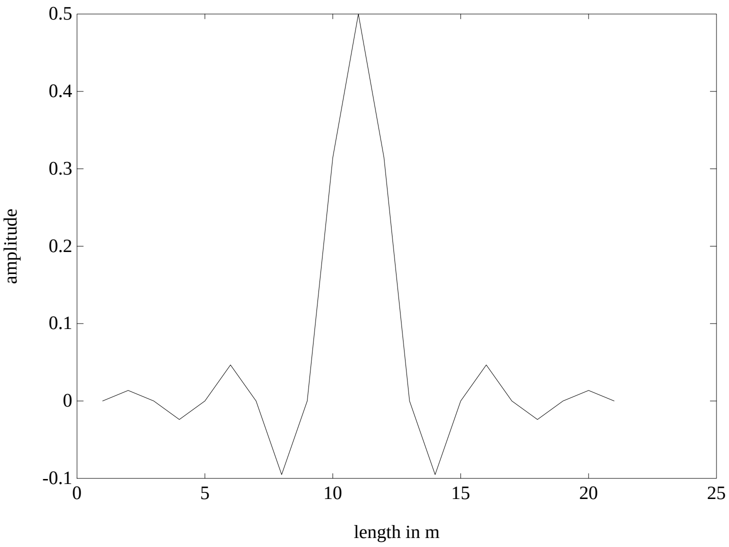


Figure 4: Twenty-one coefficient symmetric 1-D filter in the space domain corresponding to Fig 3. These are the filter coefficients h_n used in the Chebyshev filter structure shown in Figure 2. gopal1-park [CR]

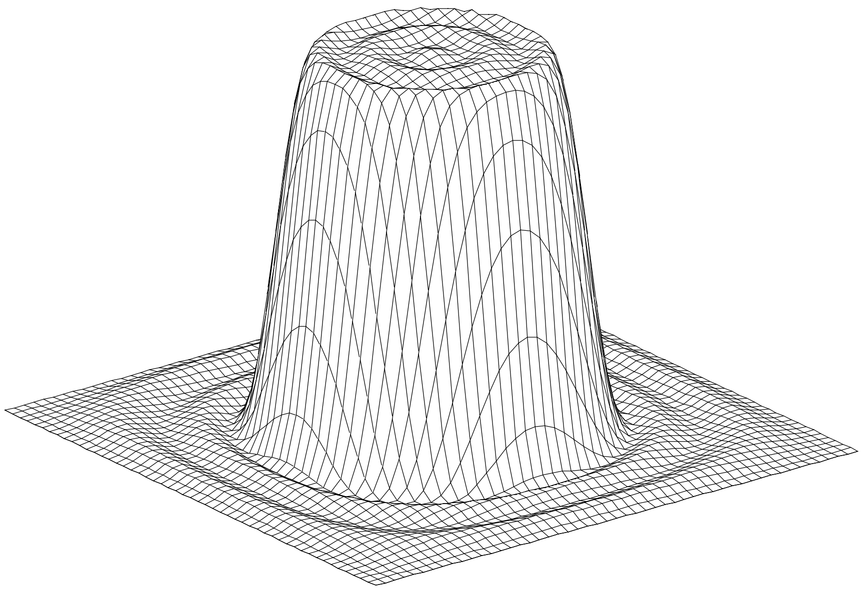
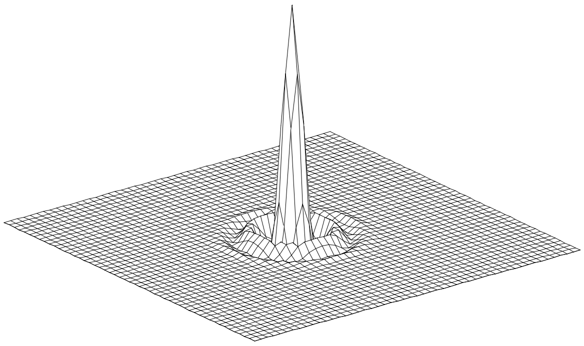


Figure 5: Circularly symmetric 2-D filter corresponding to filter in Figure 4, designed using the first order McClellan transformation. The response is in the $k_x - k_y$ domain. The two horizontal axes are the normalized wavenumbers k_x and k_y and they go from -0.5 to 0.5 cycles/s. gopal1-fil2D [CR]

Figure 6: Output of Lowpass Filter in the x-y domain. The filter implementation is in the x-y domain. The modified McClellan filter in space-domain is used to do the transformation. gopal1-lowpout [CR]



3-D Wavefield extrapolation filter

The wavefield extrapolator in three dimensions is given by Equation 8. The extrapolator for a given temporal frequency ω is a circularly symmetric operator in $(k_x - k_y)$ domain.

$$D(k_x, k_y) = \exp\left\{ i\frac{\Delta z}{\Delta x}\left[\left(\frac{\omega\Delta x}{v}^2\right) - k_x^2 - k_y^2\right]^{1/2}\right\}, \tag{8}$$

where v is the velocity while k_x and k_y are the normalized wavenumbers such that any distance quantity is measured in terms of the horizontal sampling interval Δx. Throughout this paper k refers to the normalized wavenumber.

