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Coupled wave propagation

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ABSTRACT

Coupled wave propagation phenomena have been observed in laboratory measurements for a long time; however in field experiments observations have been less convincing. In this paper coupled wave equations allow us to model such coupling by combining elastic stress and strain, electric field strength and displacement, entropy and temperature. For piezoelectric media slowness surfaces show changes in propagation velocity and particle motion when coupling is included in the computation. Average properties of such media can be computed using the Schoenberg and Muir group theory. Possible applications of coupled wave propagation, besides earthquake prediction and mineral exploration, might be the 3D monitoring of oil fields during enhanced oil recovery. Accurate rock parameter estimation seems to be sensitive to coupling effects.

INTRODUCTION

Russian researchers have been looking at phenomena of linked mechanical and electromagnetical properties; their goal has been to find anomalies in electromagnetic properties (primarily currents and radiation) induced by mechanical anomalies (propagating seismic waves). In mathematical physics the notion of generalized forces, coordinates, and work functions is a powerful and mathematically elegant tool to describe the dynamic state of a physical system. Rigorous mathematical treatments given in Lanczos (1970) and Landau et al (1984) show that the state of a physical system can be described by complete sets of dependent and independent generalized variables. These variables usually do not have geometrical euclidian meaning; rather they describe the system in an abstract space called *phase space*. The equations describing the evolution of the system take on canonical form in these coordinates. Consequently mathematical concepts like Lagrange's variational principle or Hamilton's principle also take canonical form. In this paper I combine the ideas of elastic stresses, electric field strengths, and entropy into a generalized stress theory (the

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stresses being dependent variables) in order to describe the dynamic state of a solid body. Such generalized stress can be linearly related to generalized displacement, which combines elastic strains, electric displacement, and temperature. This concept implicitly incorporates important thermodynamic principles such as Onsager's principle. It is a macroscopic description of rock properties and consequently allows us to assign some average properties to the medium. I omit the consideration of porous media, since it exceeds the scope of linear theory.

Uniform Symbols

Classical mathematical physics uses various scalar, vector and tensor notations, depending on the discipline, but for simplicity and uniformity I will use matrices operators (differential or physical), and vectors for the object or outcome of operators. Thus, for example, a second rank tensor with index range three is represented as a 3x3 matrix. Similarly, an even higher order tensor such as the elastic stiffness can be written in matrix form. The matrix representation is a widely accepted tool for complex mathematical descriptions and seems to give a more pictorial description of the physical processes. Moreover, besides the ease of representation, numerical applications benefit from efficient matrix algebra.

GENERALIZED LINEAR RELATIONSHIP

In linearized theory, stress quantities and strain quantities obey a linear law. The components of the stress variables are linear combinations of the strain variables. The quantities that relate those components are tensors. We conveniently use matrix notation (Nye, 1985), which allows us to express the relationships in a comprehensive form. Therefore the generalized "Hooke's Law" is

$$\begin{pmatrix} \sigma_\mathbf{1} \\ \sigma_\mathbf{2} \\ \sigma_\mathbf{3} \end{pmatrix} = \begin{pmatrix} \utilde{\mathbf{c_{11}}} & \utilde{\mathbf{c_{12}}} & \utilde{\mathbf{c_{13}}} \\ \utilde{\mathbf{c_{21}}} & \utilde{\mathbf{c_{22}}} & \utilde{\mathbf{c_{23}}} \\ \utilde{\mathbf{c_{31}}} & \utilde{\mathbf{c_{32}}} & \utilde{\mathbf{c_{33}}} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_\mathbf{1} \\ \epsilon_\mathbf{2} \\ \epsilon_\mathbf{3} \end{pmatrix} \tag{1}$$

In this equation we identify $\sigma_\mathbf{1}$ with the elastic stress tensor, $\sigma_\mathbf{2}$ with the electric stress (field strength), and $\sigma_\mathbf{3}$ with the change in thermal stress (temperature). The term ϵ_1 corresponds to the elastic strain tensor, $\epsilon_\mathbf{2}$ to the electrical displacement and $\epsilon_\mathbf{3}$ to the change in thermal flow (entropy). The coefficients of the block matrices $\utilde{\mathbf{c_{ij}}}$ $(i, j = 1, 2, 3)$ relate generalized "stress" vector $\sigma_\mathbf{i}$ $(i = 1, 2, 3)$ with the generalized "strain" vector $\epsilon_\mathbf{j}$ $(j = 1, 2, 3)$. The generalized stiffness c is a matrix of complex numbers, which thus accounts for absorption effects. Equation 1 can be viewed as generated by partial derivatives of a more general *state function*, which takes the form

$$\Phi = U - \epsilon_\mathbf{1}\,\sigma_\mathbf{1} - \epsilon_\mathbf{2}\,\sigma_\mathbf{2} - \epsilon_\mathbf{3}\,\sigma_\mathbf{3} \tag{2}$$

where U is the internal energy. This gives the block matrix coefficients in terms of the partial derivatives

$$\underset{\sim}{\mathbf{c_{ii}}} = -\frac{\partial \Phi}{\partial \sigma_{\mathbf{i}}}|_{\sigma_{\mathbf{j}},\sigma_{\mathbf{k}}} \tag{3}$$

$$\underset{\sim}{\mathbf{c_{ij}}} = -\frac{\partial \Phi}{\partial \sigma_{\mathbf{i}} \partial \sigma_{\mathbf{j}}}|_{\sigma_{\mathbf{k}}} \tag{4}$$

The important point is that all stresses not used in the partial derivatives are kept constant. Doing so requires that in actual experiments the boundary conditions be properly applied and maintained during measurements.

