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The inertia tensor in elastic wave modeling

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ABSTRACT

The homogeneous equivalent to a fine-layered, heterogeneous elastic medium may need a frequency-dependent, anisotropic inertia tensor if it is to be dynamically equivalent in a useful sense. The elastic and inertial matrices that make up this equation would likely not commute, and there would be no Christoffel-like equation. This difficulty is resolved by a local transformation of the field variable from displacement to an energy form. This inertia tensor is shown, by analogy, to be physically reasonable.

BACKGROUND

This Short Note is part of a continuing effort to extend equivalent medium theory from the quasi-static and kinematic to the dynamic. In other words to extend its area of practical application. Backus (1962) taught us how to add fine-layered isotropic and transverse-isotropic materials, and Schoenberg & Muir (1989) extended this result to arbitrary anisotropy and fractures, and introduced a group theory and calculus to systematize the process. But however useful these results are, they do not speak to the problem of wavelet broadening, which was addressed so eloquently by O'Doherty & Anstey (1971).

There is a general understanding by workers in this area that a useful notion is to model the loss of energy from the coherent field to the scattered field (which does not place at 0 Hz) by replacing the elastic wave equation in Helmholtz form,

$$\nabla^T C \nabla u + \omega^2 \rho u = 0, \tag{1}$$

by a standard form of the visco-elastic equation:

$$\nabla^T C(i\omega) \nabla u + \omega^2 \rho u = 0. \tag{2}$$

That is, the elastic matrix (stiffness tensor) C is replaced by a frequency-dependent form, $C(i\omega)$.

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Further metamorphosis of the elastic wave equation

I believe it is useful to extend the visco-elastic equation in two directions:

- Replace the scalar mass density, ρ, with a 3-by-3 matrix representation, R, of a (generally anisotropic) 2nd order inertia tensor.
- Allow frequency-dependence in the elements of the matrix R.

Why do I want to do this? A simple, one-dimensional answer is that velocity and impedance can only be independently controlled by access to both elasticity and density. This is certainly true at the frequency origin, and may also be true just away from the origin.

So, the scalar density in equation (2) is replaced by an inertia matrix, R,

$$\nabla^T C(i\omega)\nabla u + \omega^2 R u = 0, \tag{3}$$

and then this inertia matrix is made frequency dependent:

$$\nabla^T C(i\omega)\nabla u + \omega^2 R(i\omega) u = 0. \tag{4}$$

A COMMUTATION PROBLEM

So long as R is isotropic, then it will commute with $\nabla^T C \nabla$ and there will be a Christoffel-like representation. If R is not isotropic then the following trick will get around the problem. First, premultiply the equation by $R^{-1/2}$, which I define as the symmetric (but possibly complex) matrix whose eigenvalues have a positive real part, and are the inverse of the square-root of the eigenvalues of the matrix R, which, for physical reasons (causality, no energy gain), are also constrained. So equation (4), dropping the $(i\omega)$ notation, can be written as,

$$R^{-1/2}\nabla^T C \nabla R^{-1/2} R^{1/2} u + \omega^2 R^{-1/2} u = 0, \tag{5}$$

and this can be further simplified by replacing the displacement field variable, u, by an energy-like field variable, w, defined equal to $R^{1/2}u$. Our final form is now,

$$R^{-1/2}\nabla^T C \nabla R^{-1/2} w + \omega^2 \ddot{w} = 0. \tag{6}$$

This new field variable is quite similar to the energy flux variable introduced by Clint Frasier and reported in Aki & Richards (1980), although our motivations are somewhat different.

An analogy: the lively box

Is the idea of a mass density model that is direction and frequency dependent reasonable? Consider a rigid box, and inside it a weight suspended between springs and constrained so that it, the weight, can move only in the direction of the springs. To a person shaking this box in any direction orthogonal to the spring alignment it is just a simple massy object, not dependent on frequency. Shaking it in the spring direction is quite a different matter – the weight will be quite dependent on the frequency of shaking. Is this a good analogy? It has one thing in common with our model: at 0Hz it becomes isotropic. We want our dynamic model to have the same static properties as our previous model.

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