Stanford Exploration Project, Report 77, November 17, 1997, pages 341–349

A result in equivalent medium theory

Francis Muir[1]

keywords: *modeling, scattering, elastic, density*

ABSTRACT

A point Green's function for a heterogeneous, one-dimensional elastic material is expanded in a Taylor series and examined term by term. As expected, the first-order term gives the kinematic result predicted by Schoenberg/Muir and like theories. However, the second-order term of this expansion (subject to a minor stationarity constraint on the heterogeneity) is unexpected in that it depends on first-order averages alone. A consequence is that ensemble average responses for materials with the same statistical properties show no second-order transmission loss. Energy transferred from the direct to the scattered field must involve higher order terms in the Taylor series expansion. It follows that there is no simple 'central limit' asymptotic solution, and no equivalent homogeneous visco-elastic medium.

INTRODUCTION

There are at least two quite distinct applications of Equivalent Medium Theory in seismic exploration, and these applications drive two rather different lines of research. The first is a scaling operation. Sonic logs may be sampled one hundred times more densely than is required for modeling seismic data, and this might be too expensive to contemplate in case of 3-D models. Both (Folstad and Schoenberg, 1992) and (Karrenbach, 1993) have shown how the Schoenberg-Muir calculus can be used to 'filter' fine-sampled data and then resample at a suitably coarser depth interval.

In the other application, instead of substituting fine heterogeneity for a coarser one, we are interested in coming up with a homogeneous equivalent model. A typical situation would be where we want to think of our earth model as being blocky, but we want the block models to reflect the frequency-dependent loss due to scattering. It is this second application that drives my work in this paper.

[1]**email:** not available

DEVELOPMENT

The symbols used in the development are listed in the Appendix to this paper.

The building blocks

The field variable transformation

$$\begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \surd(\rho/s) & -\surd(\rho/s) \end{pmatrix} \begin{pmatrix} v^+ \\ v^- \end{pmatrix} \tag{1}$$

and its inverse

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \surd(s/\rho) \\ 1 & -\surd(s/\rho) \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix} \tag{2}$$

and the translation equation

$$\begin{pmatrix} v^+(z) \\ v^-(z) \end{pmatrix} = \begin{pmatrix} e^{-i\omega z\surd(\rho s)} & 0 \\ 0 & e^{+i\omega z\surd(\rho s)} \end{pmatrix} \begin{pmatrix} v^+(0) \\ v^-(0) \end{pmatrix} \tag{3}$$

concatenate to form a basic building block

$$\begin{pmatrix} v(z) \\ \sigma(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega z\surd(\rho s)) & -i\surd(s/\rho)\sin(\omega z\surd(\rho s)) \\ -i\surd(\rho/s)\sin(\omega z\surd(\rho s)) & \cos(\omega z\surd(\rho s)) \end{pmatrix} \begin{pmatrix} v(0) \\ \sigma(0) \end{pmatrix} \tag{4}$$

which may also be written

$$\begin{pmatrix} v(z) \\ \sigma(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega z\lambda) & -i\xi^{-1}\sin(\omega z\lambda) \\ -i\xi\sin(\omega z\lambda) & \cos(\omega z\lambda) \end{pmatrix} \begin{pmatrix} v(0) \\ \sigma(0) \end{pmatrix} \tag{5}$$

and in case slowness is a constant, then we can re-write this equation in a $\mathcal{Z}$–transform notation

$$\begin{pmatrix} v(z) \\ \sigma(z) \end{pmatrix} = \frac{1}{2\mathcal{Z}^{1/2}} \begin{pmatrix} (1+\mathcal{Z}) & \xi^{-1}(1-\mathcal{Z}) \\ \xi(1-\mathcal{Z}) & (1+\mathcal{Z}) \end{pmatrix} \begin{pmatrix} v(0) \\ \sigma(0) \end{pmatrix} \tag{6}$$

If we denote the RHS operator above by $\mathcal{P}(\mathcal{Z})$, note the property

$$\mathcal{P}^2(\mathcal{Z}) = \mathcal{P}(\mathcal{Z}^2) \tag{7}$$

Taylor series expansion

The operator on the RHS of Equation 4 can be expanded as a power series in $-i\omega$

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - i\omega z \begin{pmatrix} 0 & s \\ \rho & 0 \end{pmatrix} - \frac{\omega^2 z^2}{2} \begin{pmatrix} \rho s & 0 \\ 0 & \rho s \end{pmatrix}$$

$$+$$

$$i\frac{\omega^3 z^3}{6} \begin{pmatrix} 0 & \rho s^2 \\ \rho^2 s & 0 \end{pmatrix} + \frac{\omega^4 z^4}{24} \begin{pmatrix} \rho^2 s^2 & 0 \\ 0 & \rho^2 s^2 \end{pmatrix} \tag{8}$$

which may be considered as the Fourier transform of the first five terms of a Taylor series expansion of a time-domain convolutional kernel.