WAVE EQUATIONS IN TERMS OF DISPLACEMENTS

Given a linear relationship between "generalized" displacements and "generalized" forces, we can derive wave equations for elastic, electromagnetic and thermal quantities. The derivation of the individual wave equations follows a common procedure: first, finding a conservation law for the generalized stress-like and strain-like quantities, and, second, finding a constitutive relationship (linear law) between those quantities. Here the constitutive law can also include alien stress- or strain-like quantities. Using the conservation and constitutive relationships, I can find a wave equation for each type of quantity. It is in fact a coupled system of equations which have to be solved simultaneously.

The advantages of using elastic displacements as observable quantities are the speed of computation and the formal similarity to the conventional Christoffel equation. However, if, instead of displacements, elastic strains are used to describe wave propagation, the wave equation resembles more the generalized linear relationship between stresses and strains. In a few instances boundary conditions may be easier to specify in terms of strain variables. However this method, given in the appendix, is more costly.

Elastic field equations

The derivation of the coupled elastic wave equations is similar to that of the uncoupled equations. In an elastic medium forces have to be in balance. This implies that the divergence of the elastic stress tensor $\underset{\sim}{\nabla} \cdot \sigma_{\mathbf{1}}$ has to equal the product of mass density ρ and the second temporal derivative of the particle displacement $\mathbf{u}$. We can also include a forcing term $\mathbf{F}$ and get

$$\underset{\sim}{\nabla}\ \sigma_{\mathbf{1}} = \rho \frac{\partial^2}{\partial t^2} \mathbf{u_1} - \mathbf{F}. \tag{5}$$

The elastic strain ϵ_1 can be calculated by taking spatial derivatives $\underset{\sim}{\nabla}$ of the displacement $\mathbf{u}$, as follows:

$$\epsilon_1 \;=\; \underset{\sim}{\nabla}^{\mathbf{T}} \; \mathbf{u_1} \tag{6}$$

Next we relate the components of the elastic stress σ_1 to the components of the elastic strain ϵ_1, the electric displacement ϵ_2 and the first temporal derivative of the thermal displacement $\frac{\partial}{\partial t}\epsilon_3$ in the following equation:

$$\sigma_1 \;=\; \underset{\sim}{\mathbf{c_{11}}} \; \epsilon_1 \;+\; \underset{\sim}{\mathbf{c_{12}}} \; \epsilon_2 \;+\; \underset{\sim}{\mathbf{c_{13}}} \; \frac{\partial}{\partial t}\epsilon_3 \tag{7}$$

Substituting Equation 7 into 5 gives a wave equation in terms of elastic displacements (omitting the source term):

$$\underset{\sim}{\nabla} \; (\; \underset{\sim}{\mathbf{c_{11}}} \;\; \underset{\sim}{\nabla}^{\mathbf{T}} \mathbf{u_1} \;+\; \underset{\sim}{\mathbf{c_{12}}} \; \epsilon_2 \;+\; \underset{\sim}{\mathbf{c_{13}}} \; \frac{\partial}{\partial t}\epsilon_3) \;=\; \rho \; \frac{\partial^2}{\partial t^2}\mathbf{u_1} \tag{8}$$

This is a coupled wave equation for elastic displacements, since it includes effects produced by electric and thermal displacements as well.

Electromagnetic field equations

In deriving the electromagnetic coupled field equations we make the assumption that within the medium the divergence of the electric stress field is zero ($\mathbf{div}\;\sigma_2 = 0$), i.e. that there are no “batteries” buried in the medium. Making this assumption is equivalent to setting the forcing term in Equation 5 to zero. We note that the magnetic fields and magnetic induction are related by the magnetic permeability μ ($\mathbf{B} = \underset{\sim}{\mu}\,\mathbf{H}$). Since there is no magnetic monopole, divergence of the magnetic field is also zero ($\mathbf{div}\;\mathbf{B}$=0). If we allow electric current to flow, the current density $\mathbf{j}$ is related to the electric stress field by the conductivity $\underset{\sim}{\kappa_{\mathbf{e}}}$ as follows:

$$\mathbf{j} \;=\; \underset{\sim}{\kappa_{\mathbf{e}}} \; \sigma_2 \tag{9}$$

We can use Maxwell’s equation to describe the relation ship between electric and magnetic fields:

$$\mathbf{curl}\;\sigma_2 \;=\; -\frac{\partial}{\partial t}\mathbf{B} \tag{10}$$

$$\mathbf{curl}\;H = \; \frac{\partial}{\partial t}\epsilon_2 \;+\; \mathbf{j} \tag{11}$$

The linear law relating electric stress components to components of elastic strain, electric and thermal displacements is now

$$\sigma_2 \;=\; \underset{\sim}{\mathbf{c_{21}}} \;\; \underset{\sim}{\nabla}^{\mathbf{T}} \mathbf{u_1} \;+\; \underset{\sim}{\mathbf{c_{22}}} \; \epsilon_2 \;+\; \underset{\sim}{\mathbf{c_{23}}} \;\; \frac{\partial}{\partial t}\epsilon_3 \tag{12}$$

Consequently, in terms of electric displacements, we end up with the coupled electromagnetic field equations

$$-\mathbf{curl}\ \underset{\sim}{\mu}^{-\mathbf{1}}\ \mathbf{curl}\,(\underset{\sim}{\mathbf{c_{21}}}\ \underset{\sim}{\nabla}^{\mathbf{T}}\,\mathbf{u_1}\ +\ \underset{\sim}{\mathbf{c_{22}}}\ \epsilon_{\mathbf{2}}\ +\ \underset{\sim}{\mathbf{c_{23}}}\ \frac{\partial}{\partial t}\epsilon_{\mathbf{3}})\ =\ \frac{\partial^2}{\partial t^2}\epsilon_{\mathbf{2}}\ +\ \underset{\sim}{\kappa_{\mathbf{e}}}\ \frac{\partial}{\partial t}\sigma_{\mathbf{2}} \tag{13}$$

