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Numerical analysis of $\omega - $ x migration

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ABSTRACT

For the constant velocity 15-degree space-frequency $(\omega - x)$ time migration algorithm, the parabolic approximation of the exact one-way wave equation causes undermigration, and the Crank-Nicolson numerical method that is used to solve the diffraction equation causes overmigration. However, the modified wave equation that the numerical solution actually satisfies for the diffraction equation shows that the trace spacing interval plays a larger role than the depth step size of downward continuation in determining the accuracy of the numerical solution.

INTRODUCTION

Analyses of finite-difference migration errors of the approximations to the semi-circular dispersion relation by various one-way wave equations have been given by Claerbout (1976, 1985). However, the effects caused by sampling in time, space, and depth and the interaction between them have been largely ignored. Lynn (1976) performed an analysis of group velocity errors in 15- and 45-degree migration that included the effect of trace spacing but not that of depth step. Beaumont et al. (1987) presented a quantitative analysis of the phase errors of finite-difference migration. A systematic numerical analysis, however, is absent in the literature. This paper presents a numerical analysis of the constant velocity 15-degree space-frequency $(\omega - x)$ time migration algorithm as a useful step toward a systematic numerical analysis of finite-difference migration algorithms. The numerical analysis procedures, I presented, can be extended to other finite-difference migration algorithms that are implemented in space-frequency $(\omega - x)$ and space-time $(t - x)$ domain. I agree with Nichols (1992) that by optimizing the numerical parameters used in $\omega - x$ downward continuation, we can use the 15-degree scheme to migrate higher angles accurately. The following derivations largely follow Claerbout (1985) in geophysics, and Moin (1992) in numerical analysis.

This paper first explains how the parabolic approximation of the exact one-way

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wave equation causes undermigration. It then shows how the numerical method based on the eigenvalue structure of the diffraction partial differential equation and the Crank-Nicolson method causes overmigration. Finally, it derives the modified wave equation that the numerical solution actually satisfies for the diffraction equation.

UNDERMIGRATION CAUSED BY PARABOLIC APPROXIMATION

Migration is based on the one-way wave equation

$$\frac{\partial P}{\partial z} = i k_z P \tag{1}$$

which has the analytic solution

$$P(z + \Delta z) = e^{i k_z \Delta z} P(z) \tag{2}$$

where

$$k_z = \frac{\omega}{v} \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}} \tag{3}$$

and P is the wavefield, z is the depth coordinate, ω is the frequency, v is the velocity, and k_x and k_z are the horizontal and vertical wavenumbers. For small k_x, k_z can be expanded in a series such as

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \ldots \tag{4}$$

Retaining the first two terms in the expansion results in the 15-degree equation

$$\hat{k_z} = \frac{\omega}{v} - \frac{v k_x^2}{2\omega} \tag{5}$$

Because the error term is negative, the approximate $\hat{k_z}$ is always greater than the physical k_z. Migration is the process of dephasing (Mo, 1992), which means that, imaging condition (time $t = 0$) is satisfied when the entire exploding reflector upward propagation phase is exhausted. For $|k_x| > 0$, that is, for waves with propagation angles greater than zero, less distance is traveled when the imaging operation is performed. And the larger k_x is, the larger the error term, which results in a smaller propagation distance for the migration response. This explains why the migration impulse response swarms around the vertical axis (Claerbout, 1985, Figure 4.2-2; Yilmaz, 1987, Figure 4-90). Thus the parabolic approximation of the exact one-way wave equation causes undermigration – the specification error. This phenomenon was defined by Claerbout (1985) as anisotropy dispersion.

OVERMIGRATION CAUSED BY THE CRANK-NICOLSON METHOD

Equation (5) can be transformed into the space-frequency domain, resulting in the 15-degree $\omega - x$ time migration equation

$$\frac{\partial P}{\partial \tau} = -i\omega P - \frac{v^2}{-i\omega 2}\frac{\partial^2 P}{\partial x^2} \tag{6}$$

