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Anelliptic approximations for TI media

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ABSTRACT

Homogeneous scalar isotropy can be completely specified by a single velocity. Elliptical anisotropy with a vertical symmetry axis requires two velocities, vertical and horizontal. For some problems these two velocities may not be enough. In particular, because vertical scale is unknown for surface-recorded data the vertical velocity in elliptical anisotropy gains nothing over isotropy. We introduce two successive scalar anisotropic approximations beyond elliptical anisotropy which can be used when more independent parameters are needed but the full complexity of transverse isotropy is unnecessary. Both approximations take the form of simple rational polynomials. We call them *an*elliptic approximations to indicate that although they are not elliptical, they do share some of elliptical anisotropy's useful properties. We show examples of how both anelliptic approximations can be used to fit transversely isotropic wave modes with acceptable accuracy. The first anelliptic approximation is specified by three parameters: vertical velocity, surface NMO velocity, and true horizontal velocity. The first two velocity parameters determine the near-vertical paraxial behavior, which is elliptic. The second anelliptic approximation is specified by four parameters: vertical velocity, surface NMO velocity, true horizontal velocity, and borehole NMO velocity. The first anelliptic approximation occurs as a special case. To a good approximation, both of the anelliptic approximations have identical forms in the group-velocity and dispersion-relation domains and converting between the two domains is trivial. The anelliptic velocity parameters are convenient and robust because of their close relationship to standard geophysical measurements. We also show how Dix's equations for stack-moveout velocities can be interpreted in terms of a paraxial layer group, and how this formulation naturally leads to an anelliptic extension of Dix's equations. The anelliptic Dix equations encompass nonhyperbolic moveout both as a result of intrinsic anisotropy and ray bending at layers.

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INTRODUCTION

If we had unlimited resources we could use the full elastic wave equation with a complete set of elastic parameters to do seismic processing; in practice this is an expensive and unlikely proposition at best. At the other extreme, mundane processing assuming scalar P-wave isotropy may be unnecessarily limiting. It is useful to look for some middle ground between these two extremes.

Scalar elliptical anisotropy with a vertical axis of symmetry often finds use as a "first step" approximation beyond scalar isotropy, even though it is not (in general) a solution of the elastic wave equation. Elliptical anisotropy is useful because of its close relationship to standard "isotropic" geophysical techniques. Hyperbolic moveout corresponds to elliptical wavefronts in the Earth; NMO velocity analysis measures horizontal velocity, whereas time-to-depth conversion depends on vertical velocity. Elliptical anisotropy is computationally convenient as well because of its unique symmetry property; an elliptical dispersion relation implies an elliptical impulse response, and vice-versa. Thus there is a simple analytical form in both domains of interest (dispersion relation and impulse response) and converting between them is easy.

Unfortunately, elliptical anisotropy with a vertical axis is a useful approximation only in certain restricted cases. It is adequate only if the true anisotropy does not depart too far from ellipticity over the range of (probably near-vertical) propagation paths sampled. More importantly, if all sources and receivers lie in a plane (such as for surface seismic data), elliptical anisotropy is kinematically indistinguishable from isotropy; the extra parameter disappears into the unconstrained vertical dimension.

We could fix this last problem by using some more complicated type of scalar anisotropic system, for example the weak-anisotropy group-velocity scheme of Byun et al. (1989) or the qP wavetype of exact transverse isotropy with a vertical symmetry axis. If we proceed this way, though, we lose some or all of the nice symmetry properties that make elliptical anisotropy so convenient. It is better to realize that we never expected to find pure elliptical anisotropy in the Earth anyway; it was always just a convenient approximation. It is good enough if we can design a three- or four-parameter scalar anisotropic system that *in approximation* retains the useful properties of (two-parameter) scalar elliptical anisotropy.

We propose a family of nested *an*-elliptic forms. Each successively more complex member in the series contains the previous ones as special cases; the first in the series is elliptical anisotropy with a vertical symmetry axis. Like elliptical anisotropy, each member in the series has a simple relationship to NMO and true vertical and horizontal velocity, and to a good approximation has the same simple analytical form in both domains of interest, group velocity and phase slowness.

ELLIPTICAL ANISOTROPY

Waves propagating in a homogeneous medium can be described and extrapolated in either the time-space "group" or frequency-wavenumber "phase" domains. In the standard scalar isotropic case life is simple; there is one soundspeed characteristic of the medium, and it is the same whether we are talking about rays (group velocity) or plane waves (phase velocity). The scalar isotropic ray (or group-velocity) equation in polar coordinates is just

$$V(\phi_r) = V_{\text{iso}}; \tag{1}$$

the corresponding dispersion relation (phase-velocity equation) is

$$\omega^2 = V_{\text{iso}}^2 (k_z^2 + k_x^2). \tag{2}$$

With anisotropy the propagation velocity depends on the propagation direction, and the group and phase velocities and directions are no longer the same. Convenient closed-form "group" and "phase" equations like equations (**??**) and (**??**) are usually not simultaneously available. Even worse, the number of parameters required to describe the medium can be intractably large.

Scalar elliptical anisotropy does not suffer from these difficulties. There are only two velocities, horizontal and vertical. It is true that the ray equation for scalar elliptical anisotropy appears a bit complicated at first glance:

$$V(\phi_r) = \frac{1}{\sqrt{V_z^{-2}\cos^2(\phi_r) + V_x^{-2}\sin^2(\phi_r)}}, \tag{3}$$

where ϕ_r is the group-propagation (ray) direction (with $\phi_r = 0$ being vertical). If we re-express equation (**??**) in terms of *slowness squared* $M = 1/V^2$, however, it simplifies considerably:

$$M(\phi_r) = M_z \cos^2(\phi_r) + M_x \sin^2(\phi_r). \tag{4}$$

The dispersion relation for elliptical anisotropy is similarly simple:

$$\omega^2 = W_z k_z^2 + W_x k_x^2, \tag{5}$$

where W_z is the vertical velocity squared and W_x is the horizontal velocity squared. Another form of equation (**??**) displays the fundamental nature of elliptical anisotropy more clearly. If you divide equation (**??**) through by $|\mathbf{k}| = k = k_z^2 + k_x^2$, you get

$$W(\phi_w) = W_z \cos^2(\phi_w) + W_x \sin^2(\phi_w), \tag{6}$$

where $W = \omega^2/k^2$ is phase-velocity squared and ϕ_w is the wave-propagation (phase) direction, with $\phi_w = 0$ being vertical. The identity of form between equations (**??**) and (**??**) is not accidental. An ellipse is merely a stretched circle, and elliptical anisotropy is merely stretched isotropy. The Fourier similarity theorem tells us that a "stretch" (linear transformation) in the (x, z) domain corresponds to an "inverse stretch" (another linear transformation) in the (k_x, k_z) domain. Equations (**??**) and (**??**) look the same because they are both equations for stretched circles, i.e., ellipses.

Figure **??** shows how elliptical anisotropy can be used as a first-order paraxial approximation in both the group and phase domains. Although the group-velocity and phase-slowness representations of this transversely isotropic medium look very different, the first-order paraxial approximation is elliptical in both domains. (The TI medium is Greenhorn Shale from Jones and Wang (1981).)

This paraxial ellipse is also of considerable practical importance. The familiar isotropic moveout equation is

$$T(x)^2 = T(0)^2 + (x/V_{\mathrm{NMO}})^2, \tag{7}$$

$$T(0)^2 = h^2/V_z^2, \tag{8}$$

where $T(x)$ is the traveltime at offset x, V_{NMO} is the moveout velocity, h is the vertical layer thickness, and V_z is the vertical velocity. (This is one-way traveltime.) In the isotropic case we only have one velocity, and set $V_{\mathrm{NMO}} = V_z = V_{\mathrm{iso}}$. Converting from isotropy to elliptical anisotropy with a vertical axis is as simple as replacing z with a new stretched coordinate $z' \propto z$. Because the z coordinate does not occur in equation (**??**) it remains completely unchanged in the new coordinate system. The rescaled z coordinate does trivially affect equation **??** by scaling the numerator and denominator equally, leaving $T(0)$ unchanged. (h and V_z scale with z, because they involve the same vertical length units.) The moveout equation for elliptical anisotropy thus has the same form as for isotropy; we merely have to let V_{NMO} and V_z become independent. The paraxial moveout velocity V_{NMO}, unaffected by the stretch in z, must be the horizontal velocity V_x of the paraxial ellipse. Because h and V_z can be scaled together with no change in the recorded traveltimes, the vertical scale in elliptical anisotropy evidently can't be determined from surface kinematics alone (Dellinger and Muir, 1988).