Entropy field equations

The entropy flow h is related to the spatial gradient of the thermal stress (temperature) σ_3 by the thermal conductivity κ_h giving the constitutive relation

$$\mathbf{h}\ =\ -\,\underset{\sim}{\kappa_{\mathbf{h}}}\ \mathbf{grad}\ \sigma_{\mathbf{3}} \tag{14}$$

The entropy flow has to obey the conservation equation

$$\mathbf{div}\ \mathbf{h}\ =\ -\frac{\partial}{\partial t}\epsilon_{\mathbf{3}} \tag{15}$$

which means that heat is only created by letting entropy flow in the medium. There are no other heat sources present. The linear law relating changes in elastic strain, electric and thermal displacement to changes in thermal stress is

$$\frac{\partial}{\partial t}\sigma_{\mathbf{3}}\ =\ \underset{\sim}{\mathbf{c_{31}}}\ \frac{\partial}{\partial t}\underset{\sim}{\nabla}^{\mathbf{T}}\,\mathbf{u_1}\ +\ \underset{\sim}{\mathbf{c_{32}}}\ \frac{\partial}{\partial t}\epsilon_{\mathbf{2}}\ +\ \underset{\sim}{\mathbf{c_{33}}}\ \frac{\partial}{\partial t}\epsilon_{\mathbf{3}} \tag{16}$$

By combining Equations 14, 15 and 16 we find the coupled wave equation

$$\mathbf{div}\ \{\ \underset{\sim}{\kappa_{\mathbf{h}}}\ \mathbf{grad}\ (\underset{\sim}{\mathbf{c_{31}}}\ \frac{\partial}{\partial t}\underset{\sim}{\nabla}^{\mathbf{T}}\,\mathbf{u_1}\ +\ \underset{\sim}{\mathbf{c_{32}}}\ \frac{\partial}{\partial t}\epsilon_{\mathbf{2}}+\ \underset{\sim}{\mathbf{c_{33}}}\ \frac{\partial}{\partial t}\epsilon_{\mathbf{3}})\ \}\ =\ \frac{\partial^2}{\partial t^2}\epsilon_{\mathbf{3}} \tag{17}$$

THE COUPLED SYSTEM

Since Equations 8, 13 and 17 cannot be solved independently, we can, by combining them, derive a coupled system of partial differential equations. In order to investigate the nature of its solutions we can set up an analog to the Christoffel equation (Auld, 1981) by looking at plane wave solutions in a homogeneous medium. To do so, we can Fourier transform the system of equations in space and time ($\frac{\partial}{\partial t} = -i\ \omega$; $\frac{\partial}{\partial x} = i\,k_x$; *etc.*). This operation results in an algebraic system of equations for the unknown displacements

$$\begin{pmatrix} -\rho^{-1}\ \underset{\sim}{\mathbf{K}} \\ \underset{\sim}{\mathbf{R}}\ \ \underset{\sim}{\mu}^{-\mathbf{1}}\ \ \underset{\sim}{\mathbf{R}}\ \ -i\ \omega\ \underset{\sim}{\kappa_{\mathbf{e}}} \\ -\mathbf{Div}\ \underset{\sim}{\kappa_{\mathbf{h}}}\ \mathbf{Grad} \end{pmatrix} \begin{pmatrix} \underset{\sim}{\mathbf{c_{11}}}\ \underset{\sim}{\mathbf{K}}^{\mathbf{T}} & \underset{\sim}{\mathbf{c_{12}}} & -\,i\ \omega\ \underset{\sim}{\mathbf{c_{13}}} \\ \underset{\sim}{\mathbf{c_{12}}}\ \underset{\sim}{\mathbf{K}}^{\mathbf{T}} & \underset{\sim}{\mathbf{c_{22}}} & -i\ \omega\,\underset{\sim}{\mathbf{c_{23}}} \\ -i\ \omega\ \ \underset{\sim}{\mathbf{c_{31}}} & -\,i\ \omega\ \ \underset{\sim}{\mathbf{c_{32}}} & -i\ \omega\ \ \underset{\sim}{\mathbf{c_{33}}} \end{pmatrix} \begin{pmatrix} \mathbf{u_1} \\ \epsilon_{\mathbf{2}} \\ \epsilon_{\mathbf{3}} \end{pmatrix}\ =\ \omega^2 \begin{pmatrix} \mathbf{u_1} \\ \epsilon_{\mathbf{2}} \\ \epsilon_{\mathbf{3}} \end{pmatrix} \tag{18}$$

The spatial derivative operators in the form of matrices $\underset{\sim}{\mathbf{K}}$ and $\underset{\sim}{\mathbf{R}}$ are given in the Appendix. Dividing Equation 18 by the square magnitude of the spatial wave number vector $|\mathbf{k}|^2$, we get the eigenvalue equation

$$(\underset{\sim}{\mathbf{L}} \;\; \underset{\sim}{\mathbf{C}})\, \epsilon \;=\; v^2\, \epsilon \tag{19}$$

which is the "generalized" Christoffel equation. The eigenvalues ($v^2 \;=\; \frac{\omega^2}{\mathbf{k}^2}$) of this equation are propagation velocities of plane waves in the medium. The eigenvectors are directly related to the "particle motions". Since $\underset{\sim}{\mathbf{L}} \;\; \underset{\sim}{\mathbf{C}}$ is dependent on direction cosines, we can expect that propagation velocity and particle motion are dependent on the propagation direction. The dispersion relations become frequency dependent when we include heat effects or if the medium is electrically conducting. At the zero frequency limit, the contributions from heat and conductivity vanish. This result is the same as if the medium were now purely piezoelectric.

