Stanford Exploration Project, Report 75, November 17, 1997, pages 379–394

Estimation of elastic specular reflectivity by Kirchhoff prestack depth migration: WKBJ least-squares theory

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keywords: *elastic, least squares, reflectivity, displacement, ray tracing*

ABSTRACT

Estimation of elastic specular reflectivity from seismic reflection data is important in that it may provide crucial information about subsurface physical properties related to lithology, rock physics, pore-fluid content, and physical states. I derive a generalized least-squares solution for elastic reflectivity in terms of local seismic wavefield displacement vectors. Solutions for the local wavefields are derived as representation integrals over surface source and receiver displacement fields, weighted by a WKBJ approximate Green's tensor. The resulting integral solution is a vector analogy of the Kirchhoff-Rayleigh-Sommerfeld integrals for scalar waves. In particular, I give explicit formulas for the least-squares evaluation of the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}_1$ elastic specular reflection coefficients.

INTRODUCTION

Obtaining multiparameter elastic subsurface images from seismic reflection data is a crucial step toward seismic reservoir characterization. High resolution maps of elastic properties may be indicative of lithology, subsurface physical states (e.g., pressure and temperature), and pore-fluid content. Information about such physical material properties may be useful in hydrocarbon exploration (including direct detection), reservoir management (infill drilling and field extension), and monitoring time-variable reservoir production and EOR processes (Nur, 1989).

Seismic reflectivity is intrinsically related to subsurface physical material properties by their elastic impedance contrasts. Estimation of elastic specular reflection coefficients can therefore be viewed as a preprocessing step prior to elastic parameter inversion. I derive elastic vector integral solutions for the nine-component elastic reflectivity matrix. This approach is somewhat similar in spirit to that of Parsons (1986), which is based on the work of Beylkin (1985) and Bleistein (1987), except

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that I consider elastodynamic vector wave propagation as opposed to acoustics, and find a least-squares solution as opposed to a "direct" integral inversion.

In a related work, Tarantola (1986) proposed a nonlinear strategy to perform full waveform inversion of seismic reflection data directly for elastic parameters, essentially bypassing the reflectivity estimation step. This approach was implemented by Crase et al. (1990) on a marine data set, and was *extremely* computer intensive. Furthermore, because this one-step approach offers no intermediate processed steps to check for quality control, their results may be somewhat ambiguous in terms of the reliability of the elastic impedance results, robustness of multiple suppression, and sensitivity to the initial background velocity model.

Beydoun and Mendes (1989) greatly reduced the magnitude of complexity of the full nonlinear inverse problem by assuming a linear relation between elastic parameter perturbations and seismic reflection data under the Born approximation. Although their method is much more tractable than that of Tarantola et al., it is not clear that the Born approximation is valid for seismic reflection data, in which impedance contrasts may be large and nonlinear scatterer-to-scatterer interaction (i.e., reflections) may be significant.

Since elastic reflectivity, and not elastic parameters themselves, is linearly related to seismic reflection data amplitudes, it is natural to pose the inverse problem as first inverting for elastic specular reflection coefficients, and then subsequently performing a much-simplified nonlinear inversion of Zoeppritz-like reflectivity, in order to estimate elastic parameters. Cunha (1992) developed an elastic reverse time migration method based on finite differences to forward model and continue elastic displacement wavefields. Estimates of elastic angle-dependent reflectivity were obtained by performing the appropriate numerical correlations of attributes of the continued wavefields. Although potentially promising, possible disadvantages to Cunha's method include its computational expense, and significant potential for instability in the presence of noise due to its mandatory shot gather domain implementation.

As a result, I seek to formulate the inverse problem in the space-time constant offset domain, which offers the potential for enhanced numerical inversion stability. The integral solutions are compressed by the use of ray-valid WKBJ Green's tensors, thus reducing the number of floating point operations ultimately required to perform wavefield continuation and correlation. Finally, since the Green's function integral kernels have a physical ray-based interpretation, it may be easier to "tweak" the inversion process based on physical intuition, as opposed to black box processing by finite differences or iterative gradient techniques.

This paper proceeds as follows. I first pose the reflectivity estimation as a formal least-squares inverse problem, which I then solve by stationary point analysis. Next, the elastodynamic wave equation is introduced, and a representation integral solution is presented by use of Betti's Theorem. The representation integrals are evaluated by assuming WKBJ approximate Green's tensors, which are in turn evaluated by the ray theoretic eikonal and transport equations. The representation integrals are substituted back into the generalized least-squares solution to obtain expressions for elastic specular reflectivity. I give explicit integral expressions for the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}_1$ reflectivities as surface integrals over the recorded vector displacement wavefields.

THEORY

A least-squares formalism

Consider a point $\mathbf{x}$ in the subsurface. I define the reflectivity matrix $\mathcal{R}(\mathbf{x};\mathbf{x}_s,\mathbf{x}_r)$ at the point $\mathbf{x}$ by the following relation between the incident displacement vector field $\mathbf{u}^s(\mathbf{x},t;\mathbf{x}_s)$ due to a source at $\mathbf{x}_s$, and the reflected displacement vector field $\mathbf{u}^r(\mathbf{x},t;\mathbf{x}_r)$ due to a receiver at the position $\mathbf{x}_r$:

$$\mathbf{u}^r = \mathcal{R}\mathbf{u}^s \ , \tag{1}$$

where all quantities are evaluated locally at the point $\mathbf{x}$. The reflectivity matrix $\mathcal{R}$ is of order (3x3) and, in general, is a scattering matrix which relates $\mathbf{u}^s$ to $\mathbf{u}^r$ and contains refractions, diffractions and reflections. However, by anticipating that $\mathbf{u}^s$ will be simulated as a direct primary incident field, and $\mathbf{u}^r$ will be reconstructed from refraction-muted surface observations as a single-conversion primary reflected field, the interpretation of $\mathcal{R}$ as a generalized reflectivity matrix is justified.