THE FIRST ANELLIPTIC APPROXIMATION

In cases such as cross-borehole tomography where true vertical and horizontal scale measurements are available, elliptical anisotropy can be useful as a "first step" away from isotropy. If only surface kinematic data are available, though, equation (**??**) is enough for doing NMO and assuming elliptical anisotropy accomplishes nothing more than making explicit the depth ambiguity that was always there. To add another velocity parameter to this isotropic (really elliptically anisotropic) case, we need to add some sort of an *an*elliptic parameter.

Note that equations (**??**), (**??**), and (**??**) share a common polynomial form:

$$f = \text{z-term}^2 + \text{x-term}^2. \tag{9}$$

We add an *an*elliptic parameter to equation (**??**) by generalizing it to

$$f = \frac{\text{z-term}^4 + (1+F)\,\text{z-term}^2\,\text{x-term}^2 + \text{x-term}^4}{\text{z-term}^2 + \text{x-term}^2}. \tag{10}$$

Varying the cross term F lets us vary the behavior away from perfect ellipticity in between the coordinate axes. If $F \equiv 1$ the numerator becomes a perfect square and equation (**??**) reverts to the original elliptic form in equation (**??**) for all propagation directions.

Why this particular generalization? This anelliptic form needs only one more parameter beyond elliptical anisotropy (or isotropy for the surface survey case), retains the properties that make elliptical anisotropy so convenient (in approximation), and makes a good approximation to exact transverse isotropy, especially in the phase-slowness domain. (See Figure 2.)

Following this template, equation (**??**) (the ray equation) becomes

$$M(\phi_r) = \frac{\Big(M_z C\Big)^2 + (1 + M_{\mathrm{NMO}}/M_x)\Big(M_z C\Big)\Big(M_x S\Big) + \Big(M_x S\Big)^2}{M_z C + M_x S}, \tag{11}$$

where $S = \sin^2(\phi_r)$, $C = \cos^2(\phi_r)$, and, as before, M indicates slowness squared ($M_x = 1/V_x^2$, $M_{\mathrm{NMO}} = 1/V_{\mathrm{NMO}}^2$, and $M_z = 1/V_z^2$). Note this equation explicity separates the true horizontal velocity V_x from the paraxial NMO velocity V_{NMO}.

We can similarly extend equation (**??**) (the dispersion relation) to

$$W(\phi_w) = \frac{\Big(W_z C\Big)^2 + (1 + W_{\mathrm{NMO}}/W_x)\Big(W_z C\Big)\Big(W_x S\Big) + \Big(W_x S\Big)^2}{W_z C + W_x S}, \tag{12}$$

where $S = \sin^2(\phi_w)$, $C = \cos^2(\phi_w)$, and as before W indicates velocity squared ($W_x = V_x^2 = 1/M_x$, $W_{\mathrm{NMO}} = V_{\mathrm{NMO}}^2 = 1/W_{\mathrm{NMO}}$, and $W_z = V_z^2 = 1/M_z$).

Note the symmetry between equations **??** and **??**; as for elliptical anisotropy, converting from the representation in one domain to the other is as simple as replacing each of the velocity parameters with its reciprocal.

Consistency

Although replacing each of the velocity parameters with its reciprocal is certainly convenient, the resulting pair of equations are unfortunately *inconsistent*: equation (**??**) implies a ray surface which does not satisfy equation (**??**), and vice versa. This is not a problem in practice. We don't require exact consistency because these equations are only meant as approximations anyway. The original elliptical forms, equations (**??**) and (**??**), are exactly consistent everywhere. Because equations (**??**) and (**??**) approximate the original elliptical equations paraxially around the z axis, our two *an*elliptic equations must also be *approximately consistent* paraxially around the z

axis. Similarly, for propagation along the x axis the two new equations are each governed by the true horizontal velocity V_x. As a result they must be exactly consistent on the x axis as well and approximately consistent nearby.

Figure 3 shows the approximations in the group-velocity and phase-slowness domains for the qP surface of Greenhorn Shale (Jones and Wang, 1981), a transversely isotropic medium. As shown by the four plots, the approximations are reasonably consistent even away from the axial "control points" for the qP surface, despite the strongly non-elliptic anisotropy. (If the first anelliptic approximation in the two domains of interest were perfectly consistent with each other, the thick dashed line would follow the same trajectory relative to the thin continuous line in both columns. As it is, at 45° the approximation in the group domain underestimates the group velocity by 1.7% and the approximation in the phase domain overestimates the group velocity by .6%.) The three axial constraints seem to be enough to tie the two approximations to the underlying TI medium, and thus to each other.

The fit is not so good for the corresponding qSV surface, shown in Figure 4. This is not surprising as there is no way a single-valued ray equation can track a triplication. The approximation can still be used in the phase-slowness domain in such cases, but there is no hope that the phase- and group-domain approximations can be very consistent if the approximation in either domain is anywhere concave. Mathematically, for there to be no concavity in either domain the "F" factor in equation (**??**) (or F_z and F_x factors in equation (**??**)) should satisfy the inequality

$$3/7 < F < 7/3. \tag{13}$$

THE SECOND ANELLIPTIC APPROXIMATION

The first anelliptic approximation has a certain lack of symmetry. There are two vertical control parameters, V_z and V_{NMO}, but only a single horizontal control parameter V_x. If our data includes sources and receivers separated both horizontally and vertically, it makes sense to use an approximation that is symmetric between x and z. We can do this by generalizing to the following template:

$$f = \frac{\text{z-term}^6 + (2+F_x)\,\text{z-term}^4\,\text{x-term}^2 + (2+F_z)\,\text{z-term}^2\,\text{x-term}^4 + \text{x-term}^6}{(\text{z-term}^2 + \text{x-term}^2)^2}. \tag{14}$$

If $F_z \equiv F_x$ equation (**??**) reduces to the first anelliptic form given by equation (**??**); if $F_z \equiv F_x \equiv 1$ equation (**??**) reduces to the original elliptical form given by equation (**??**).

Following this newer template, equation (**??**) (the ray equation) becomes

$$M(\phi_r) = \tag{15}$$

$$\frac{\left(M_zC\right)^3 + (2 + M_{x\mathrm{NMO}}/M_x)\left(M_zC\right)^2\left(M_xS\right) + (2 + M_{z\mathrm{NMO}}/M_z)\left(M_zC\right)\left(M_xS\right)^2 + \left(M_xS\right)^3}{(M_zC + M_xS)^2},$$

where $S = \sin^2(\phi_r)$, $C = \cos^2(\phi_r)$, and as before M indicates slowness squared ($M_x = 1/V_x^2$, $M_{x\mathrm{NMO}} = 1/V_{x\mathrm{NMO}}^2$, $M_{z\mathrm{NMO}} = 1/V_{z\mathrm{NMO}}^2$, and $M_z = 1/V_z^2$).

Similarly, equation (**??**) (the dispersion relation) becomes

$$W(\phi_w) = \tag{16}$$

$$\frac{\Big(W_zC\Big)^3 + (2 + W_{x\mathrm{NMO}}/W_x)\Big(W_zC\Big)^2\Big(W_xS\Big) + (2 + W_{z\mathrm{NMO}}/W_z)\Big(W_zC\Big)\Big(W_xS\Big)^2 + \Big(W_xS\Big)^3}{(W_zC + W_xS)^2},$$

where $S = \sin^2(\phi_w)$, $C = \cos^2(\phi_w)$, and as before W indicates velocity squared ($W_x = V_x^2$, $W_{x\mathrm{NMO}} = V_{x\mathrm{NMO}}^2$, $W_{z\mathrm{NMO}} = V_{z\mathrm{NMO}}^2$, and $W_z = V_z^2$).

Note that the xNMO subscript indicates moveout velocity measured for near-vertical propagation; i.e., $W_{x\mathrm{NMO}}$ is the square NMO velocity we use every day in surface-to-surface data processing. (If an x subscript seems confusing for a paraxial measurement about the vertical, remember that in an elliptic world it is a horizontal velocity that surface moveout measures.) The zNMO subscript indicates moveout velocity measured for near-horizontal propagation, such as might be found in a cross-borehole experiment.

ANELLIPTIC PARAMETERS FOR TI MEDIA

The anelliptic approximations are useful in their own right for fitting measured phase- and group-velocity surfaces, but it is also useful to know how to relate these approximations to a real anisotropic symmetry system. Because the approximations are two-dimensional, the appropriate symmetry system is transverse isotropy (TI). The equations for TI media are most easily written in terms of phase-velocity squared, suggesting we follow the notation of equation (**??**).