CHARACTERISTIC SURFACES

A wave traveling in a homogeneous medium described by Equation 18 propagates with a velocity corresponding to one of the eigenvalues of the system. The propagation velocity depends on the prescribed direction. Such a wave generally has three elastic displacement components, three electric displacement components, and the scalar entropy component. We can only classify the waves into pure elastic, electric, and thermal wavetypes if there is no coupling between the fields. This means that the matrix $\underset{\sim}{\mathbf{C}}$ only consists of the diagonal block matrices $\underset{\sim}{\mathbf{c_{ii}}}$ and is otherwise zero. If the offdiagonal terms are non-zero, field coupling is introduced. For any propagation direction we obtain three waves carrying energy mainly in their elastic displacement components; secondly we get two wavetypes which consist mostly of electric displacements and which cannot have purely longitudinal components; thirdly we obtain a wave type which is primarily thermal (a thermal diffusion wave). Plotting inverse propagation velocity versus direction results in a slowness surface which characterizes the physical properties of the medium. Take for example an electrically non-conductive but piezoelectric medium; in this case the propagation velocity does not depend on frequency and one 3D surface characterizes the propagation of one wavetype completely. If the medium is allowed to be electrically conductive, we need a 3D slowness surface for each frequency in order to describe propagation completely.

In the uncoupled case, elastic and electromagnetic particle motions have an indeterminate relationship; there is no interaction between them. As soon as coupling is in effect, electric "particle motion" exhibits a 90 degree phase shift, which can be explained by the fact that a propagating electromagnetic field is generated only by a temporal change in the electromagnetic field variables.

In the literature it is very hard to find complete sets of generalized stiffness constants that are measured consistently with the correct boundary conditions applied.

Thus in the following paragraphs I show results mainly for piezoelectric materials. Two materials are especially attractive to investigate: quartz and Lead-Titanate-Zirconate (PZT2). Firstly quartz since it is a naturally occurring rock and known for its wave coupling effects. Secondly Lead-Titanate-Zirconate (PZT2), that resembles shales in its "manufacturing" process and composition. The generalized stiffness constants for these two materials are listed in the appendix.

"Elastic" slowness surfaces

The computation of characteristic surfaces as shown by Dellinger (1989) can be extended to the coupled wave propagation case. Figure **??** shows slowness surfaces for the uncoupled purely elastic wavetypes in the medium PZT2. It is transversely isotropic with the z-axis as symmetry axis. In appearance it resembles Greenhorn shale, which is shown in Figure 1. For the PZT2 medium the piezoelectric constants are given in the literature; and we get a very interesting result, Figure 2, when we include these effects in our calculations. Instead of showing a well-behaved picture similar to that of shale's behaviour, the two slowest wavetypes change shape radically. The slowness increases by a factor of 4 in the direction of the z-axis. This behavior directly reflects the magnitude of the piezoelectric stress coefficients, which show a strong coupling of the electric field, especially into strain components in the z-direction.

Consequently we can ask ourselves what the slowness surfaces for Greenhorn shale would look like, if we measure the piezoelectric stress constants.

"Electromagnetic" slowness surfaces

The electromagnetic slowness surfaces for PZT2 on the other hand, do not show any anomalous behavior. PZT2 is a strongly birefringent material. Thus we can clearly see the two wavetypes propagating at speeds comparable to that of electromagnetic waves. Both slowness surfaces touch at the vertical axis (symmetry axis). The inside surface has an electric displacement direction which is orthogonal to that on the outside surface (a manifestation of Maxwell's equations). The shape is ellipsoidal according to the structure of the electric permittivity and magnetic permeability tensors. Including the piezoelectric stress coefficients in the calculation results in only minor effects on those two wavetypes.

In quartz the propagation velocities of the two electromagnetic waves are very close; they are only distinguishable by looking at both the outside and the inside slowness surface. The inside surface has an electric displacement direction which is orthogonal to that on the outer surface (a manifestation of Maxwell's equations). The shape is ellipsoidal according to the structure of the electric permittivity and magnetic permeability tensors. Again, for the coupled propagation, the changes are very small.

Cross-coupled slowness surfaces

So far we have been looking at waves whose displacement type corresponds to their wave propagation speed. In these instances the waves propagate either at elastic speeds, exhibiting elastic particle motions, or at electromagnetic speeds, exhibiting electric displacements. Cross-coupled waves differ in their behaviour from those "ordinary" wavetypes: their particle motion is not congruent with their propagation velocity category. For such cross-coupled waves we get in general a much smaller particle motion magnitude compared with "ordinary" wavetypes. In Figure 5 slowness surfaces for waves traveling in quartz at elastic speeds are shown. The values of the speeds are the same as in Figure 4, however, this time electric particle motions are displayed. The general appearance resembles that of the purely elastic surfaces; the striking difference is that there is no purely longitudinal particle motion. All the particle motions have transverse components. Slowness surfaces of waves traveling in quartz at speeds comparable to that of electromagnetic waves maintain their shape. However the particle motions are not purely transversal any more, but also include longitudinal motions (introduced by coupling).

BOUNDARY CONDITIONS

In the description of the dynamic states of solid bodies, boundary conditions play a very important role from the theoretical as well as the experimental point of view. The dynamic state function can be reduced in the number of independent variables (degrees of freedom) by keeping specific variables fixed. This concept is intimately related to the principle of hidden variables in a dynamic system. Hidden variables do not appear explicitly in dynamic equations but are rather transformed into given boundary conditions. The simplest example is an elastic solid; its behavior differs, depending upon whether we specify an isothermal or an adiabatic boundary. Entropy and temperature do not appear explicitly in the constitutive equations, but rather in the statement that the temperature or entropy remains constant. The values of the elastic constants are different in both cases. Translating this notion into a more general formulation means that we keep some of either the generalized stresses or the generalized strains constant (or zero). By doing so, we specify two extremes of boundary conditions imposed on the wave propagation. Any other boundary condition can be generated from linear combinations of these two extremes. Consequently, since the description of a dynamic solid body is far from being completely specified by the three physical properties – elastic, electric, and thermal quantities – we can also include magnetic state variables. If we do not, however, we keep such variables implicitly constant.