Now consider the local plane-wave unit polarization vectors $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, where $\mathbf{e}_1(\mathbf{x},t)$ is the local P-wave propagation direction, $\mathbf{e}_2(\mathbf{x},t)$ the S_1 direction, and $\mathbf{e}_3(\mathbf{x},t)$ the S_2 direction. The displacement vectors $\mathbf{u}$ can be expressed in this same ray-centered coordinate system without loss of generality:

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \ ,$$

where u_1 is the P-wave displacement amplitude, u_2 is the S_1 displacement amplitude, and u_3 is the S_2 displacement amplitude. In this local ray-coordinate frame, the elements of $\mathcal{R}$ are associated with generalized non-specular elastic reflection coefficients:

$$\mathcal{R} = \begin{pmatrix} \grave{P}\acute{P} & \grave{S}_1\acute{P} & \grave{S}_2\acute{P} \\ \grave{P}\acute{S}_1 & \grave{S}_1\acute{S}_1 & \grave{S}_2\acute{S}_1 \\ \grave{P}\acute{S}_2 & \grave{S}_1\acute{S}_2 & \grave{S}_2\acute{S}_2 \end{pmatrix} \ . \tag{2}$$

The generalized non-specular reflection coefficients are related to the plane-wave specular Zoeppritz coefficients as discussed by Frazer and Sen (1985), and will be examined in more detail later in this paper. It is apparent that the generalized reflection coefficients within $\mathcal{R}$ can be "isolated" by the appropriate vector dot products. For example, the $\grave{P}\acute{P}$ coefficient can be isolated as

$$\mathbf{u}^r \cdot \mathbf{e}_1^r = \grave{P}\acute{P}\, \mathbf{u}^s \cdot \mathbf{e}_1^s \ , \tag{3}$$

where $\mathbf{e}_1^s$ is the local P-wave polarization vector of the incident displacement vector $\mathbf{u}^s$, evaluated at the subsurface point $\mathbf{x}$, and $\mathbf{e}_1^r$ is the local P-wave polarization vector of the reflected displacement vector $\mathbf{u}^r$, evaluated at the subsurface point $\mathbf{x}$. Please refer to Figure **??** to see the appropriate polarization vector geometry. Similar dot product relations can be stated for the other reflection coefficients of $\mathcal{R}$ using the appropriate vector directions $\mathbf{e}_k(\mathbf{x}, t)$. For example, the $\grave{P}\acute{S}_1$ coefficient can be isolated as

$$\mathbf{u}^r \cdot \mathbf{e}_2^r = \grave{P}\acute{S}_1\, \mathbf{u}^s \cdot \mathbf{e}_1^s \ . \tag{4}$$

I will define the reflected scalar function on the left of (4) as Φ_i^r and the incident scalar function on the right of (4) as Φ_j^s:

$$\Phi_j^s(\mathbf{x}, t; \mathbf{x}_s) = \mathbf{u}^s \cdot \mathbf{e}_j^s$$

and

$$\Phi_i^r(\mathbf{x}, t; \mathbf{x}_r) = \mathbf{u}^r \cdot \mathbf{e}_i^r \ . \tag{5}$$

If I can forward model numerous source wavefields $\mathbf{u}^s$ and I have numerous reconstructions of the reflected wavefield $\mathbf{u}^r$ from multiple surface shot gather observations, then (5) (and all of its nine converted-wave counterparts) can be rearranged into an optimization problem to estimate the isolated reflection coefficient, hereafter called $\mathcal{R}_{ij}$. The objective function J for this problem has the general l_p form

$$J = \|W_d(\Phi_i^r - \mathcal{R}_{ij}\Phi_j^s)\|_p^2 + \|W_m \mathcal{R}_{ij}\|_p^2 \tag{6}$$

where W_d and W_m are arbitrary weighting functions in data and model space respectively. The weighting functions in this form are nonconvolutional, and therefore correspond to diagonal covariance functions $\mathbf{W}^T\mathbf{W}$ in the usual least-squares theory sense. Note that adding the second term to (6) will add stability to the final least-squares solution, and will ensure that $\mathcal{R}_{ij}$ will be a minimum energy model in the l_p sense, which is appropriate for a desired spiky reflectivity solution. Choosing the l_2 norm and integrating over the implicit functional dependencies, one obtains:

$$\begin{aligned} J(\mathcal{R}_{ij}) &= \int_t \int_{\mathbf{x}} W_d^2(\mathbf{x}, t) \left\{ \Phi_i^r(\mathbf{x}, t) - \mathcal{R}_{ij}(\mathbf{x}) \Phi_j^s(\mathbf{x}, t) \right\}^2 \, d\mathbf{x}\, dt \\ &+ \int_{\mathbf{x}} W_m^2(\mathbf{x}) \mathcal{R}_{ij}^2(\mathbf{x}) \, d\mathbf{x} \ . \end{aligned} \tag{7}$$

To extremize J and thus find an "optimal" solution for $\mathcal{R}_{ij}$, I perform the following stationary point analysis. Perturb J by an arbitrary small amount $\delta\mathcal{R}_{ij}(\mathbf{x})$:

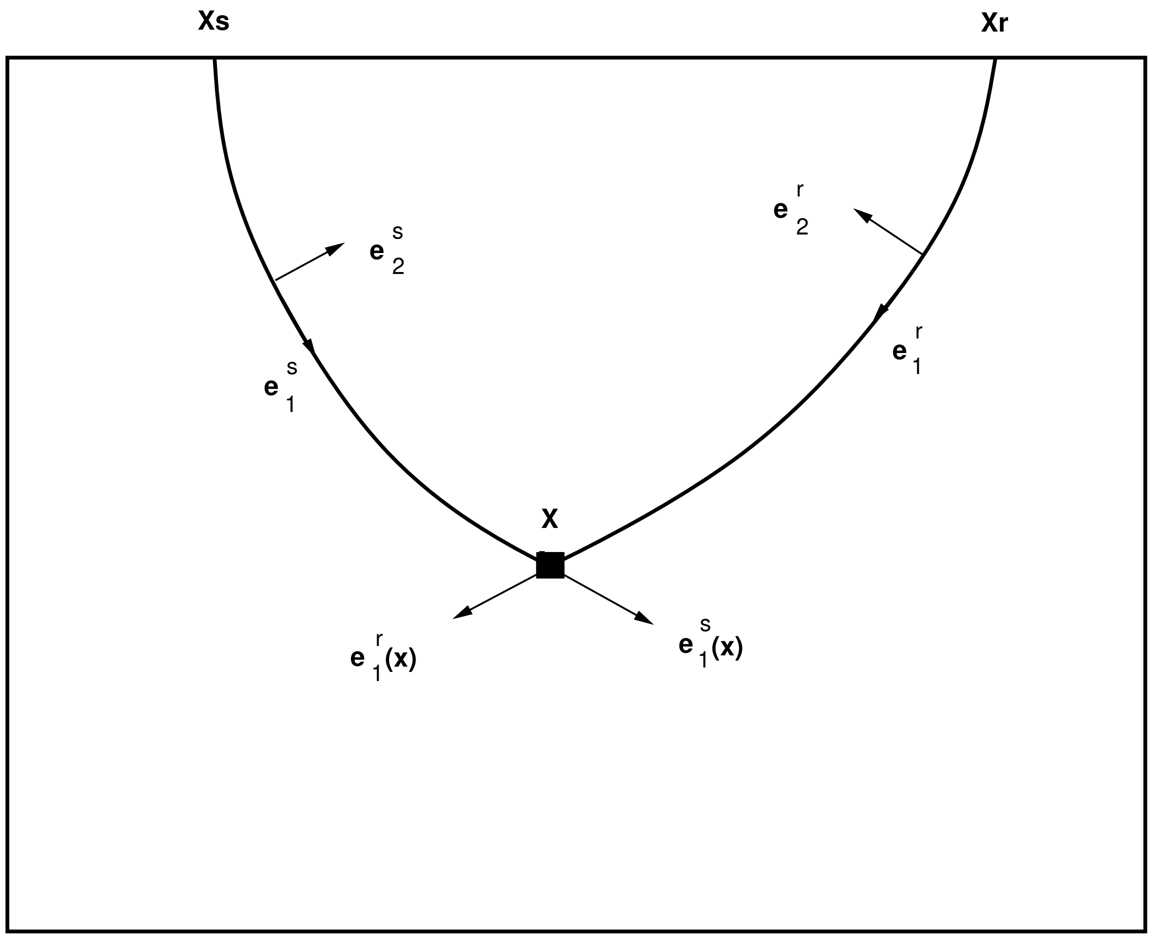

Figure 1: The elastic polarization vector geometry. The source position is at $\mathbf{x}_s$, the receiver position is at $\mathbf{x}_r$ and the subsurface reflection point is at $\mathbf{x}$. The polarization vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the local P, S_1 and S_2 plane-wave directions, and functions of $(\mathbf{x}, t)$. The $\mathbf{e}_2$ polarization vectors for S_2 waves point out of the plane of this page. david2-geometry [NR]

$$
\begin{aligned}
J(\mathcal{R}_{ij} + \delta\mathcal{R}_{ij}) \; &= \; \int_t \int_{\mathbf{x}} W_d^2 \left\{ \Phi_i^r - (\mathcal{R}_{ij} + \delta\mathcal{R}_{ij}) \Phi_j^s \right\}^2 \; d\mathbf{x} \; dt \\
&+ \; \int_{\mathbf{x}} W_m^2 (\mathcal{R}_{ij} + \delta\mathcal{R}_{ij})^2 \; d\mathbf{x} \; . \qquad (8)
\end{aligned}
$$

A stationary point of J with respect to $\mathcal{R}_{ij}$ occurs when

$$
\lim_{\delta\mathcal{R}_{ij} \to 0} \left\{ J(\mathcal{R}_{ij} + \delta\mathcal{R}_{ij}) - J(\mathcal{R}_{ij}) \right\} = 0 \; .
$$

Neglecting terms of order $\delta\mathcal{R}_{ij}^2$,

$$
\int_{\mathbf{x}} \left\{ \int_t W_d^2 (\mathcal{R}_{ij} \Phi_j^s \Phi_j^s - \Phi_i^r \Phi_j^s) \; dt + W_m^2 \mathcal{R}_{ij} \right\} \; \delta\mathcal{R}_{ij}(\mathbf{x}) \; d\mathbf{x} \; = \; 0 \; . \qquad (9)
$$