Let $W(\phi_w)$ be the phase-velocity squared measured at phase angle ϕ_w from the vertical, and let $S = \sin^2(\phi_w)$ and $(1 - S) = \cos^2(\phi_w)$. Express the elastic constants in units of phase-velocity squared as well, so $W_{ij} = C_{ij}/\rho$. Then we have for the TI SH wavetype

$$W(\phi_w) = W_{44}(1 - S) + W_{66}S \quad . \tag{17}$$

Equation (**??**) is linear because SH waves in TI media are exactly elliptically anisotropic; by matching coefficients in equations (**??**) and (**??**) we find the required anelliptic parameters:

$$W_{\mathrm{SH},x} = W_{\mathrm{SH},x\mathrm{NMO}} = W_{66} \quad ; \tag{18}$$

$$W_{\mathrm{SH},z} = W_{\mathrm{SH},z\mathrm{NMO}} = W_{44} \quad .$$

The TI qP-qSV wavetype is rather more complicated:

$$W(\phi_w) = \frac{1}{2}\Big((W_{33} + W_{55})(1 - S) + (W_{11} + W_{55})S$$

$$\pm\sqrt{((W_{33}-W_{55})(1-S)-(W_{11}-W_{55})S)^2+4(W_{13}+W_{55})^2(1-S)S}\ \Big), \qquad (19)$$

where the + sign is for qP, − for qSV. We can find W_z or W_x by substituting $S=0$ or $S=1$, respectively, into equation (**??**). To find $W_{x\mathrm{NMO}}$ or $W_{z\mathrm{NMO}}$, however, we first have to fit a paraxial ellipse about the z or x axes, respectively. We have already seen that elliptical anisotropy is linear in these coordinates. To find the equation for the paraxial ellipse about the z axis we therefore linearize about $S=0$, obtaining

$$W_{\text{z-paraxial}}(S) = W\Big|_{S=0} + \frac{dW}{dS}\Big|_{S=0} S \quad . \qquad (20)$$

Next, remembering that $W_{x\mathrm{NMO}}$ is the horizontal velocity for the paraxial approximation about the z axis, we set $S=1$ (horizontal propagation) in equation (**??**), obtaining

$$W_{x\mathrm{NMO}} = W\Big|_{S=0} + \frac{dW}{dS}\Big|_{S=0} \quad . \qquad (21)$$

Similarly,

$$W_{z\mathrm{NMO}} = W\Big|_{S=1} - \frac{dW}{dS}\Big|_{S=1} \quad . \qquad (22)$$

For the TI qP wavetype we obtain:

$$W_{\mathrm{P},x} = W_{11} \qquad (23)$$

$$W_{\mathrm{P},x\mathrm{NMO}} = W_{55} + \frac{(W_{13}+W_{55})^2}{W_{33}-W_{55}}$$

$$W_{\mathrm{P},z} = W_{33}$$

$$W_{\mathrm{P},z\mathrm{NMO}} = W_{55} + \frac{(W_{13}+W_{55})^2}{W_{11}-W_{55}}.$$

For the TI qSV wavetype we obtain:

$$W_{\mathrm{SV},x} = W_{\mathrm{SV},z} = W_{55} \qquad (24)$$

$$W_{\mathrm{SV},x\mathrm{NMO}} = W_{11} - \frac{(W_{13}+W_{55})^2}{W_{33}-W_{55}}$$

$$W_{\mathrm{SV},z\mathrm{NMO}} = W_{33} - \frac{(W_{13}+W_{55})^2}{W_{11}-W_{55}}.$$

Approximating TI dispersion relations

Now that we have related transverse isotropy to our anelliptic approximations we show an example of using these approximations to do Fourier-domain migration. (For a real data example using the first anelliptic approximation for Fourier-domain imaging see Gonzalez et al. (1991).)

Figure 5 shows the result of diffracting, then migrating, a model consisting of five point diffractors arranged in a diamond. Scalar Stolt diffraction and time-migration with an anisotropic dispersion relation were used to do the calculations; the dispersion relation is for the qP (left column) and qSV (right column) wavetypes of the TI medium Greenhorn Shale (Jones and Wang, 1981). The result is not perfect because of the offset truncation of the modeled hyperboloids. The right-hand depth scale on the time migrations uses the true vertical velocities to convert from time to depth.

Figure 6 shows the results of time-migrating the modeled qP and qSV hyperboloids from Figure 5 using isotropic dispersion relations. The velocities used are the usual ones determined from moveout, $W_{x\mathrm{NMO}}$. Even for the less extreme qP wavetype (left column) the isotropic migration manages to properly focus only the nearest-offset part of the hyperboloids.

There is a more subtle error in the isotropic migration as well. The true vertical scale cannot be determined from surface data alone, but isotropy makes no allowance for this ambiguity and just assumes the vertical velocity is the same as the moveout velocity. As a result, the right-hand depth scale for the qSV waves is off by nearly a factor of two. (The depth scale for the qP waves also mislocates the diffractors, but only by about 5%.)

Figure 7 shows the results of using the first anelliptic approximation to time-migrate the modeled TI hyperboloids in Figure 5. The approximation is meant to be fit using only surface data, so as in Figure 6 the unknowable vertical velocity has been set equal to the moveout velocity. Unlike the isotropic approximation in Figure 6, though, the first anelliptic approximation does a good job of focusing the hyperboloids, especially for the less extreme qP wavetype.

In Figure 8 we improve the fit to the second anelliptic approximation. The main difference is that the qSV diffractors are better focused. Since fitting the second anelliptic approximation assumes access to horizontal as well as vertical data, the correct vertical velocity has been used and the right-hand depth scales are correct.

NMO

The first-anelliptic example in Figure 7 would have been more realistic if we had simply omitted the depth scale entirely. Mathematically this means rewriting equation (**??**) explicitly in terms of vertical traveltime, finding the first-anelliptic equivalent of equation (**??**). To do this, replace $M_z C$ by $T(0)^2$, S by x^2, and M_x by $F_W M_{\mathrm{NMO}}$, obtaining the first-anelliptic processing-NMO equation:

$$T(x)^2 = \frac{T(0)^4 + (F_W + 1)T(0)^2 M_{\mathrm{NMO}} x^2 + F_W^2 M_{\mathrm{NMO}}^2 x^4}{T(0)^2 + F_W M_{\mathrm{NMO}} x^2}. \tag{25}$$

The anellipticity parameter F_W controls the deviation from normal moveout; if $F_W \equiv 1$ the moveout is exactly hyperbolic. Note that one should not try to define

$F_W = M_x/M_{\rm NMO}$ using a measured value for M_x. F_W and $M_{\rm NMO}$ are both vertical paraxial measurements; there is no reason to suppose they should be strictly related to M_x, a horizontal measurement.

Dix's equation

Equation (**??**) tells us how to process a single layer. Suppose we have a stack of flat layers, each with moveout obeying equation (**??**); can we replace the stack of anelliptic layers with a single homogeneous anelliptic layer that also obeys equation (**??**), at least in paraxial approximation?

The standard method of attacking this question begins by finding a power series $T(x)$ for the stack traveltime in terms of the various layer parameters; the required relations can then be found by examining the coefficients of the first few powers of x. This method entails rather lengthy and tedious algebra (see for example Sena (1991)).

We begin by writing the traveltime through each *layer* T_i as a power series in ray parameter p. The traveltimes T_i just sum along each ray, so adding all the layer power series together gives a power series for the stack as a whole. More importantly, because the ray parameter p is constant along each ray (and thus is the same p in each layer), we can add all the layer power series together just by summing all the coefficients on each distinct power of p. The result is a power series $T(p)$ for the stack as a whole in the same form as the ones for each layer.

This equivalence of form is significant. The set of coefficients on powers of p in fact comprises an alternative way of parameterizing the moveout properties of a layer. We will call this representation the "Dix group". For the standard "isotropic" Dix approximation the Dix-group elements are the familiar zero-offset traveltime $T(0)$ and RMS velocity $T(0)V_{x\rm NMO}^2$. (For a more complete treatment of the theory of Abelian group representations for layered media and an application to the low-frequency elastic case see Schoenberg and Muir (1989).)