In practice it seldom happens that we measure all state variables in one experiment. Thus we have to be aware of the effect that the implicit boundary conditions have on the realized values. Let me illustrate this effect using the PZT2 material. Figure **??** shows pure elastic slowness surfaces. In this case we assume that the dy-

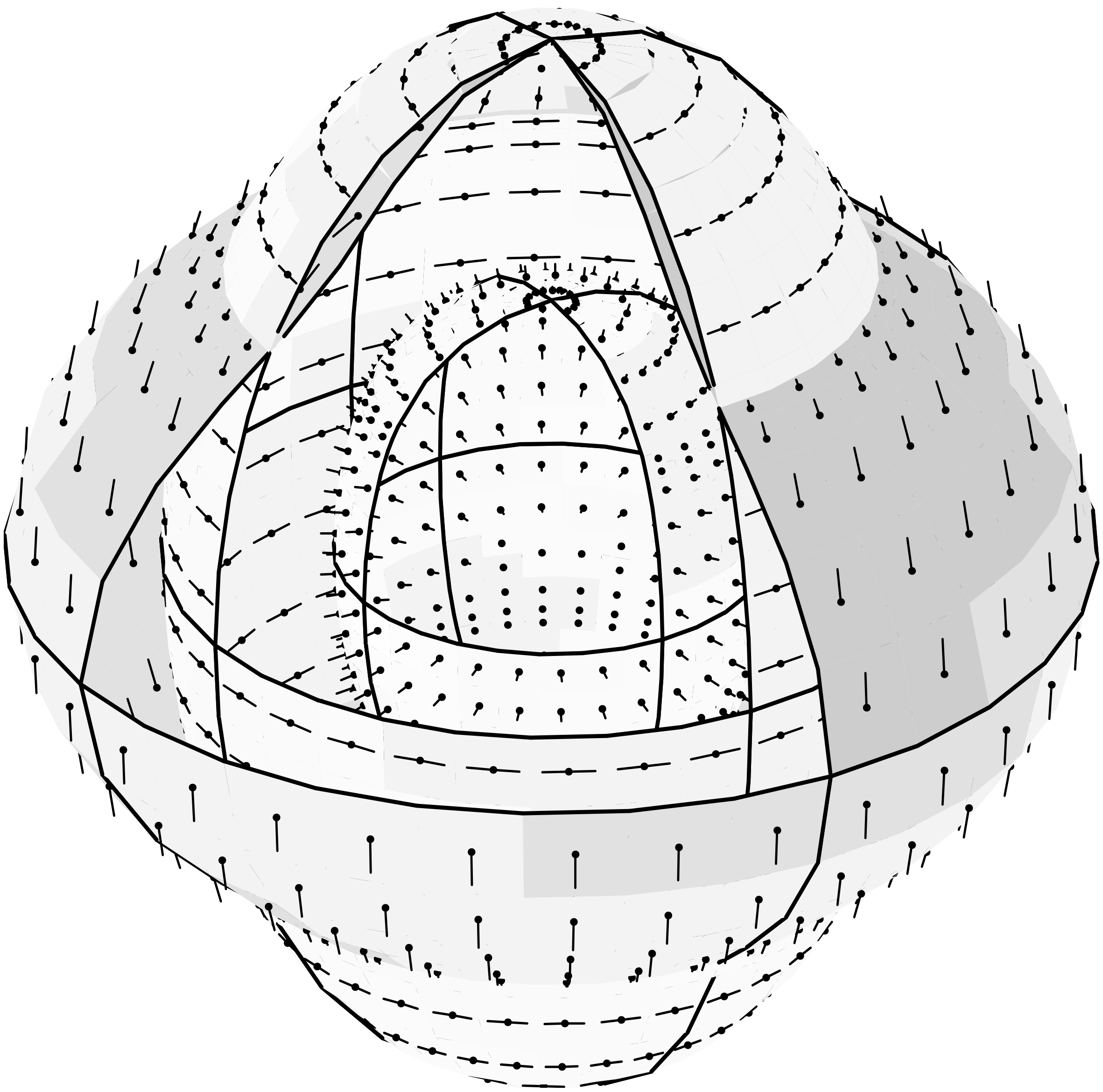

Figure 1: 3D slowness surfaces for Greenhorn shale are calculated for wavetypes with elastic speeds using elastic constants. The material has transverse isotropic symmetry and is in appearance similar to that of PZT2. martin-greenhornelastic [NR]

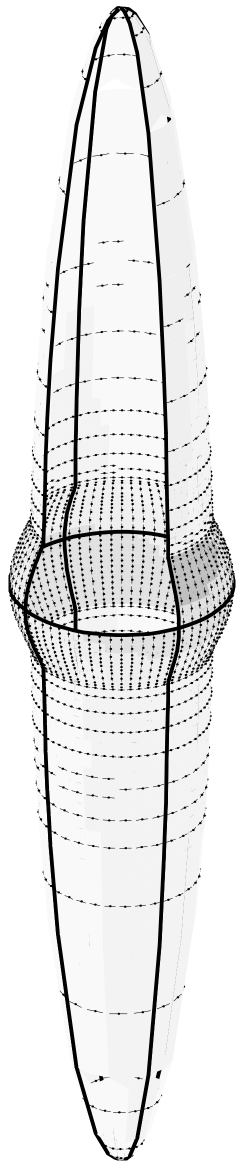

Figure 2: 3D slowness surfaces for PZT2 are calculated for wavetypes with elastic speeds using elastic and piezoelectric constants. The material has transverse isotropic symmetry. The slowness along the vertical axis is greatly increased, due to piezoelectric coupling constants (see appendix). Mainly the slowest S-wave is affected and shown here. martin-AuldPZT2piezoel [NR]