Since $\delta\mathcal{R}_{ij}(\mathbf{x})$ is *arbitrary*, then the integrand kernel $\{\cdots\}$ *must* vanish at the stationary point. This results in a least-squares reflectivity solution:

$$
\mathcal{R}_{ij}(\mathbf{x}) = \frac{\int_t W_d^2(\mathbf{x},t) \Phi_i^r(\mathbf{x},t) \Phi_j^s(\mathbf{x},t) \; dt}{\int_t W_d^2 \left(\Phi_j^s(\mathbf{x},t) \right)^2 \; dt + W_m^2(\mathbf{x})} \; . \qquad (10)
$$

Equation (10) is a weighted zero-lag correlation of the incident and reflected wavefield scalar products, normalized by the weighted minimum-threshold energy of the incident wavefield scalar product. Equation (10) is valid for all of the generalized elastic reflection coefficients of (2) by simply making the appropriate redefinitions of Φ_j^s and Φ_i^r as in (5).

The Kirchhoff elastodynamic integral solution

To evaluate the least-squares reflectivity solution (10), one requires representations for the vector wavefields $\mathbf{u}^s$ and $\mathbf{u}^r$.

A vector wavefield $\mathbf{u}$ must satisfy the elastodynamic wave equation operator $\mathcal{L}$:

$$
\mathcal{L}\mathbf{u} = \rho\ddot{\mathbf{u}} - \nabla\cdot(\mathcal{C}{:}\nabla\mathbf{u}) = \mathbf{f} \; ,
$$

subject to initial and boundary conditions:

$$
\begin{array}{lcl}
\mathbf{u}, \nabla\mathbf{u} & ; & \mathbf{x} \in S \\
[\mathbf{u}, \; \dot{\mathbf{u}}]_{t=t_o} & ; & \mathbf{x} \in V
\end{array} \; , \qquad (11)
$$

where $\rho(\mathbf{x})$ is mass density, $\ddot{\mathbf{u}}(\mathbf{x},t)$ is the second time derivative of $\mathbf{u}$, $\mathcal{C}(\mathbf{x})$ is the elastic stiffness tensor C_{ijkl}, and $\mathbf{f}(\mathbf{x},t)$ is a body force density. The surface S bounds the solution domain volume V, and the (:) symbol is a second order inner contraction. An integral solution can be obtained for $\mathbf{u}$ using Betti's Theorem (the vector equivalent of Green's Theorem for scalars), and by assuming zero initial conditions:

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \int_{t'}\int_{V'} \mathbf{f}(\mathbf{x}',t')\cdot\mathcal{G}(\mathbf{x},t;\mathbf{x}',t')\;dV'dt' \\ &+ \int_{t'}\int_{S'} \left\{\mathcal{G}^T(\mathbf{x},t;\mathbf{x}',t')\cdot\;[\mathcal{C}(\mathbf{x}'):\nabla'\mathbf{u}(\mathbf{x}',t')]\;\cdot\mathbf{n}'\right. \\ &- \left.\mathbf{u}(\mathbf{x}',t')\cdot\;[\mathcal{C}(\mathbf{x}'):\nabla'\mathcal{G}(\mathbf{x},t;\mathbf{x}',t')]\;\cdot\mathbf{n}'\right\}\;dS'dt' \qquad (12)\end{aligned}$$

as demonstrated in Aki and Richards (1980, p.29). The unit normal to the surface S is denoted as $\mathbf{n}$. Note that the integral solution (12) is the elastodynamic vector equivalent of the Kirchhoff-Rayleigh-Sommerfeld integral equation for scalar waves.

The WKBJ Green's tensor

The Green's tensor $\mathcal{G}(\mathbf{x},t;\mathbf{x}',t')$ is the displacement vector impulse response at an "observation" point $(\mathbf{x},t)$ due to an impulsive force density located at $(\mathbf{x}',t')$, and satisfies the auxiliary wave equation:

$$\mathcal{L}\mathcal{G}(\mathbf{x},t;\mathbf{x}',t') = \mathcal{I}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\;, \qquad (13)$$

where $\mathcal{I}$ is the identity matrix. I will assume the following WKBJ form for the Green's tensor:

$$\begin{aligned}&\mathcal{G}(\mathbf{x},t;\mathbf{x}',t') \sim \\ &\quad A\delta(t-t'-\tau)\mathbf{e}_1^n(\mathbf{x})\mathbf{e}_1^n(\mathbf{x}') + B\delta(t-t'-\phi)\left[\mathbf{e}_2^n(x)\mathbf{e}_2^n(\mathbf{x}') + \mathbf{e}_3^n(\mathbf{x})\mathbf{e}_3^n(\mathbf{x}')\right]\;, \qquad (14)\end{aligned}$$

where $\mathbf{e}_k^n$ are the polarization vectors evaluated at either endpoint along the ray joining $\mathbf{x}'$ to $\mathbf{x}$, and where $\{A,B,\tau,\phi\}$ all have the functional dependency $A(\mathbf{x};\mathbf{x}')$. I refer the reader to Aki and Richards (1980) for a constant velocity overview of the Green's tensor, and to Morse and Feshbach (1953) for a review of the WKBJ method of solution to partial differential equations. Note that the form (14) of $\mathcal{G}$ implicitly requires an isotropic elastic background propagation model since the two shear components have the same arrival time ϕ and amplitude B, although all of the theory previous to this point has been valid for general anisotropy. The traveltimes τ and ϕ satisfy the P and S eikonal equations respectively,