Group representations are useful because in the group domain two or more layers can be combined into a single homogeneous equivalent simply by adding corresponding elements. More importantly, in the group domain individual layers can be *removed* from a stack by subtraction. In practice these manipulations require transforming from the standard layer parameters to the group representation, adding and subtracting as needed, and then transforming back. This is exactly what we are doing when we perform interval-velocity analysis using Dix's equations: The layer parameters are vertical traveltime and interval velocity, and the group parameters are total vertical traveltime and RMS velocity.

The fundamental power series $T(p)$ can be conveniently calculated in the following way:

1. Expand $T(x)$ as a power series about $x = 0$.

2. Differentiate the power series $T(x)$ with respect to x, giving the power series $p(x) = \frac{dT}{dx}(x)$.

3. Revert this power series for $p(x)$ to obtain a series for $x(p)$ in terms of powers of p. (See Knuth, (1981)).

4. Compose the power series for $x(p)$ with the power series for $T(x)$, obtaining a series for $T(p)$ in powers of p.

(Each step in this algorithm corresponds to a basic command in the symbolic algebra program Mathematica (TM) (Wolfram, 1988).) If we do this for equation (**??**), we obtain:

$$T(p) = T(0)\Big(1 + \frac{1}{2}(pV_{x\mathrm{NMO}})^2 + \frac{3}{8}(1 + 4F_W - 4F_W^2)(pV_{x\mathrm{NMO}})^4 + \tag{26}$$

$$\frac{5}{16}(1 + 12F_W + 12F_W^2 - 56F_W^3 + 32F_W^4)(pV_{x\mathrm{NMO}})^6 + \ldots\Big).$$

As expected, from the p^0 term we find the first Dix-group element

$$T_{\mathrm{stack}}(0) = \sum_i T_i(0) \quad , \tag{27}$$

"vertical traveltimes add". From the p^2 term we find the second Dix-group element,

$$T_{\mathrm{stack}}(0){V_{x\mathrm{NMO}}}^2_{\mathrm{stack}} = \sum_i T_i(0){V_{x\mathrm{NMO}}}_i^2 \quad , \tag{28}$$

which is just Dix's familiar RMS-velocity equation.

In the standard Dix case the first two powers of p exhaust the available free layer parameters and the coefficient on p^4 cannot also be made consistent. (If *all* powers of p could be made consistent, the equivalence would be exact, not paraxial.) For the first anelliptic approximation the F_W parameter yet remains, so the p^4 coefficient in equation (**??**) defines a new *anelliptic* Dix-group element that extends the original pair:

$$T(0)V_{x\mathrm{NMO}}^4(1 + 4F_W - 4{F_W}^2) \quad . \tag{29}$$

This third Dix-group element does not invalidate Dix's original equations for $T(0)$ and $V_{x\mathrm{NMO}}$ in any way. In fact, before we can use equation (**??**) to find the stack value of F_W we first have to find the stack values of $T(0)$ and $T(0)V_{x\mathrm{NMO}}^2$ using equations (**??**) and (**??**).

Note if $F_W \neq 1$ the calculated stack moveout is *non*hyperbolic. Although nonhyperbolic moveout can be caused by intrinsically anisotropic layers within the stack, more generally it will also be due to ray bending at layer boundaries not accounted for by Dix's equations (i.e., the first two Dix-group elements). Although the anelliptic Dix-group element in equation (**??**) also cannot account for ray bending exactly, it does guarantee to do a better job (at least paraxially) than the first two Dix-group elements alone could.

Figure 9 shows an example. On the left is synthetic surface data computed for a simple layered model. All of the layers are isotropic (layer $F_W \equiv 1$) except for the layer extending from traveltime 175 to 225, which is strongly anisotropic (layer $F_W = .6$). Mathematically, identifying the the zero-offset traveltime of each event corresponds to applying the first Dix-group element. In the center we have performed standard hyperbolic moveout; mathematically, this corresponds to applying the first two Dix-group elements. As expected, at near offset all the reflection events have been flattened. On the right the third Dix-group element has also been applied; as a result all but one of the events are now flattened out to a higher offset. (The topmost event is unchanged because it was already exactly hyperbolic.)

Note in particular that the event at traveltime 175 is slightly improved, even though the slight nonhyperbolic moveout on that event (stack $F_W = .96$) is entirely due to the isotropic layering above it. The improvement is most marked for the reflector at the bottom of the anisotropic layer, at time 225. This event is strongly nonhyperbolic (stack $F_W = .72$), mostly because of intrinsic anisotropy.

What if we are given a surface dataset without any layer parameters; is there any way to distinguish nonhyperbolic moveout caused by intrisic anisotropy from that due to layering? For one, elliptically anisotropic layering can only cause F_W to decrease from unity, so if we do find a stack hyperboloid displaying $F_W > 1$ we can suspect intrinsic anisotropy as a possible cause. More generally, after we have done conventional interval velocity analysis using Dix-group elements one and two, there is no theoretical reason we should not be able to continue the process to find interval anelliptic parameters using Dix-group element three.

Of course, our three-term Dix-group analysis is still paraxial. If the hyperboloids being fit in the gather extend out too far in offset the higher-order terms in equation (**??**) cannot be ignored. Our anelliptic "velocity" analysis is guaranteed to work out to at least as large an offset as the standard Dix approximation, so at worst we lose nothing. (If the goal is simply better stacking and imaging there is no problem in any case.)

It is worth explicitly noting that the crucial factor in equation (**??**) is a *dimensionless* quantity, ray parameter times NMO velocity ($pV_{x\mathrm{NMO}}$). Preliminary synthetic tests indicate that the paraxial approximation is generally good out to at least $pV_{x\mathrm{NMO}} \approx .5$. (The moved-out events in Figure 9 have been cut off at $pV_{x\mathrm{NMO}} = .75$, well past the point at which the two-term power series begins to fail.) Should the paraxial approximation become a problem the standard fallback to exact raytracing is equally applicable to the anelliptic case.

We have not presented a "processing NMO equation" analogous to equation (**??**) for the second anelliptic approximation because there is no reason to do so. Instead of attempting to use the meaningless (for NMO geometry) "z_{NMO}" term paraxially, it would be better to find a different approximation more natural for near-vertical propagation.

Anelliptic NMO for TI media

Equations (**??**) and (**??**) relate TI elastic constants and NMO velocities, but to perform and interpret anelliptic moveout we will also need equations relating TI elastic constants and the anelliptic parameter F_W. For the SH wavetype $F_W \equiv 1$, because SH waves in TI media are exactly elliptically anisotropic. For the qP and qSV wavetypes we have to begin by deriving a general form. We proceed by constructing a power series for $T(p)$ in the same form as equation (**??**). That done, we can just match coefficients on like powers of p.

Start with the equation for the traveltime through a layer,

$$T(\phi_r) = \frac{h}{V_r(\phi_r)\cos(\phi_r)}, \tag{30}$$

where $T(\phi_r)$ is the traveltime through a layer of thickness h, $V_r(\phi_r)$ is the *group* velocity, and ϕ_r is the *group* propagation direction. After some algebra to convert from group to phase variables equation (**??**) becomes

$$T(S) = \frac{h}{\sqrt{W(S)}\sqrt{1-S}\Big(1 - SW'(S)/W(S)\Big)}, \tag{31}$$

where T is the *group* traveltime through a layer of thickness h, S is the square of the sine of the *phase* propagation direction, $W(S)$ gives the squared *phase* velocity as a function of S, and W' is the first derivative of W. (See equation (**??**) for $W(S)$ in the TI case.)

The ray parameter p can also be expressed as a simple function of S and $W(S)$:

$$p^2(S) = \frac{S}{W(S)}. \tag{32}$$

Expand this equation and equation (**??**) to get power series for $p^2(S)$ and $T(S)$, respectively.

Finally, we revert $p^2(S)$ and composite the resulting series $S(p^2)$ with the series $T(S)$, obtaining

$$T(p) = T(0)\Big(1p^0 + \frac{1}{2}W_{x\mathrm{NMO}}p^2 + \frac{3}{8}\Big(W_{x\mathrm{NMO}}^2 + 2W(0)W''(0)\Big)p^4 + \ldots\Big). \tag{33}$$

We have used the identity $W(0) + W'(0) = W_{x\mathrm{NMO}}$ (refer back to equation (**??**)) to emphasize the similarity between equations (**??**) and (**??**). The term $W(0)W''(0)$ is also invariant under changes of vertical scale (as we knew it must be from the symmetry of the problem).