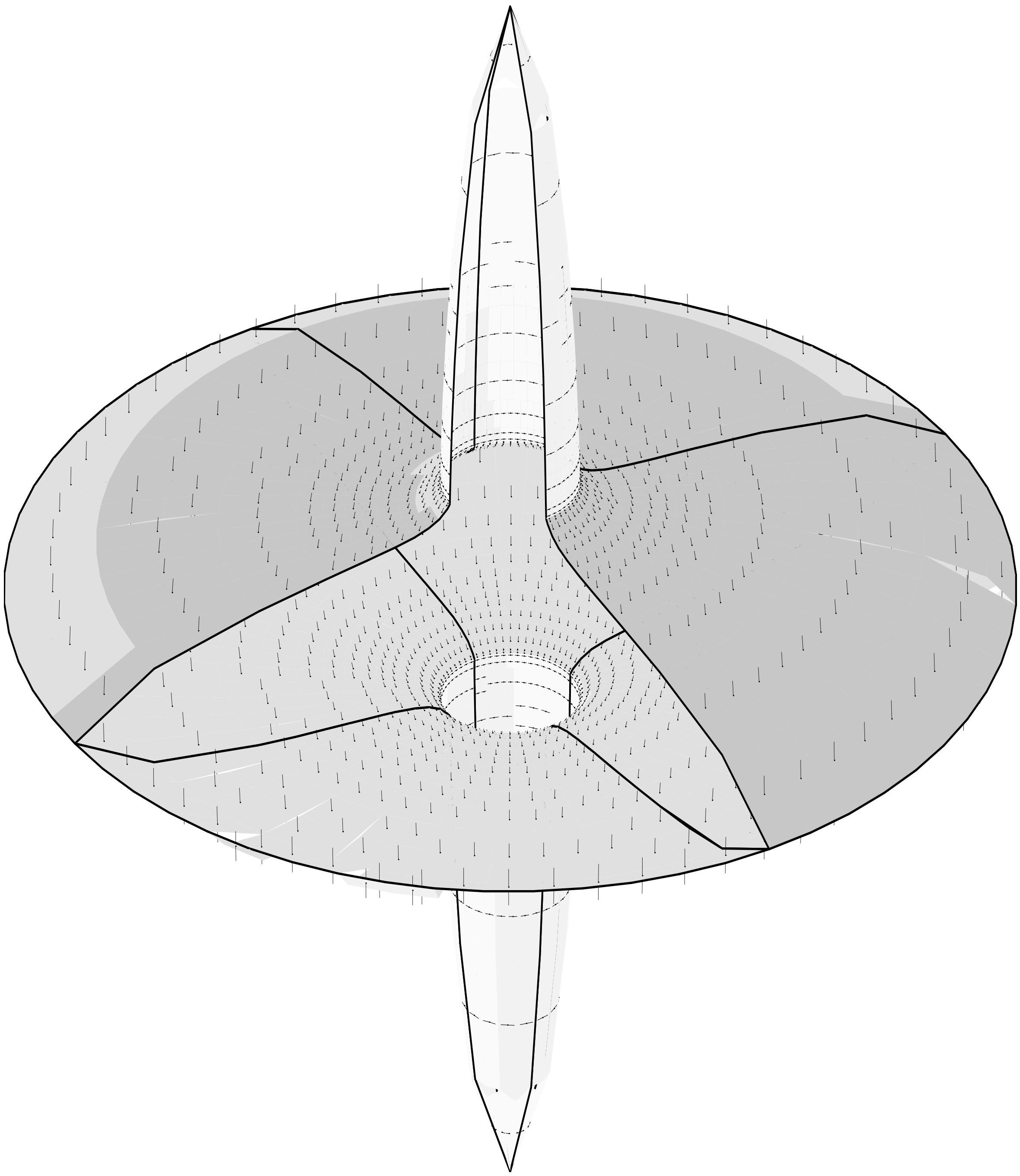

Figure 3: 3D slowness surfaces for PZT2 are calculated for wavetypes with elastic speeds using elastic and piezoelectric constants. The material has transverse isotropic symmetry. The slowness along the vertical axis is still greatly increased, while additionally the slowness in the symmetry plane is increased to about the same magnitude, due to piezoelectric coupling and the influence of the boundary condition ($\sigma_2 = 0$). Mainly the slowest S-wave is affected and shown here. martin-AuldPZT2stiff [NR]