This partial differential equation can be solved by splitting it (Claerbout, 1985) into the diffraction term

$$\frac{\partial P}{\partial \tau} = -i\frac{v^2}{\omega 2}\frac{\partial^2 P}{\partial x^2} \tag{7}$$

and the lens term

$$\frac{\partial P}{\partial \tau} = -i\omega P \tag{8}$$

The lens equation can be solved accurately by analytic exponentiation. Here I consider the diffraction equation, which is a partial differential equation (PDE). A PDE can be readily converted to a system of ordinary differential equations (ODEs) by using finite-difference approximations for derivatives in all but one of the dimensions. To do so, I discretize coordinate x with $N+1$ uniformly spaced grid points, as follows:

$$x_j = x_{j-1} + \Delta x, j = 0, 1, 2, \ldots, N \tag{9}$$

Using the second-order central-difference scheme for the second derivative in x results in

$$\frac{\partial P_j}{\partial \tau} = \alpha\frac{P_{j+1} - 2P_j + P_{j-1}}{\Delta x^2}, j = 1, 2, \ldots, N-1 \tag{10}$$

where $P_j = P(\tau, x_j)$ and $\alpha = -i\frac{v^2}{\omega 2}$. With a further assumption of the boundary condition $P(\tau, 0) = P(\tau, L) = 0$, I obtain a system of ODEs that can be written in matrix form as

$$\frac{d\vec{P}}{d\tau} = A\vec{P} \tag{11}$$

where P_j's are the elements of the vector $\vec{P}$, and A is the $(N-1)*(N-1)$ tri-diagonal matrix

$$A = \frac{\alpha}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix} \tag{12}$$

This system of ODEs can be solved using any of the numerical methods, such as the Runge-Kutta formulas or multi-step methods.

However, when dealing with systems, we need to be concerned about stiffness. To investigate numerical stiffness, we must study the eigenvalue structure of the matrix

A. The eigenvalues of A obtained from a known formula for the eigenvalues of a tri-diagonal matrix with constant entries are

$$\lambda_j = \frac{\alpha}{\Delta x^2}(-2 + 2\cos\frac{\pi j}{N}), j = 1, 2, \ldots, N-1 \tag{13}$$

It is important to keep in mind that all the eigenvalues of A are non-zero and purely imaginary, since α is purely imaginary. The eigenvalue with the smallest magnitude is

$$\lambda_1 = \frac{\alpha}{\Delta x^2}(-2 + 2\cos\frac{\pi}{N}) \tag{14}$$

For large N, the series expansion for $\cos\frac{\pi}{N}$,

$$\cos\frac{\pi}{N} = 1 - \frac{1}{2!}(\frac{\pi}{N})^2 + \frac{1}{4!}\frac{\pi}{N}^4 + \ldots \tag{15}$$

converges rapidly. Retaining the first two terms in the expansion results in

$$\lambda_1 \approx -\frac{\pi^2\alpha}{N^2\Delta x^2} \tag{16}$$

Also, for large N

$$\lambda_{N-1} \approx -\frac{4\alpha}{\Delta x^2} \tag{17}$$

Therefore, the ratio of the eigenvalue with the largest modulus to the eigenvalue with the smallest modulus is

$$\left|\frac{\lambda_{N-1}}{\lambda_1}\right| \approx \frac{4N^2}{\pi^2} \tag{18}$$

Thus for seismic migration, typically $N \geq 500$ traces, this is an extremely stiff system of ODEs. There is severe contradiction in choosing the pseudodepth (time τ) step size of downward extrapolation, so that explicit methods, for example, explicit Euler and Runge-Kutta methods are unworkable.

Standard decoupling using the matrix of eigenvectors S yields

$$\frac{d\vec{Q}}{d\tau} = \Lambda\vec{Q} \tag{19}$$

where $\Lambda = SAS^{-1}$ is a diagonal matrix with the eigenvalues of A on the diagonal, S is the matrix whose columns are the eigenvectors of A, and $Q = S\vec{P}$. Since the equations are uncoupled, the following solution can be readily obtained:

$$Q_j(\tau) = Q_j(0)e^{\lambda_j\tau} \tag{20}$$

Thus, purely imaginary eigenvalues result in the oscillation of the solution that is expected for the wave equation.

The Crank-Nicolson method

The most popularly used numerical method of solving a stiff system of ODEs such as (11) is the Crank-Nicolson method, chosen because of its unconditional stability and good accuracy. First let us look at the Crank-Nicolson (trapezoidal) method for a single first-order ODE. Applying the Crank-Nicolson method to the model equation

$$\frac{dy}{dt} = \lambda y \tag{21}$$

yields

$$\frac{y_{j+1} - y_j}{\Delta t} = \frac{\lambda}{2}(y_{j+1} + y_j) \tag{22}$$

or

$$y_{j+1} = \frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} y_j \tag{23}$$