Many layers

Since this translational operator is now in terms of parameters v and σ which are constant across layer boundaries, we can form the n-layer transformation by forming the product of the n individual layer transformations. In the case of the first three terms of our truncated power series approximation of Equation 8, this leads to

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$-i\omega z \begin{pmatrix} 0 & \sum_{j=1}^{n} s_j \\ \sum_{j=1}^{n} \rho_j & 0 \end{pmatrix}$$

$$-\frac{\omega^2 z^2}{2} \begin{pmatrix} \sum_{j=1}^{n} s_j \rho_j + 2 \sum_{j=2}^{n} s_j \sum_{k=1}^{j-1} \rho_k & 0 \\ 0 & \sum_{j=1}^{n} \rho_j s_j + 2 \sum_{j=2}^{n} \rho_j \sum_{k=1}^{j-1} s_k \end{pmatrix} \tag{9}$$

Lag formulation

Without changing the outcome, the double summations in Equation 9 can be rewritten in terms of lags (l), and the n-layer transformation operator now looks like:

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$-i\omega z \begin{pmatrix} 0 & \sum_{j=1}^{n} s_j \\ \sum_{j=1}^{n} \rho_j & 0 \end{pmatrix}$$

$$-\frac{\omega^2 z^2}{2} \begin{pmatrix} \sum_{j=1}^{n} s_j \rho_j + 2 \sum_{l=1}^{n-1} \sum_{j=1}^{n-l} s_{j+l} \rho_j & 0 \\ 0 & \sum_{j=1}^{n} \rho_j s_j + 2 \sum_{j=1}^{n-1} \sum_{j=1}^{n-l} \rho_{j+l} s_j \end{pmatrix} \tag{10}$$

A symmetric/anti-symmetric reformulation

Now set

$$\mathcal{S} = \sum_{j=1}^{n} s_j \rho_j + \sum_{l=1}^{n-1} \sum_{j=1}^{n-l} (s_{j+l} \rho_j + \rho_{j+l} s_j)$$

that is, $\mathcal{S}$ is the sum of products of s and ρ over all instances and lags, and

$$\mathcal{A} = \sum_{l=1}^{n-1} \sum_{j=1}^{n-l} (s_{j+l} \rho_j - \rho_{j+l} s_j)$$

so that $\mathcal{S}$ and $\mathcal{A}$ are, respectively, the symmetric and anti-symmetric components of the principal diagonal of the second-order matrix. The n-layer transformation of Equation 10 is now

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - i\omega z \begin{pmatrix} 0 & \sum_{j=1}^{n} s_j \\ \sum_{j=1}^{n} \rho_j & 0 \end{pmatrix} - \frac{\omega^2 z^2}{2} \begin{pmatrix} \mathcal{S} + \mathcal{A} & 0 \\ 0 & \mathcal{S} - \mathcal{A} \end{pmatrix} \tag{11}$$

A stationarity condition

The anti-symmetric term, $\mathcal{A}$, is a scalar which accounts for linear trends in the addable material constants. $\mathcal{A}$ can be removed from the problem by removing any such trends from ρ and s without altering their sum values. With this, Equation 11 becomes

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - i\omega z \begin{pmatrix} 0 & \sum_{j=1}^{n} s_j \\ \sum_{j=1}^{n} \rho_j & 0 \end{pmatrix} - \frac{\omega^2 z^2}{2} \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{pmatrix} \tag{12}$$

which can be rewritten

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - i\omega Z \begin{pmatrix} 0 & <s> \\ <\rho> & 0 \end{pmatrix} - \frac{\omega^2 Z^2}{2} \begin{pmatrix} <\rho><s> & 0 \\ 0 & <\rho><s> \end{pmatrix} \tag{13}$$

THE RESULT

Dynamic equivalence

Comparing Expression 8 with Expression 13, it is evident that they are identical in form. That is, any heterogeneous sequence of layers has the same wave behavior – to second order in $i\omega$ – as a particular homogeneous material of the same total thickness

> *provided that lines of best fit to the elastic parameters have zero slope with respect to the direction of layering.*

The startling thing for me is that the second-order term says nothing about the second-order structure of the layers, and so contains no information about the loss of energy from the main pulse to the coda. If we wish to see how energy is lost to the coda we must examine higher-order terms in the Taylor series expansion.

A matrix symmetry property

At this point it is useful to recognize a group property of some 2-by-2 matrices. If $F(a,b)$ represents any 2-by-2 matrix, a function of a and b, with the property that

$$F_{11}(a,b) = F_{22}(b,a)$$

and

$$F_{12}(a,b) = F_{21}(b,a)$$

then products of two such matrices will also have the property. That is, matrices with this property form a group under matrix multiplication. If we substitute $/rho$ for a and s for b, then our basic building block operator (Equation 8) has this property, and so do the terms in the Taylor series expansion of this operator, and so do any linear combinations of products of these terms. It follows that the terms in the Taylor series expansion of our product forms will also have the property, and this will greatly simplify the development.

Discussion

We know that heterogeneity causes scattering, and we have discovered that this scattering is not apparent in the transmissivity at least up to and including the second-order term in $i\omega$. A straight-forward consequence is that no visco-elastic model can simulate scattering loss – it has to be a less lossy system. Nor can the equivalent dynamic model be a Kjartannsson model since this would have the wrong static (non-elastic) properties. Is there necessarily any dynamically equivalent model? No. In the end we could always replace our complex heterogeneities with simpler ones. As an example, we could substitute bimodal models for multi-modal data.

REFERENCES

Folstad, P. G., and Schoenberg, M., 1992, Use of equivalent anisotropic medium theory to calculate synthetic seismograms: SEG Summer Research Workshop on AVO, Soc. Expl. Geophys., Expanded Abstracts.

Karrenbach, M., 1993, Relating measurements on different scales: SEP–**77**, 303–318.

APPENDIX: SYMBOLS

The basic material properties and other parameters are

Uniform layer thickness z

Number of layers n

Total thickness $Z = nz$

Mass density of j'th layer ρ_j

Average mass density $<\rho>$

Mass density perturbation of jth layer $\Delta\rho_j$

Compliance of jth layer s_j

Average compliance $<s>$

Compliance perturbation of jth layer Δs_j

Slowness λ, defined equal to $\sqrt{(\rho s)}$

Impedance ξ, defined equal to $\sqrt{\rho/s})$

Velocity field v

Traction field σ

The two one-way velocity fields v^+ and v^-