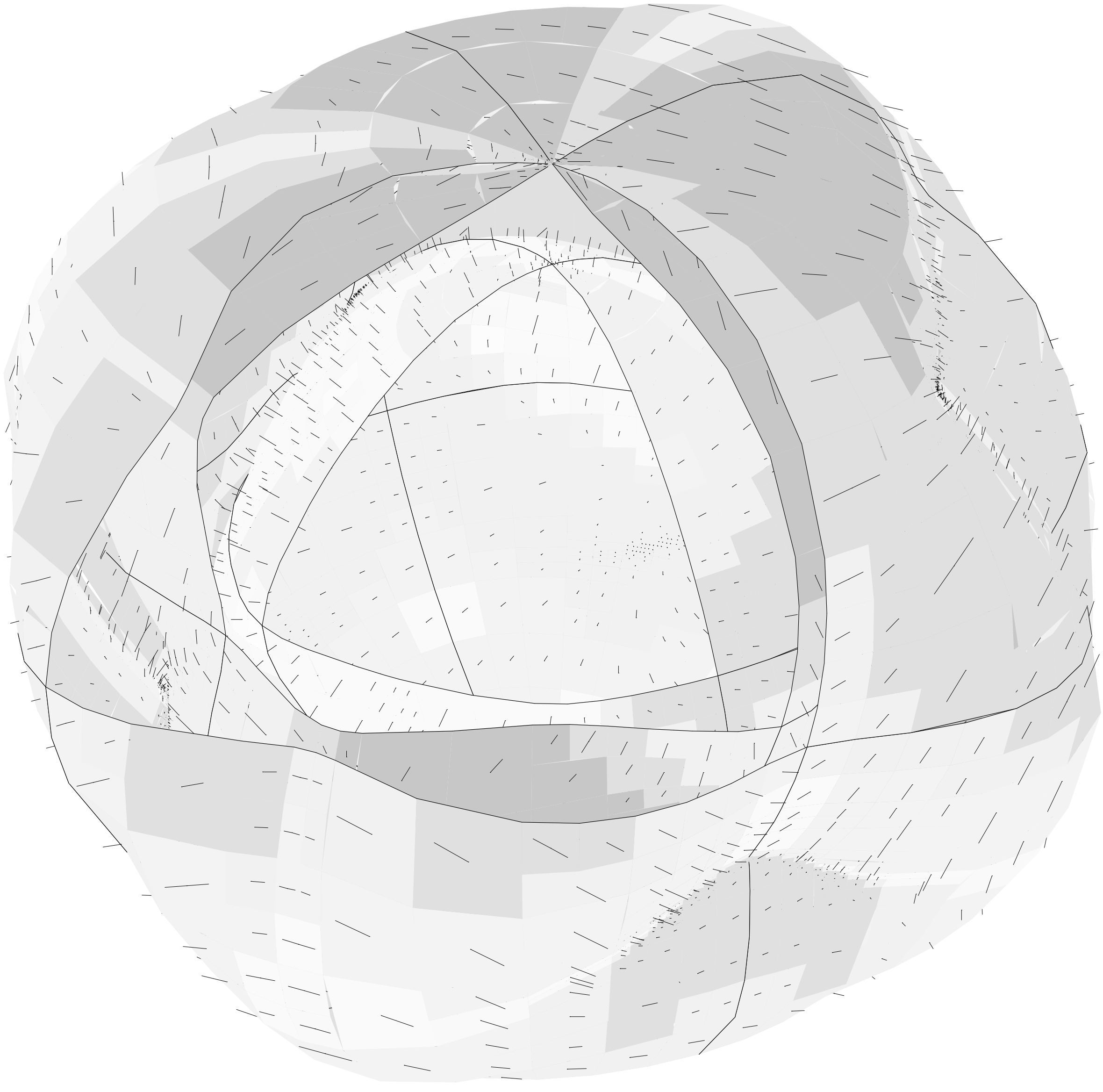

Figure 4: 3D slowness surfaces for quartz are calculated for wavetypes with elastic speeds using elastic and piezoelectric constants. The material has hexagonal symmetry, which causes the two slowness surfaces for the s-waves to be of complicated shape. The difference between purely elastic and piezoelectric wave propagation is in the range of a couple percent deviation in slowness and particle motion. martin-AuldQUARTZpiezoel [NR]