Equating the p^4 terms in equations (**??**) and (**??**) we obtain the desired equation for the paraxial value of F_W:

$$F_W = \frac{1}{2}\left(1 + \sqrt{2 - \frac{W_{x\mathrm{NMO}}^2 + 2W(0)W''(0)}{W_{x\mathrm{NMO}}^2}}\right) \quad . \tag{34}$$

Although we could substitute equation (**??**) into equation (**??**) directly, the resulting hash of elastic constants is best avoided. The answer is much simpler in terms of $\delta_{33} = C_{33} - C_{55}$, $\delta_{11} = C_{11} - C_{55}$, and $\chi = C_{13} + C_{55}$. Using these variables, for the TI qP mode we obtain

$$W_{x\mathrm{NMO}}^2 + 2W(0)W''(0) = \tag{35}$$

$$W_{55}^2 + \chi^2 \frac{-3\chi^2\delta_{33} + 4\delta_{11}\delta_{33}^2 - 4\chi^2 W_{55} + 4\delta_{11}\delta_{33}W_{55} + 2\delta_{33}^2 W_{55}}{\delta_{33}^3},$$

and for the qSV mode

$$W_{x\mathrm{NMO}}^2 + 2W(0)W''(0) = \tag{36}$$

$$W_{55}^2 + (\chi^2 - \delta_{11}\delta_{33}) \frac{\chi^2\delta_{33} - \delta_{11}\delta_{33}^2 + 4\chi^2 W_{55} - 2\delta_{33}^2 W_{55}}{\delta_{33}^3}.$$

These equations and equation (**??**) allow us to find the layer F_W for a given set of TI elastic constants. (Note these equations are not weak-anisotropy approximations.) Unfortunately, given a layer F_W (perhaps found by anelliptic three-term "velocity" analysis) these equations do not appear to put simple constraints on the corresponding layer elastic constants.

CONCLUSIONS

We have presented a set of three nested scalar anisotropic approximations; beyond isotropy there is standard elliptical anisotropy, then the first anelliptic approximation, and then the second anelliptic approximation. Each subsequent approximation adds one more velocity parameter and includes the previous as a special case.

There are two velocity parameters for elliptical anisotropy, horizontal and vertical. The first anelliptic approximation adds to these a third, horizontal NMO velocity (the standard moveout velocity recorded by surface surveys). The second anelliptic approximation adds a fourth, vertical NMO velocity (the moveout velocity measured in a cross-borehole survey).

All members of the family share elliptical anisotropy's convenient unity of form and ease of transformation between the group-velocity and phase-slowness domains. Although the anelliptic representations in the two domains are not exactly consistent, the disparity should be of about the same magnitude as the mismatch between the anelliptic approximation and the TI wave mode being approximated. Trouble can occur for extremely anisotropic media because the single-valued anelliptic approximations cannot follow triplications. In this case the approximations should not be used in the group-velocity domain.

The approximations can be successfully used in the phase-slowness domain even in such extreme cases, as demonstrated by the qSV examples in Figures 7 and 8. These results suggest that the anelliptic approximations should find wide use replacing TI dispersion relations in modeling, migration, and inversion programs.

There are several reasons to use the anelliptic approximations instead of exact transverse isotropy. For one, there are fewer parameters to find values for, because the equations are more simple. Most importantly, the anelliptic velocity parameters should be more natural and robust, because the velocity parameters used in the anelliptic approximations (horizontal and vertical true velocities and moveout velocities) are the most important ones in the natural independent coordinate sets for the sorts of surface-to-surface and cross-borehole geometries favored in geophysics.

We have derived the theoretical basis for a first-anelliptic extension of Dix's equations. How well our anelliptic analysis can work in practice remains to be determined.

ACKNOWLEDGMENTS

Thanks to Doug Jones for letting Joe use his copy of Mathematica on his workstation in Illinois for the two months that the only Sun licensed to run Mathematica at SOEST was dead. This is a University of Hawai'i at Mānoa School of Ocean and Earth Science and Technology contribution; it has been submitted to the Journal of Seismic Exploration.

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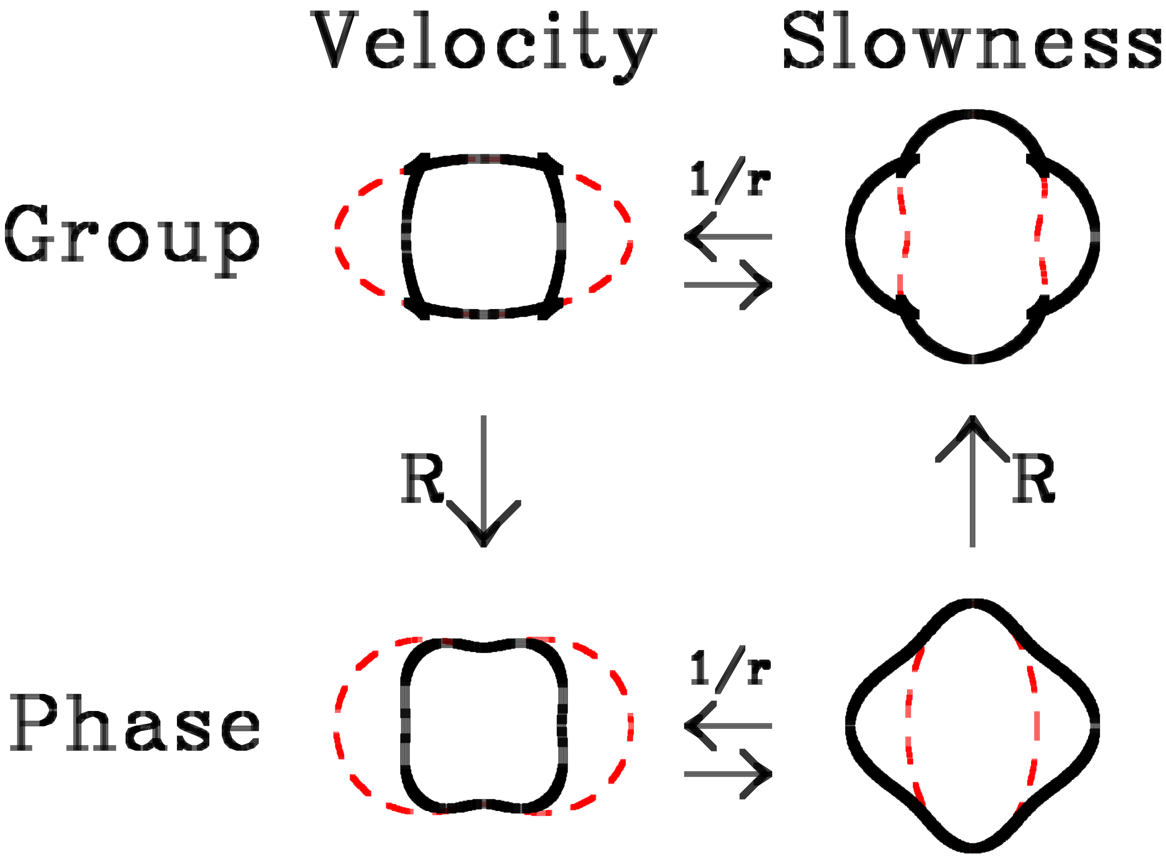

Figure 1: Plots of group velocity (impulse response), group slowness, phase velocity, and phase slowness (dispersion relation) for the qSV mode of Greenhorn Shale (thick solid line) and the elliptically anisotropic vertical-paraxial approximation (thin dashed line). The four plots are connected by only two transformations, here labeled "$1/r$" and "R". The transformation "$1/r$" (invert the radial coordinate) is its own inverse, as is the combined transformation "$R \circ 1/r$" that goes from group velocity to phase slowness (and vice versa). The combined transformation also maps ellipses onto ellipses. (Garmany, 1989) and (Dellinger, 1991) $\boxed{\text{martin2-Group-four}}$ [NR]