$$\begin{aligned}|\nabla\tau(\mathbf{x})|^2 &= \alpha^{-2}(\mathbf{x}) \\ |\nabla\phi(\mathbf{x})|^2 &= \beta^{-2}(\mathbf{x})\;, \qquad (15)\end{aligned}$$

and the WKBJ amplitudes A and B satisfy the P and S transport equations respectively,

$$A(\mathbf{x})\nabla^2\tau(\mathbf{x}) + 2\nabla\tau(\mathbf{x}) \cdot \nabla A(\mathbf{x}) = 0$$

$$B(\mathbf{x})\nabla^2\phi(\mathbf{x}) + 2\nabla\phi(\mathbf{x}) \cdot \nabla B(\mathbf{x}) = 0 \ . \tag{16}$$

The P and S-wave propagation velocities are denoted as $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ respectively. Incorporating general anisotropy in the background model would require modification of the Green's tensor (14) to have two coupled shear traveltimes ϕ_1, ϕ_2 and amplitudes B_1, B_2, given by S_1–S_2 coupled eikonal and transport equations (Červený et al., 1977). Since the WKBJ Green's tensor satisfies the zeroth order asymptotic ray eikonal and transport equations, the quantities A, B, τ, and ϕ can be evaluated by numerical ray techniques, providing the migration background model is ray valid (smooth velocity gradients on the order of a seismic wavelength within a sequence of possibly discontinuous parameterized layers).

The incident wavefield

Given the elastodynamic integral solution (12) and the WKBJ Green's tensor (14), a closed form solution for the incident wavefield $\mathbf{u}^s(\mathbf{x}, t)$ can be derived. I assume the source can be represented as a body point force, and thus evaluated by the volume integral portion only of (12):

$$\mathbf{u}^s(\mathbf{x}, t) = \int_{t'} \int_{V'} \mathbf{f}(\mathbf{x}', t') \cdot \mathcal{G}(\mathbf{x}, t; \mathbf{x}', t') \ dV' dt' \ . \tag{17}$$

A spatially compact impulsive body force may take the general form:

$$\mathbf{f}(\mathbf{x}', t') = \left[A_o w_1(t') \mathbf{e}_1^{s'} + B_{o2} w_2(t') \mathbf{e}_2^{s'} + B_{o3} w_3(t') \mathbf{e}_3^{s'} \right] \ \delta(\mathbf{x}' - \mathbf{x}_s) \ , \tag{18}$$

where $\mathbf{e}_k^{s'}$ are all evaluated at the source location $\mathbf{x}' = \mathbf{x}_s$. The amplitude $A_o(\mathbf{e}_1^{s'})$ is the scale and radiation pattern of the P source displacement at $\mathbf{x}_s$, and may vary as a function of the take-off angle which can be obtained from $\mathbf{e}_1^{s'}$, and in static strength as a function of shot location $\mathbf{x}_s$. The terms B_{o2} and B_{o3} are the equivalent S_1 and S_2 factors of A_o. The term $w_1(t'; \mathbf{x}_s)$ is the P waveform, and may vary with source location $\mathbf{x}_s$. The terms $w_2(t')$ and $w_3(t')$ are the S_1 and S_2 waveforms respectively, and may differ from w_1 in both phase and frequency content as a function of $\mathbf{x}_s$.

The body force (18) can be substituted into the integral solution (17) for the incident wavefield. The dot product $\mathbf{f} \cdot \mathcal{G}$ is evaluated from (18) and (14), for $\mathbf{e}_k^n = \mathbf{e}_k^s$, as

$$\mathbf{f}\cdot\mathcal{G} = \\ [A_o w_1 A\delta(t-t'-\tau)\mathbf{e}_1^s \\ +B_{o2}w_2 B\delta(t-t'-\phi)\mathbf{e}_2^s + B_{o3}w_3 B\delta(t-t'-\phi)\mathbf{e}_3^s]\,\delta(\mathbf{x}'-\mathbf{x}_s)\ , \tag{19}$$

where the remaining polarization vectors $\mathbf{e}_k^s$ are all evaluated at the observation point $\mathbf{x}$. Substitution of (19) into (17) and performing the t' integration results in:

$$\mathbf{u}^s(\mathbf{x},t) = \int_{V'} [A_o A w_1(t-\tau)\mathbf{e}_1^s + B_{o2}Bw_2(t-\phi)\mathbf{e}_2^s + B_{o3}Bw_3(t-\phi)\mathbf{e}_3^s]\,\delta(\mathbf{x}'-\mathbf{x}_s)\,dV'\ . \tag{20}$$

The final volume integration over V' yields a compact form for the incident wavefield solution:

$$\mathbf{u}^s(\mathbf{x},t) = A_o A_s w_1(t-\tau_s)\mathbf{e}_1^s + B_{o2}B_s w_2(t-\phi_s)\mathbf{e}_2^s + B_{o3}B_s w_3(t-\phi_s)\mathbf{e}_3^s\ , \tag{21}$$

where the notation A_s means $A(\mathbf{x};\mathbf{x}_s)$ and τ_s means $\tau(\mathbf{x};\mathbf{x}_s)$, i.e., the value at $\mathbf{x}$ due to a source at $\mathbf{x}_s$. I remind you again that the polarization vectors $\mathbf{e}_k^s$ in (21) are to be explicitly evaluated at each subsurface location $\mathbf{x}$. To evaluate (21), the WKBJ amplitudes and traveltimes $\{A_s, B_s, \tau_s, \phi_s\}$, and the polarization vectors $\mathbf{e}_k^s(\mathbf{x},t)$, need to be raytraced from each source position $\mathbf{x}_s$ to each subsurface position $\mathbf{x}$, by numerically solving systems (15)–(16) with a rayracing or finite difference algorithm.