Thus the solution of (21) can be written as

$$y_j = \sigma^j y_0 \tag{24}$$

where y_0 is the initial condition and

$$\sigma = \frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} \tag{25}$$

For the migration wavefield extrapolation equation (19), the eigenvalue $\lambda = i\beta$ ($\beta > 0$) is purely imaginary. Thus

$$\sigma = |\sigma| e^{i\theta} \tag{26}$$

Because $|\sigma| = 1$, the Crank-Nicolson method as applied to migration wavefield extrapolation does not generate any amplitude error. This result agrees with the one-way migration wavefield extrapolation that is a dampless sinusoidal wave propagation. However, there is phase error. The numerical solution is

$$\theta = 2 \arctan \frac{\beta \Delta t}{2} \tag{27}$$

$$y_j = \sigma^j y_0 \tag{28}$$

$$y_j = \sigma^j y_0 = e^{i\theta j} y_0 \tag{29}$$

But the exact solution of the ODE (21) is

$$y(x) = e^{i\beta t} y_0 = e^{i\beta \Delta t j} y_0 \tag{30}$$

The phase error is

$$PE = \beta \Delta t - 2 \arctan \frac{\beta \Delta t}{2} = \frac{(\beta \Delta t)^3}{12} + \dots \tag{31}$$

The numerical solution lags behind the exact solution. Thus applying the Crank-Nicolson method to the diffraction equation causes overmigration – the experimental error. And the eigenvalues with larger moduli have larger degrees of overmigration.

THE MODIFIED WAVE EQUATION

Since the numerical solution is an approximation of the exact solution, it does not exactly satisfy the continuous partial differential equation at hand, but it satisfies a *modified* equation. In this case, I use the Crank-Nicolson method to solve the diffraction equation

$$\frac{\partial P}{\partial \tau} = \alpha \frac{\partial^2 P}{\partial x^2} \tag{32}$$

$$\frac{P^{(n+1)} - P^{(n)}}{\Delta \tau} = \frac{\alpha}{2}\left[\frac{\partial^2 P^{(n+1)}}{\partial x^2} + \frac{\partial^2 P^{(n)}}{\partial x^2}\right] \tag{33}$$

$$\frac{P_j^{(n+1)} - P_j^{(n)}}{\Delta \tau} = \frac{\alpha}{2}\left[\frac{P_{j+1}^{(n+1)} - 2P_j^{(n+1)} + P_{j-1}^{(n+1)}}{\Delta x^2} + \frac{P_{j+1}^{(n)} - 2P_j^{(n)} + P_{j-1}^{(n)}}{\Delta x^2}\right] \tag{34}$$

By Taylor series expansion of every term about P_j^n, I obtain the modified wave equation that the numerical solution actually satisfies:

$$\frac{\partial P}{\partial \tau} = \alpha \frac{\partial^2 P}{\partial x^2} + \epsilon \tag{35}$$

$$\epsilon = \frac{\alpha \Delta x^2}{12}\frac{\partial^4 P}{\partial x^4} + \frac{\alpha}{24}\left[\frac{\Delta x^4}{30} + \Delta x^2 \Delta \tau \alpha + \Delta \tau^2 \alpha^2\right]\frac{\partial^6 P}{\partial x^6} + \ldots \tag{36}$$

The error term shows that the trace spacing interval Δx plays a larger role than the depth step size $\Delta \tau$ of downward continuation in determining the accuracy of the numerical solution.

DISCUSSION

In this paper, the second spatial derivative $\frac{\partial^2 P}{\partial x^2}$ is computed by the operator $T = (1, -2, 1)$. Usually it is computed by a more sophisticated scheme (Claerbout, 1985) for better accuracy. However, when that scheme is used the numerical analysis is carried out as explained in the previous two sections.

CONCLUSIONS

On the basis of the idea of migration dephasing, the effect of the parabolic approximation of the exact one-way wave equation is explained as undermigration and the effect of the Crank-Nicolson numerical method that is used to solve the diffraction equation is explained as overmigration. The modified wave equation that the numerical solution actually satisfies for the parabolic diffraction wave equation shows that the trace spacing interval plays a larger role than the depth step size of downward continuation in determining the accuracy of the numerical solution.

FUTURE RESEARCH

Building on the results presented in this paper, I plan to (1) apply the above analysis to higher-angle accurate migration schemes, and (2) develop an optimization procedure to determine migration parameters for accuracy and efficiency by making the specification error (undermigration) and the experimental error (overmigration) cancel.

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