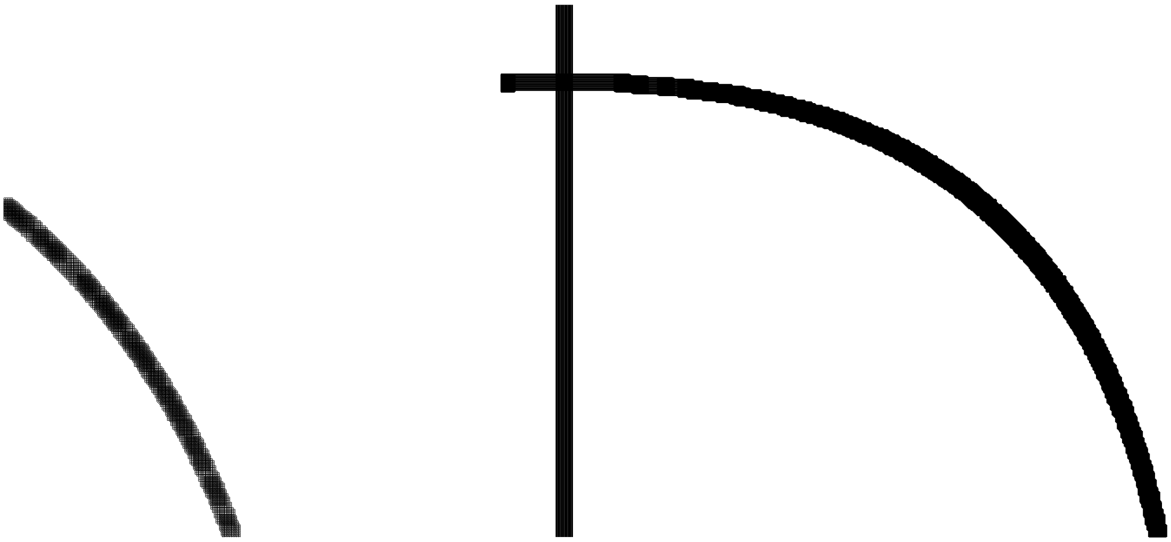

Figure 2: The curves that result for three different values of the F factor in equation (10); the asterisks mark the origin. On the left, $F = 3/7$; in the middle $F = 1$ and the curve is an ellipse; on the right $F = 7/3$. $\boxed{\text{martin2-Sample}}$ [NR]

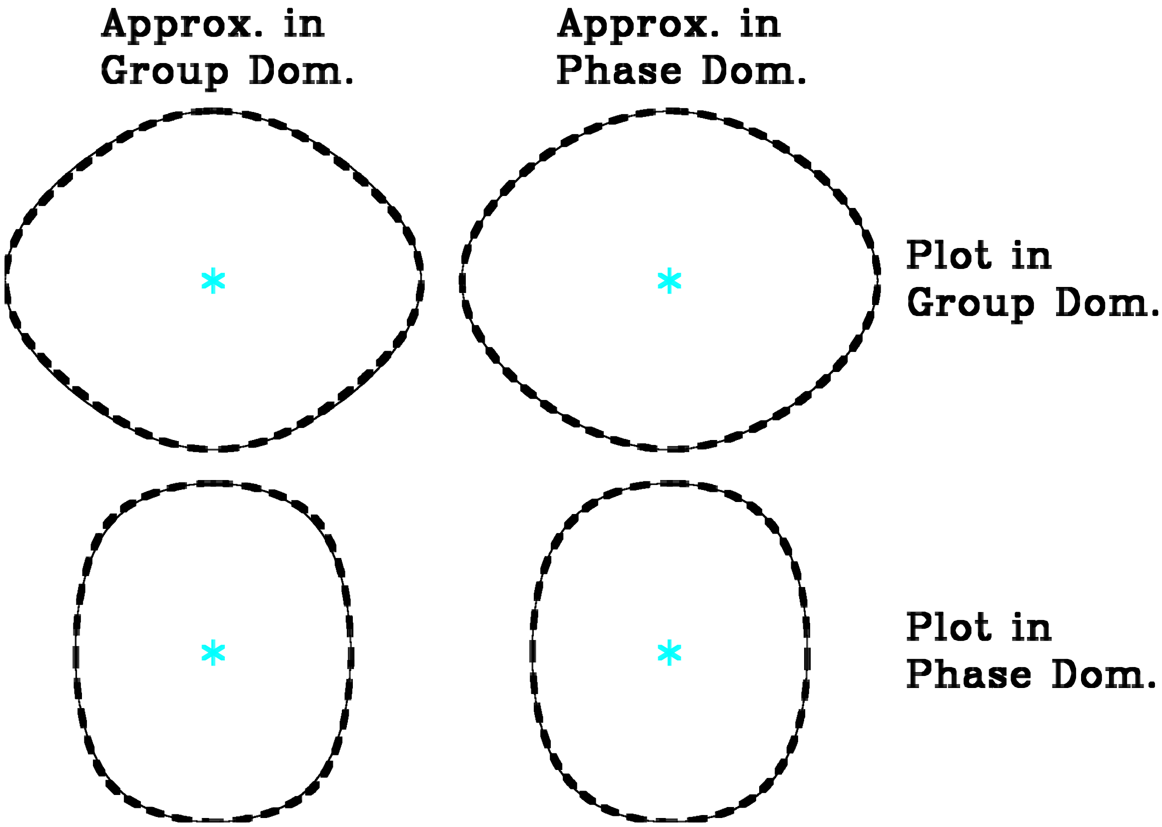

Figure 3: Group-velocity and phase-slowness plots of the qP surface of Greenhorn shale (thin solid line) and the corresponding first anelliptic approximation (thick dashed line). On the left the approximation is made in the group-velocity domain; on the right the approximation is made in the phase-slowness domain. The approximations are consistent to within about two percent. martin2-Compare.s.1 [NR]

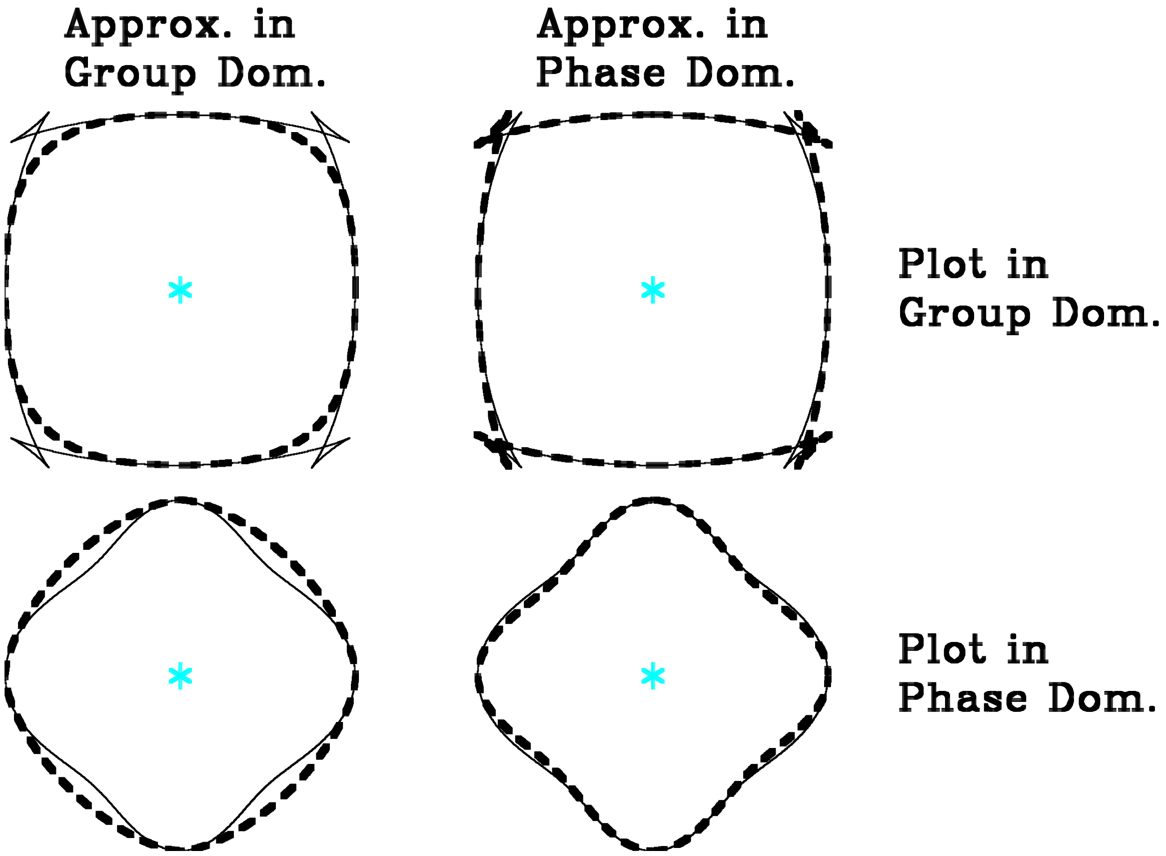

Figure 4: As in Figure 3, but for the considerably more anisotropic qSV surface. The approximations in the two domains are not consistent around the triplication. martin2-Compare.s.-1 [NR]