The circular symmetry of the operator enables it to be realized using McClellan filter. Figure **??** shows the ideal-extrapolator in 3-D for a certain temporal frequency ω. The parameters used in generating the extrapolator are $dx = dz = 12.5m$, $\omega = 40\pi rad/s$, and $v = 1000.m/s$. A tapered 1-D extrapolator was designed by applying a Gaussian taper to the ideal explicit wavefield extrapolation filter. This leads to a stable extrapolation filter as described in Nautiyal *et al*, (1993). The Gaussian taper is given by $\exp\left(-x^2/2\sigma^2\right)$ where σ is a parameter to be chosen. Figure **??** shows the amplitude spectrum for the ideal and the tapered extrapolator . The extrapolation filter in two dimensions is a symmetric filter with complex coefficients. Figure **??** shows the absolute magnitude of the filter coefficients. Using the filter coefficients for the tapered extrapolator, I generated the corresponding 3-D extrapolator (Figure **??**) at the frequency ω using the McClellan transformation. Compare this 3-D extrapolator with the ideal one shown in Figure **??**.

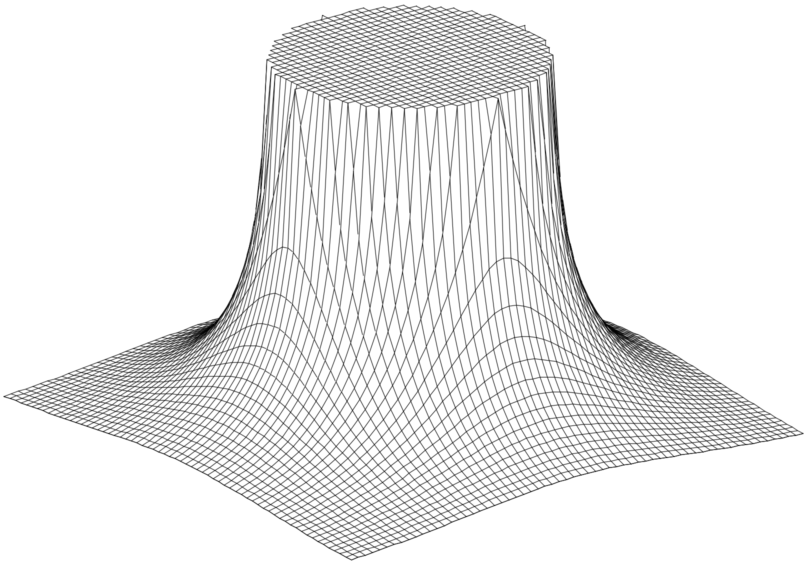


Figure 7: Ideal 3-D extrapolator in the wavenumber $(k_x - k_y)$ domain for a particular temporal frequency ω. The circular region in the center is the propagating region and the region external to it is the evanescent region. The two horizontal axes are the normalized wavenumbers k_x and k_y and they go from -0.5 to 0.5 cycles/s. gopal1-extrap3d [CR]

Figure 8: The ideal 2-D wavefield extrapolator (continuous curve) and the tapered extrapolator (dashed curve) in wavenumber domain.

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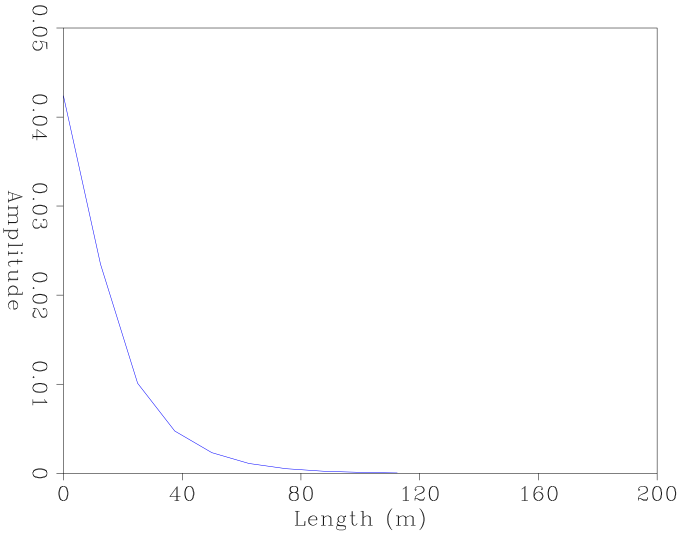