The reflected wavefield

An expression for the reflected wavefield $\mathbf{u}^r(\mathbf{x},t)$ can be obtained from the surface integrals of the general representation theorem (12):

$$\mathbf{u}^r(\mathbf{x},t) = \int_{t'}\int_{S'} \mathcal{G}^{T*}(\mathbf{x},t;\mathbf{x}',t')\cdot\ [\mathcal{C}(\mathbf{x}') : \nabla'\mathbf{u}(\mathbf{x}',t')]\ \cdot\mathbf{n}'\ dS'dt'\ , \tag{22}$$

in which I have assumed a free-surface boundary condition, such that the traction $(\mathcal{C}{:}\nabla\mathcal{G})\cdot\mathbf{n}$ vanishes on S. Since the reconstruction of the subsurface reflected wavefield requires reverse time propagation, the conjugate G^* is used. Using the WKBJ form (14), with $\mathbf{e}_k^n = \mathbf{e}_k^r$, the conjugate transpose Green's function can be evaluated as:

$$\mathcal{G}^{T*}(\mathbf{x},t;\mathbf{x}',t') \sim \\ A\delta(t-t'+\tau)[\mathbf{e}_1^r\mathbf{e}_1^{r'}]^T + B\delta(t-t'+\phi)\left([\mathbf{e}_2^r\mathbf{e}_2^{r'}]^T + [\mathbf{e}_3^r\mathbf{e}_3^{r'}]^T\right)\ , \tag{23}$$

where the primed polarization vectors $\mathbf{e}_k^{r'}$ are evaluated at the receiver positions $\mathbf{x}' = \mathbf{x}_r$, and the unprimed polarization vectors $\mathbf{e}_k^r$ are evaluated at the subsurface positions $\mathbf{x}$. Substituting (23) into (22) and performing the t' integration yields:

$$\mathbf{u}^r(\mathbf{x},t) = \int_{S'} \left\{ A_r(\mathbf{e}_1^r\mathbf{e}_1^{r'})^T \cdot \mathcal{C}:\nabla\mathbf{u}(t+\tau_r) + B_r\left[(\mathbf{e}_2^r\mathbf{e}_2^{r'})^T + (\mathbf{e}_3^r\mathbf{e}_3^{r'})^T\right]\cdot\mathcal{C}:\nabla\mathbf{u}(t+\phi_r)\right\}\cdot\mathbf{n}'\,dS'. \quad (24)$$

In the case that the background propagation model is isotropic, which has been implicitly assumed in the definition of the Green's function (although this need not be the case), then the term $\mathcal{C}:\nabla\mathbf{u}$ can be written as

$$\mathcal{C}:\nabla\mathbf{u} = \lambda\nabla\cdot\mathbf{u} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla) \ , \quad (25)$$

where $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the Lamé parameters. Thus, the solution for the reconstructed reflected vector wavefield $\mathbf{u}^r(\mathbf{x},t)$ in an isotropic ray-valid heterogeneous background medium is given by the surface integral (24) and the constitutive equation (25).

The $\grave{P}\acute{P}$ specular reflection coefficient

Now that expressions for the incident and reflected vector wavefields have been obtained in (21) and (24), the least-squares estimation of the $\grave{P}\acute{P}$ specular reflection coefficient can be evaluated from (10):

$$\grave{P}\acute{P}(\mathbf{x};\mathbf{x}_s) = \mathcal{R}_{11}(\mathbf{x}) = \frac{\int_t W_d^2(\mathbf{x},t)\Phi_1^r(\mathbf{x},t)\Phi_1^s(\mathbf{x},t)\,dt}{\int_t W_d^2\left(\Phi_1^s(\mathbf{x},t)\right)^2\,dt + W_m^2(\mathbf{x})} \ . \quad (26)$$

First, the incident wave scalar can be evaluated given (5) and (21):

$$\Phi_1^s(\mathbf{x},t;\mathbf{x}_s) = \mathbf{u}^s\cdot\mathbf{e}_1^s = A_oA_s w_1(t-\tau_s) \ . \quad (27)$$

Next, the reflected wave scalar can be evaluated given (5), (24) and (25):

$$\Phi_1^r(\mathbf{x},t;\mathbf{x}_r) = \mathbf{u}^r\cdot\mathbf{e}_1^r = \int_{S'} A_r\left\{\lambda(\nabla\cdot\mathbf{u})(\mathbf{e}_1^{r'}\cdot\mathbf{n}') + \mu\left[\mathbf{e}_1^{r'}\cdot(\nabla\mathbf{u})\cdot\mathbf{n}'\right]\right\}\,dS' \ . \quad (28)$$

Note that (28) requires knowledge of $\lambda\nabla\cdot\mathbf{u}$, the local normal stress of the displacement field at each receiver, and $\mu\nabla\mathbf{u}$, the local stress tensor at each receiver. For marine seismic data, the pressure $\lambda(\nabla\cdot\mathbf{u})$ is measured directly by hydrophones. However, for both land and marine data acquisition, the local receiver stress tensor is not routinely measured. As suggested by Keho (1986), I use a farfield WKBJ approximation of the displacement vector wavefield gradient and divergence terms such that

$$\mathbf{u}(\mathbf{x},t) \sim \mathbf{U}(\mathbf{x}) e^{iw\tau(\mathbf{x})} \ . \tag{29}$$