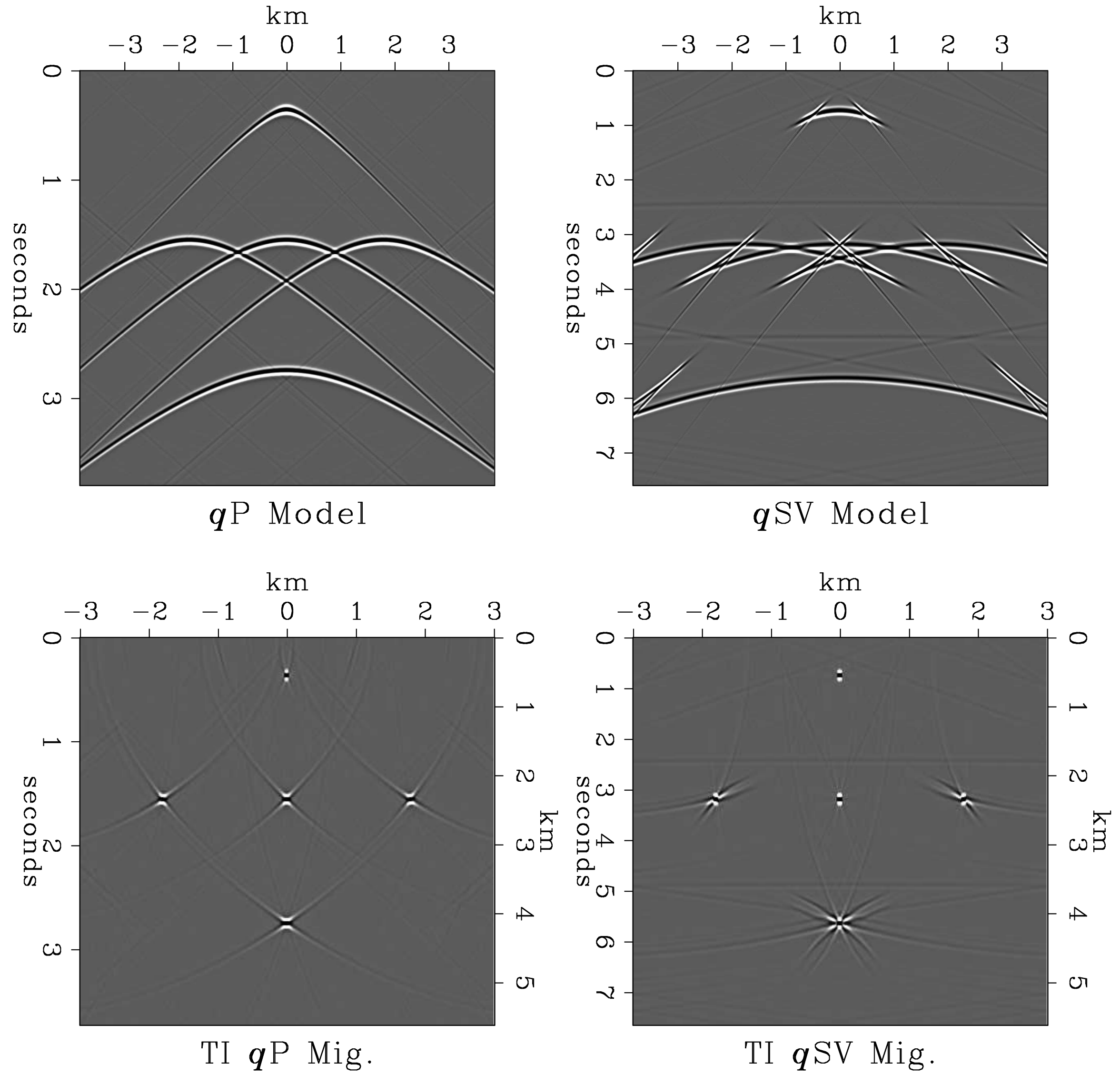

Figure 5: Scalar Stolt modeling and migration for qP and qSV waves in a TI medium. ($W_{11} = 13.72$, $W_{13} = 4.28$, $W_{33} = 9.08$, $W_{44} = 2.16$, and $W_{66} = 4.24$, all in km/s.) Top: qP and qSV models resulting from four point reflectors arranged in a diamond-shaped lattice. Bottom: Exact Stolt migrations of the modeled data. The hyperboloids do not completely collapse back to points because of the truncations at the model edges. martin2-perfect [NR]

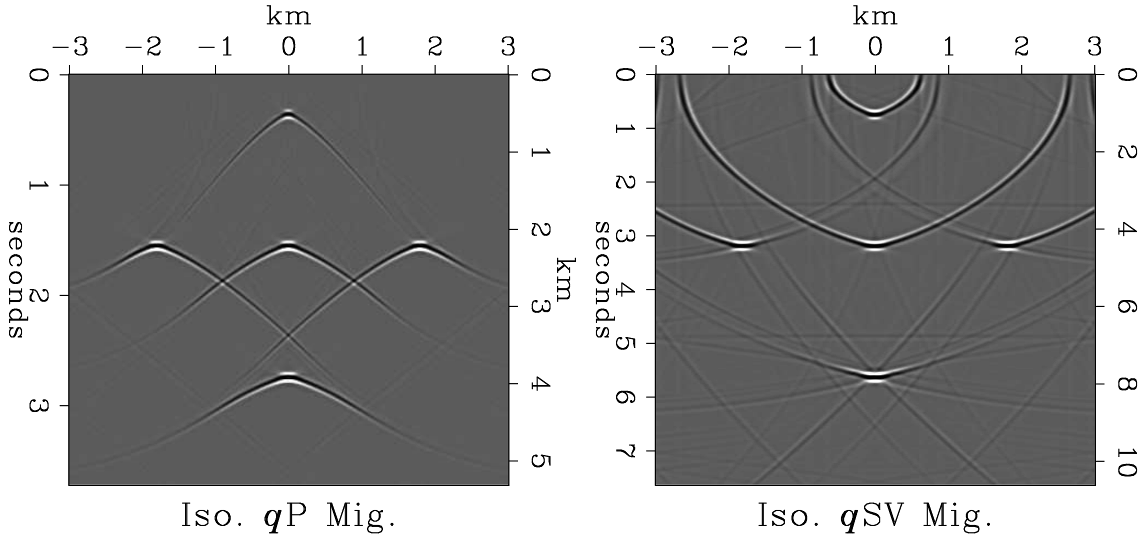

Figure 6: Approximate isotropic migrations of the exact TI models in Figure 5. The P migration velocity (left) was $V_{\mathrm{P},x\mathrm{NMO}} = 2.8554$ km/s; the S migration velocity (right) was $V_{\mathrm{SV},x\mathrm{NMO}} = 2.780$ km/s. martin2-approxI [NR]

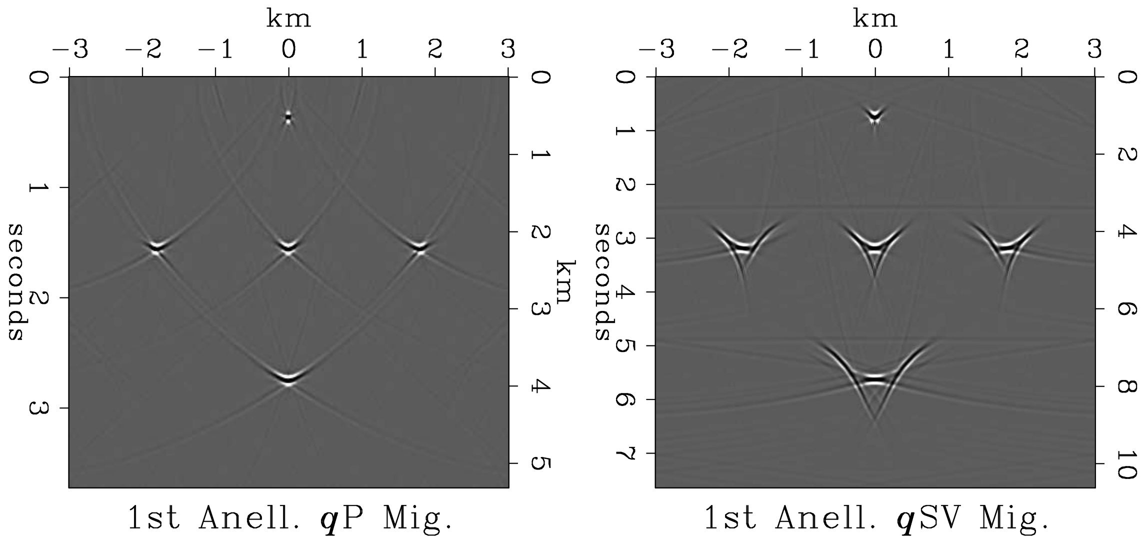

Figure 7: Approximate migrations of the exact TI models in Figure 5 using the first anelliptic approximation. For the qP waves, the migration used $V_{\mathrm{P},z} \equiv V_{\mathrm{P},x\mathrm{NMO}} = 2.8554 km/s$, and $V_{\mathrm{P},x} = 3.7041$ km/s. For the qSV waves, the migration used $V_{\mathrm{SV},z} \equiv V_{\mathrm{SV},x\mathrm{NMO}} = 2.7797$ km/s, and $V_{\mathrm{SV},x} = 1.4697$ km/s. (The unconstrained vertical velocities are set equal to the moveout velocities to conform to the usual practice; the vertical scale thus chosen has no effect on the time migration.) martin2-approxA1 [NR]

