The derivation of two partial differential equations

In this appendix, I derive two partial differential equations that are related to the calculation of the geometrical amplitudes of seismic waves.

The first partial differential equation is for the calculation of the take-off angle and its gradient fields. Let $\gamma = \gamma(x, z; x_s)$ be the take-off angle of the ray connecting a surface point x_s and a subsurface point (x, z). Because the ray has a unique take-off angle, γ is constant for any point (x, z) along the ray. Therefore, the derivative of γ with respect to the arc length of the ray is zero, which gives

$$\frac{d\gamma}{ds} = \frac{\partial \gamma}{\partial x}\frac{dx}{ds} + \frac{\partial \gamma}{\partial z}\frac{dz}{ds} = 0. \tag{1}$$

The ray equations (equation (**??**) of Chapter **??**) give

$$\begin{cases} \dfrac{dx}{ds} & = & v\dfrac{\partial \tau}{\partial x} \\ \\ \dfrac{dz}{ds} & = & v\dfrac{\partial \tau}{\partial z}. \end{cases} \tag{2}$$

Substituting the relations in equation (2) into equation (1) and dividing both sides of the equation by v yields

$$\frac{\partial \tau}{\partial x}\frac{\partial \gamma}{\partial x} + \frac{\partial \tau}{\partial z}\frac{\partial \gamma}{\partial z} = 0, \tag{3}$$

which is equation (**??**) of Chapter **??**.

Equation (3) can be rewritten as

$$\nabla \tau \cdot \nabla \gamma = 0, \tag{4}$$

where $\nabla \tau$ is the gradient of the traveltime field, $\nabla \gamma$ is the gradient of the take-off angle field, and $\cdot$ stands for the dot product of two vector fields. As shown in Figure 1, the vector $\nabla \tau$ at a point is tangential to the ray at that point and the vector $\nabla \gamma$ is tangential to the wavefronts. Therefore, equation (3) simply states the fact that in an isotropic medium, the ray and wavefront at any point are orthogonal.

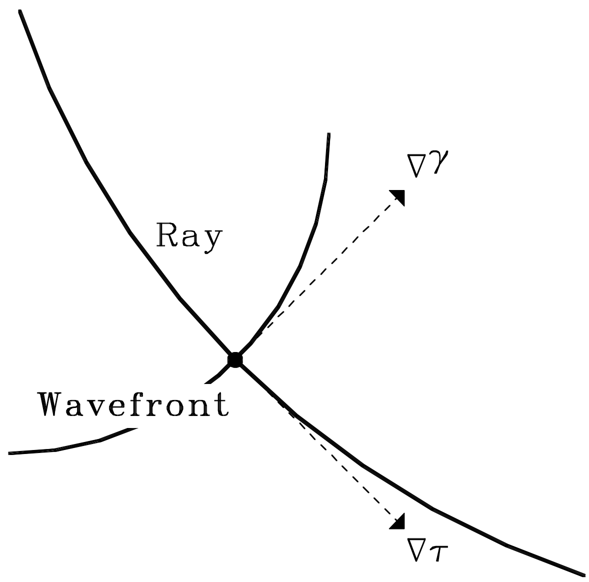

Figure 1: The Orthogonal relation of ray and wavefront in an isotropic medium. The gradient of traveltime is orthogonal to the gradient of take-off angle. pdeapda-vpraywft [ER]

The second partial differential equation is for the calculation of the geometrical spreading factor η that is defined in equation (**??**) of Chapter **??**. Here I rewrite it as follows:

$$\eta(x(s,\gamma), z(s,\gamma); x_s) = \frac{1}{v_0}\int_0^s v(x(\xi,\gamma), z(\xi,\gamma))d\xi, \tag{5}$$

where v_0 is the velocity at the source point, (s,γ) are ray-coordinates, and $x(s,\gamma)$ and $z(s,\gamma)$ are the mapping functions from ray coordinates to Cartesian coordinates. Taking the derivative of η with respect to the arc length s along a ray gives

$$\frac{d\eta}{ds} = \frac{\partial\eta}{\partial x}\frac{dx}{ds} + \frac{\partial\eta}{\partial z}\frac{dz}{ds} = \frac{v}{v_0}. \tag{6}$$

Substituting the relations in equation (2) into equation (6) and then dividing both sides of equation (6) by velocity v yield

$$\frac{\partial\tau}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\tau}{\partial z}\frac{\partial\eta}{\partial z} = \frac{1}{v_0}, \tag{7}$$

which is equation (**??**) of Chapter **??**.