Then

$$\nabla \cdot \mathbf{u} \sim \dot{\mathbf{u}} \cdot \nabla \tau \ , \tag{30}$$

and

$$\nabla \mathbf{u} \sim \dot{\mathbf{u}} \nabla \tau \ . \tag{31}$$

In this case, (28) can be approximated for land acquisition as:

$$\Phi_1^r(\mathbf{x},t;\mathbf{x}_r) = \int_{S'} A_r \left\{ \lambda (\dot{\mathbf{u}} \cdot \nabla \tau_r)(\mathbf{e}_1^{r'} \cdot \mathbf{n}') + \mu \left[\mathbf{e}_1^{r'} \cdot (\dot{\mathbf{u}} \nabla \tau_r) \cdot \mathbf{n}' \right] \right\} \, dS' \ . \tag{32}$$

Substituting (27) and (32) into (26) and performing the t integrations, one obtains

$$\mathcal{R}_{11}(\mathbf{x}) = P_1 / Q_1 \ , \tag{33}$$

such that

$$P_1 = A_o A_s W_d^2(\mathbf{x},\tau_s) \int_{S'} A_r \left\{ \lambda_r (w_1 * \dot{\mathbf{u}}(t_{sr}) \cdot \nabla \tau_r)(\mathbf{e}_1^{r'} \cdot \mathbf{n}') + \mu_r \left[\mathbf{e}_1^{r'} \cdot (w_1 * \dot{\mathbf{u}}(t_{sr}) \nabla \tau_r) \cdot \mathbf{n}' \right] \right\} \, dS' \tag{34}$$

where $\dot{\mathbf{u}}(\mathbf{x},t_{sr})$ is now evaluated at the total ray traveltime $t_{sr} = \tau_s + \tau_r$ from each source at $\mathbf{x}_s$ to any subsurface point $\mathbf{x}$ and back up to each receiver position $\mathbf{x}_r$. The primed coordinates indicate that the polarization vectors should be evaluated at the receiver positions $\mathbf{x}_r$. The subscripts on λ_r and μ_r indicate that the Lamé parameters should be evaluated at the receiver positions $\mathbf{x}_r$. Note that the $*$ symbol signifies a convolution of the surface data $\mathbf{u}$ with the estimated P-wave source wavelet w_1. The autocorrelation integral is

$$Q_1 = A_o^2 A_s^2 \int_{t'} W_d^2(\mathbf{x},t') w_1^2(t') \, dt' + W_m^2(\mathbf{x}) \ , \tag{35}$$

and is related to the energy of the incident P wavefield. Combining (33)–(35) yields:

$$\grave{P}\acute{P}(\mathbf{x};\mathbf{x}_s) = \frac{W_d^2(\mathbf{x},\tau_s) \int_{S'} A_r \left\{ \lambda_r (w_1 * \dot{\mathbf{u}}(t_{sr}) \cdot \nabla \tau_r)(\mathbf{e}_1^{r'} \cdot \mathbf{n}') + \mu_r \left[\mathbf{e}_1^{r'} \cdot (w_1 * \dot{\mathbf{u}}(t_{sr}) \nabla \tau_r) \cdot \mathbf{n}' \right] \right\} dS'}{A_o A_s \int_{t'} W_d^2(\mathbf{x},t') w_1^2(t') \, dt' + W_m^2(\mathbf{x})} \ . \tag{36}$$

Equation (36) gives the least-squares elastic wavefield integral solution for $\grave{P}\acute{P}$ specular reflectivity. It can be slightly simplified further by noting that

$$\nabla \tau_r = \mathbf{e}_1^{r'} / \alpha_r \ , \tag{37}$$

which follows from the P-wave eikonal equation in (15), where α_r is the P-wave velocity evaluated at each receiver location $\mathbf{x}_r$.

The $\grave{P}\acute{S}_1$ specular reflection coefficient

I now demonstrate the analogous derivation for the $\grave{P}\acute{S}_1$ specular reflection coefficient. Again, the least-squares solution is given by (10):

$$\grave{P}\acute{S}_1(\mathbf{x};\mathbf{x}_s) = \mathcal{R}_{21}(\mathbf{x}) = \frac{\int_t W_d^2(\mathbf{x},t)\Phi_2^r(\mathbf{x},t)\Phi_1^s(\mathbf{x},t)\ dt}{\int_t W_d^2\left(\Phi_1^s(\mathbf{x},t)\right)^2\ dt + W_m^2(\mathbf{x})} \ . \tag{38}$$

For a $\grave{P}\acute{S}_1$ reflection, the incident P-wave scalar is the same as (27):

$$\Phi_1^s(\mathbf{x},t;\mathbf{x}_s) = \mathbf{u}^s\!\cdot\!\mathbf{e}_1^s = A_o A_s w_1(t-\tau_s) \ . \tag{39}$$

However, the reflected S_1-wave scalar needs to be evaluated given (4), (24) and (25):

$$\Phi_i^r(\mathbf{x},t;\mathbf{x}_r) = \mathbf{u}^r\!\cdot\!\mathbf{e}_2^r = \int_{S'} B_r \left\{\lambda(\dot{\mathbf{u}}\!\cdot\!\nabla\phi_r)(\mathbf{e}_2^{r'}\!\cdot\!\mathbf{n}') + \mu\left[\mathbf{e}_2^{r'}\!\cdot\!(\dot{\mathbf{u}}\nabla\phi_r)\!\cdot\!\mathbf{n}'\right]\right\}\ dS' \ . \tag{40}$$

In deriving (40) I have taken the farfield WKBJ approximation of the displacement vector wavefield gradient and divergence terms given in (30)–(31).