Figure 9: The magnitude of the tapered extrapolator in space domain. The filter coefficients are complex. It is a nineteen coefficient symmetric operator. The spatial wavelet magnitude is shown for one side. gopal1-migfil [ER]

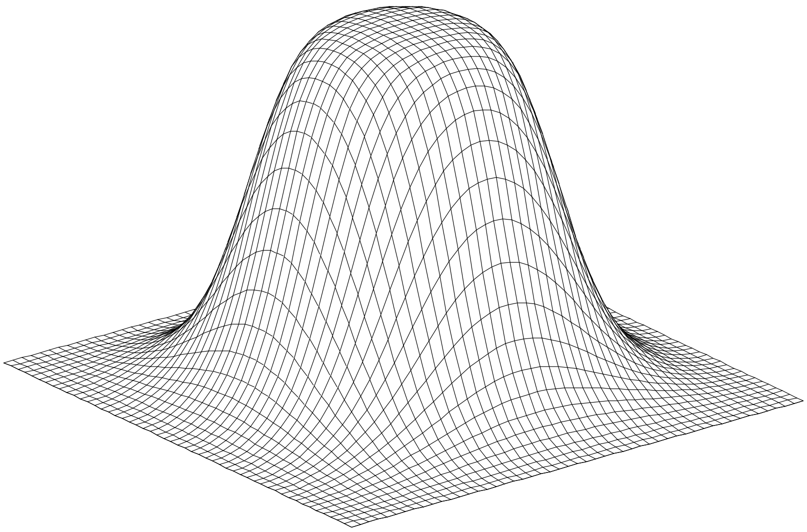

Figure 10: 3-D wavefield extrapolator in $(k_x - k_y)$ domain generated using McClellan transformation. Compare the operator to the ideal operator in Figure 7. gopal1-mclextrap3d [CR]

CONCLUSIONS

Circularly symmetric 2-D filters can be designed efficiently using the McClellan transformation. This method needs only the 1-D filter coefficients to be stored, and requires only N stages compared to N^2 for 2-D convolution filter implementation. The transformation filter has non-circular response for higher wavenumbers. This can be improved upon by using a higher order 2-D filter.

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APPENDIX A

In this Appendix, I give the Chebyshev recursion formula and the different approximations to $\cos\sqrt{k_x^2+k_y^2}$. The Chebyshev recursion formula is given by,

$$\cos nk = 2\cos\left[(n-2)k\right]\cos k - \cos\left[(n-1)k\right] \qquad \text{(A-1)}$$

The original McClellan transformation is given by

$$\cos\sqrt{k_x^2+k_y^2} \approx -\frac{1}{2}+\frac{1}{2}\cos k_x+\frac{1}{2}\cos k_y+\frac{1}{2}\cos k_x \cos k_y \tag{A-2}$$

The modified McClellan transformation is given by (taken from (Hale, 1991)),

$$\cos\sqrt{k_x^2+k_y^2} \approx -\frac{1}{2}+\frac{1}{2}(1+\cos k_x)(1+\cos k_y)-\frac{c}{2}(1-\cos(2k_x))(1-\cos(2k_y)) \tag{A-3}$$

where c is chosen, by exactly matching a particular value k along the diagonal $k_x = k_y$. $c = 0.0255$ was used here, which corresponds to $k{=}\pi/3$.

The 2-D filter in the x-y domain corresponding to the original McClellan transformation is given by,

1/8	1/4	1/8
1/4	-1/2	1/4
1/8	1/4	1/8

The 2-D filter in the x-y domain corresponding to the modified McClellan transformation is given by (taken from (Hale, 1991)),

-c/8	0	c/4	0	-c/8
0	1/8	1/4	1/8	0
c/4	1/4	-(1+c)/2	1/4	c/4
0	1/8	1/4	1/8	0
-c/8	0	c/4	0	-c/8

The 2-D filters have a quadrantal symmetry, so the number of floating point operations required for the 2-D convolution of the filter with data is reduced.