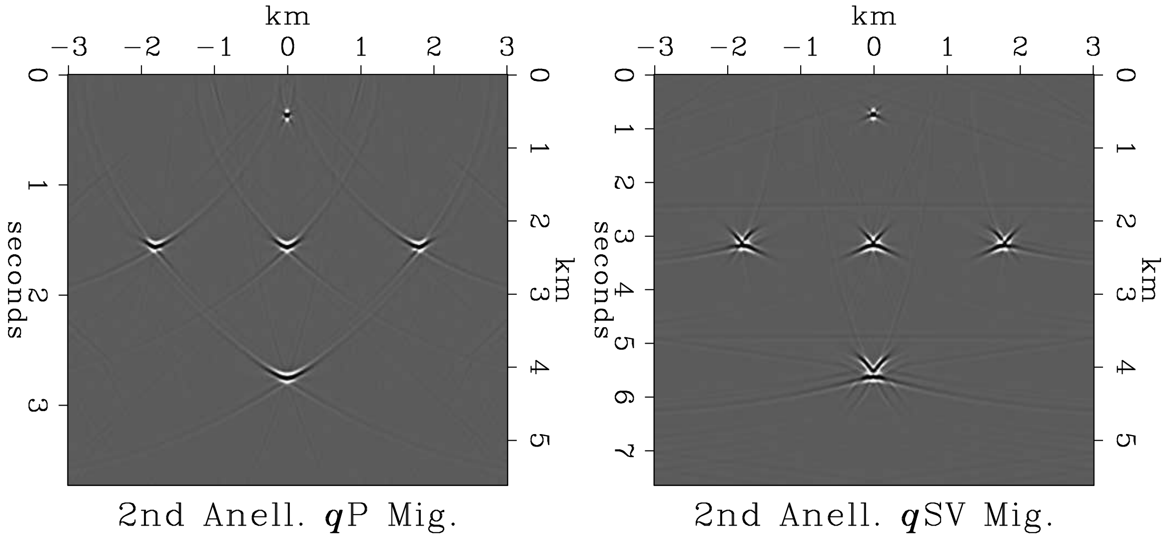

Figure 8: Approximate migrations of the exact TI models in Figure 5 using the second anelliptic approximation. For the qP waves, the migration used $V_{\mathrm{P},x\mathrm{NMO}} = 2.8554$ km/s, $V_{\mathrm{P},z} = 3.0133$ km/s, $V_{\mathrm{P},x} = 3.7041$ km/s, and $V_{\mathrm{P},z\mathrm{NMO}} = 2.3974$ km/s. For the qSV waves, the migration used $V_{\mathrm{SV},x\mathrm{NMO}} = 2.7797$ km/s, $V_{\mathrm{SV},z} \equiv V_{\mathrm{SV},x} = 1.4697$ km/s (from the symmetry of TI), and $V_{\mathrm{SV},z\mathrm{NMO}} = 2.3436$ km/s. In this migration the vertical scale is assumed to be known. martin2-approxA2 [NR]

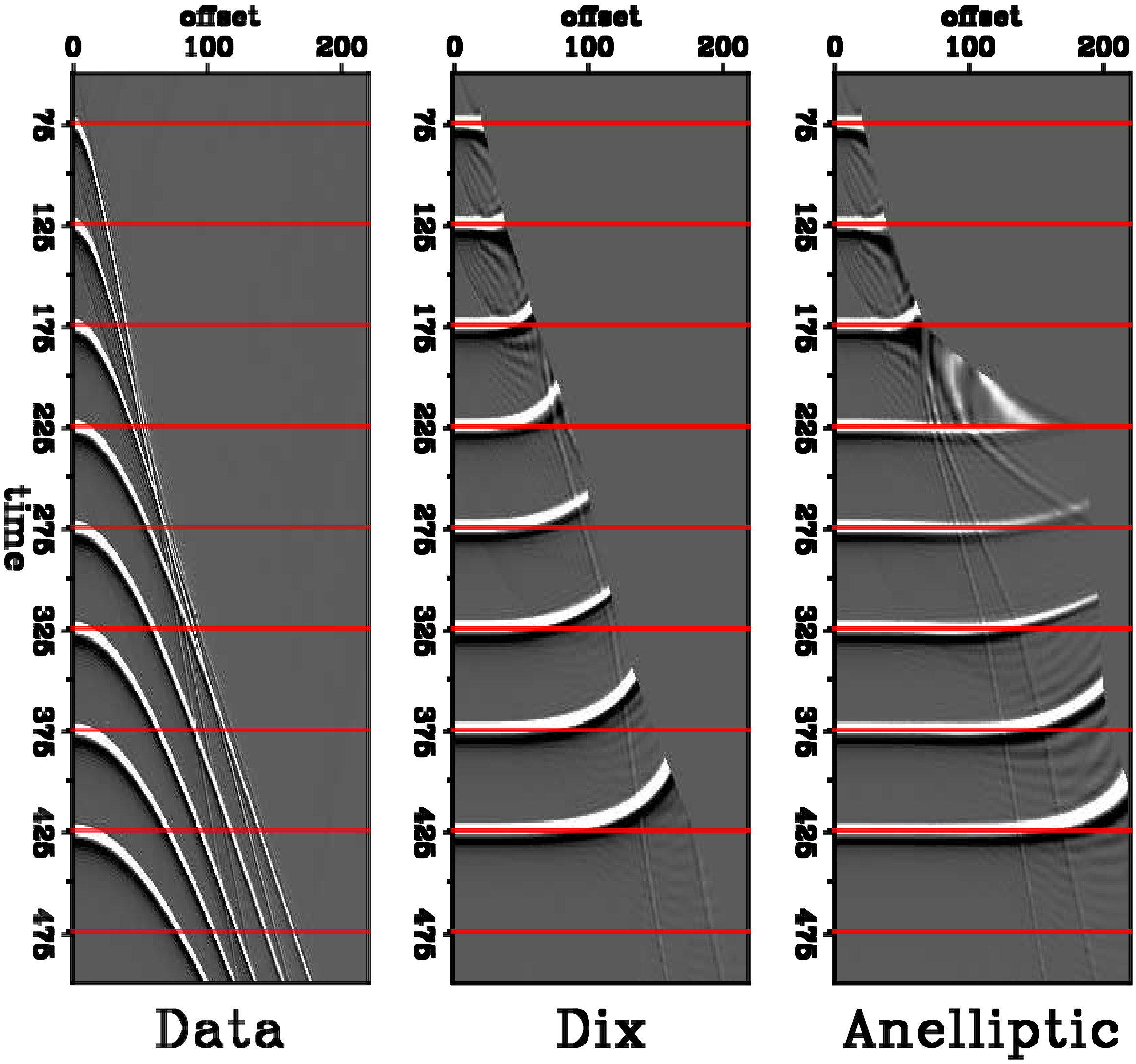

Figure 9: Anelliptic moveout removal. The model consists of 8 layers and a halfspace, the top layer 75 time-units thick and all others 50 time-units thick. The layer moveout velocities from top to bottom are .25, .3, .4, .45, .3, .3, .45, and .5. (Units are arbitrary.) All layers but the one from time 175 to 225 are isotropic. The single anisotropic layer, fourth down, corresponds to the qSV mode for elastic parameters $W_{11} = .81$, $W_{13} = .3645$, $W_{33} = .8505$, and $W_{55} = .243$. (The qSV moveout velocity is .45.) martin2-NMOfig [NR]