Substituting (39) and (40) into (38) and performing the required t integrations, one obtains

$$\mathcal{R}_{21}(\mathbf{x}) = P_2/I_1 \ , \tag{41}$$

such that

$$P_2 = A_o A_s W_d^2(\mathbf{x},\tau_s) \int_{S'} B_r \left\{\lambda_r(w_1 * \dot{\mathbf{u}}(t_{sr})\!\cdot\!\nabla\phi_r)(\mathbf{e}_2^{r'}\!\cdot\!\mathbf{n}') + \mu_r\left[\mathbf{e}_2^{r'}\!\cdot\!(w_1 * \dot{\mathbf{u}}(t_{sr})\nabla\phi_r)\!\cdot\!\mathbf{n}'\right]\right\}\ dS \tag{42}$$

Now $\dot{\mathbf{u}}(\mathbf{x},t_{sr})$ is now evaluated at the total incident P plus reflected S ray traveltime $t_{sr} = \tau_s + \phi_r$. The subscripts on λ_r and μ_r indicate that the Lamé parameters should be evaluated at the receiver positions $\mathbf{x}_r$, and the $*$ symbol still signifies a convolution of the surface data $\mathbf{u}$ with the estimated P-wave source wavelet w_1. The P-wave autocorrelation integral is the same as the $\grave{P}\acute{P}$ case:

$$I_1 = A_o^2 A_s^2 \int_{t'} W_d^2(\mathbf{x}, t') w_1^2(t') \, dt' + W_m^2(\mathbf{x}) \; . \tag{43}$$

Combining (41)–(43) yields:

$$\grave{P}\acute{S}_1(\mathbf{x}; \mathbf{x}_s) = \frac{W_d^2(\mathbf{x},\tau_s) \int_{S'} B_r \left\{ \lambda_r (w_1 * \dot{\mathbf{u}}(t_{sr}) \cdot \nabla\phi_r)(\mathbf{e}_2^{r'} \cdot \mathbf{n}') + \mu_r \left[\mathbf{e}_2^{r'} \cdot (w_1 * \dot{\mathbf{u}}(t_{sr}) \nabla\phi_r) \cdot \mathbf{n}' \right] \right\} dS'}{A_o A_s \int_{t'} W_d^2(\mathbf{x},t') w_1^2(t') \, dt' + W_m^2(\mathbf{x})} \; . \tag{44}$$

Equation (44) gives the least-squares elastic wavefield integral solution for $\grave{P}\acute{S}_1$ specular reflectivity. It also can be slightly simplified further by noting that

$$\nabla\phi_r = \mathbf{e}_2^{r'} / \beta_r \; , \tag{45}$$

which follows from the S-wave eikonal equation in (15), where β_r is the S-wave velocity evaluated at each receiver position $\mathbf{x}_r$.

Discussion

This completes the fully elastic least-squares estimation of the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}_1$ specular reflection coefficients from displacement vector wavefield seismic data. By analogy, any of the other 7 components of the elastic scattering matrix $\mathcal{R}_{ij}$ in (2) can be derived in a similar manner. Perhaps in the next theory paper I will append all 9 integral solutions, but I am most interested in the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}_1$ expressions at this point in time.

In a future paper, I will re-express the integral solutions derived here in the common-offset domain, and derive companion integral solutions for the least-squares estimation of the specular reflection angles Θ_{PP} and Θ_{PSv} directly as a weighted migration of the vector wavefield seismic data. At that point, I will discuss pertinent implementational issues.

CONCLUSION

I have posed the goal of elastic specular reflectivity estimation as a formal least-squares inverse problem, which I subsequently solve by stationary point analysis. After introducing the elastodynamic wave equation, a representation integral solution is presented by use of Betti's Theorem which is the vector analogy to the Kirchhoff-Rayleigh-Sommerfeld integrals for scalar waves. The representation integrals are evaluated by assuming WKBJ approximate Green's tensors, which are in turn evaluated by the ray theoretic eikonal and transport equations. The representation integrals are substituted back into the generalized least-squares solution to obtain expressions for elastic specular reflectivity. I give explicit integral expressions for the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}_1$ reflectivities as surface integrals over the recorded vector displacement wavefields.

ACKNOWLEDGMENTS

I would like to acknowledge my thoroughly enjoyable ongoing research relationship with Wafik Beydoun of Elf Research, UK. Although we must communicate by `email` now, we still get the occasional opportunity to brainstorm together in the same room. Many such sessions have helped me to crystallize some of the theory presented in this paper. I would also like to acknowledge Professor Doug Oldenburg, my former M.Sc. advisor at the University of British Columbia. He was very supportive of this work when it was in its formative stages back then. I would like to thank Professor Albert Tarantola (University of Paris) for originally suggesting I pose true amplitude reflectivity estimation as a formal inverse problem. We had several stimulating chats about this work during his 1990 visit to UBC, by which I was profoundly influenced. Finally, I thank Jim Berryman (Lawrence Livermore National Laboratory) and Dave Nichols for their careful and thorough reviews of this paper. They made several suggestions which improved the content of this work.

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