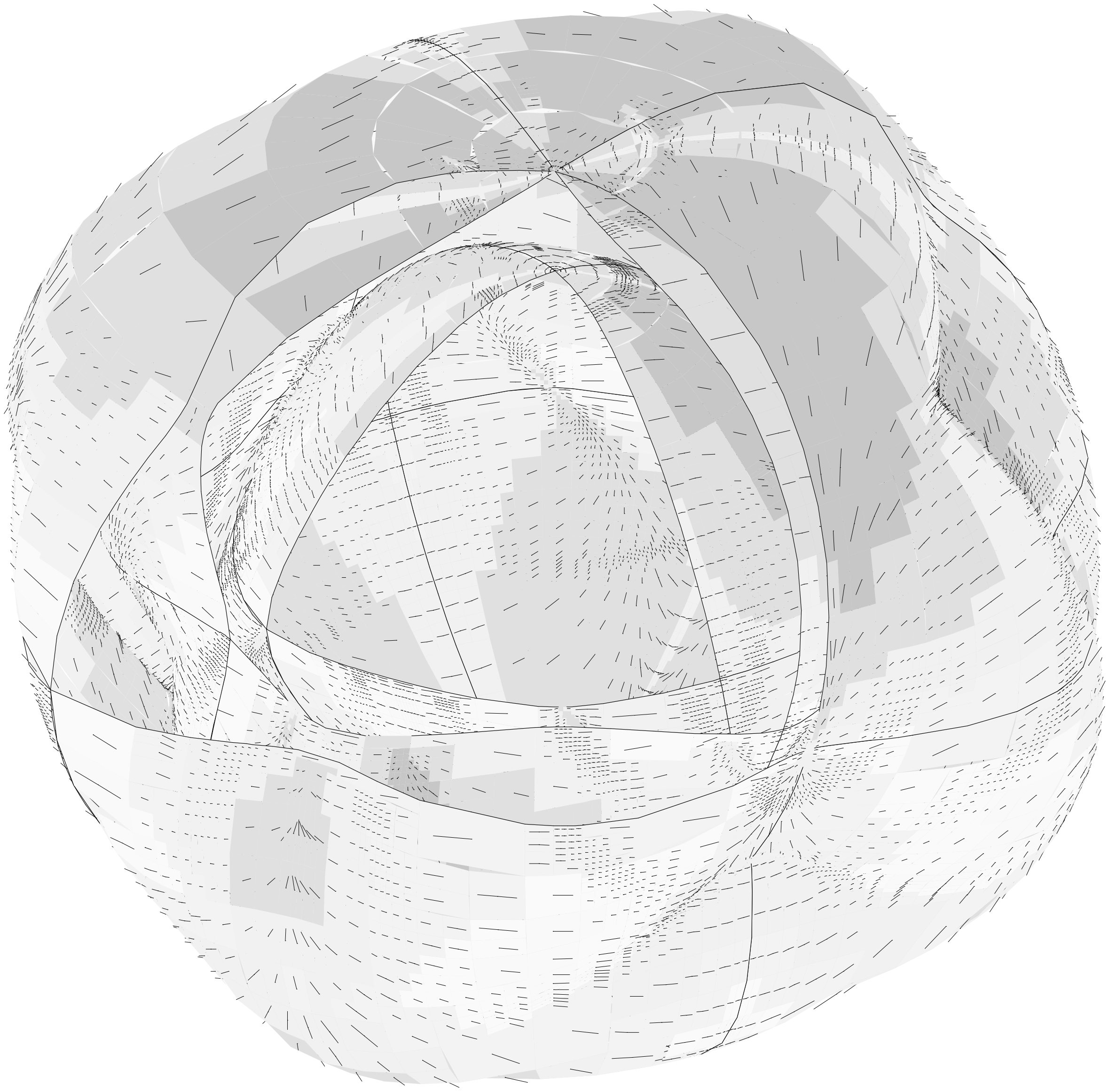

Figure 5: 3D slowness surfaces for quartz are calculated for cross-coupled wavetypes with elastic speeds and electromagnetic particle motions, using elastic and piezoelectric constants. The material has hexagonal symmetry. The slowness surfaces for these waves are similar to those of ordinary elastic waves, but now show only purely transversal particle motion. martin-AuldQUARTZpiezoelem [NR]

namic state is solely specified by the elastic constants, in other words, that all alien strains are kept at zero (boundary condition). In contrast, Figure **??** shows the elastic slowness surfaces when we specify as a boundary condition that alien stresses (electrical field strength) are kept at zero. The difference in scale between the two figures is a factor of 4. In the latter case we acknowledge the existence of a hidden state variable and incorporate its effects in form of altered elastic constants. This phenomenon in piezoelectric material is commonly termed "electrical stiffening". The elastic constants $\underset{\sim}{\mathbf{c_{11}}}$ are now modified to be $\underset{\sim}{\mathbf{c_{11}}} - \underset{\sim}{\mathbf{c_{12}^T}}\, \underset{\sim}{\mathbf{c_{22}^{-1}}}\, \underset{\sim}{\mathbf{c_{21}}}$. The shape of the two outermost slowness surfaces are altered significantly, while the innermost (p wave) does not change very much. However the intrinsic symmetry is the same in both cases. The comparison of the two extreme boundary conditions emphasizes the importance of including piezoelectrical effects, even when electromagnetic waves are not measured in an experiment.

MODELING COUPLED WAVE PROPAGATION

Modeling of coupled wave propagation represents a special problem for certain algorithms. Examining Equation 18 we expect the eigenvalues of the system to be grouped into two categories for the two propagating wave types; one in the range of speed of light in the material, the other in the range of seismic wave speeds. This grouping of eigenvalues establishes a scale difference between elastic and electromagnetic wavetypes. This discrepancy poses a problem if we want to propagate both coupled wavetypes at the same time. We basically have to propagate the wavetypes with the higher speed on a grid which is orders of magnitude smaller than the one we use for the slower speed wavetypes. Obviously a finite difference algorithm would be costly and inefficient when propagating the coupled waves on a common space-time grid. A phase shift related algorithm is much more effective, but is restricted to horizontal layers. However, in my opinion, a phase shift algorithm can easily be extended to model or migrate coupled waves in arbitrary anisotropic media. This is a promising future application. In this case the disadvantage of being restricted to horizontal layers is outweighed by the possibility of calculating coupled material constants using the group theoretical approach of Schoenberg and Muir.

COUPLING IN THE VIEW OF SCHOENBERG AND MUIR

The same group theory that applies to elastic media can also be extended to incorporate coupling effects. We can do so by replacing the elastic stiffness matrix with the generalized stiffness matrix in a form which groups coefficients related to normal and tangential components. The elastic stress and strain is thereby extended to the generalized stress and strain. Consequently, in order to calculate the group elements correctly, we have to specify the components of the generalized stress and strain vectors which are continuous across boundaries. In general, when crossing a layer boundary, those components of the generalized stress vector normal to the horizontal layers are continuous and the tangential components of the generalized strain vector are also continuous. This allows us finally to compute average properties of packets of layers efficiently, not only for elastic wave propagation but also for any coupled waves propagating in the medium.

PRACTICAL IMPORTANCE

At first glance there seems to be no need for extending conventional wave propagation measurements to include coupling effects, since just using an isotropic elastic assumption has served well for the purpose of exploration. Introducing more degrees of freedom would apparently be just a nuisance. However industry is now shifting its priorities to making more and more detailed studies of small effects that characterize temporal changes in rock properties. It is quite common to monitor producing areas with 3D surveys, where the geometry is kept the same over a large time period. Such measurements, in general, show only small changes in the amplitudes and velocities of seismic waves. These changes can be produced by a variety of causes: depletion of the reservoir, progression of a subsurface fire front, CO_2 flooding or hydro-fracturing for enhanced oil recovery. Measurements in such an environment, targeted to a certain formation, serve as a basis for accurately estimating rock properties and, more importantly, how these properties change. In this context of minute changes, we need to formulate a description of wave propagation that not only includes primary effects, such as anisotropic elasticity, but also takes secondary (coupled, not second order) effects into account. Enhanced oil recovery and monitoring could benefit from the unified description this paper offers.

Future data acquisition might include electromagnetic waves, as well as seismic waves. Existing purely elastic algorithms could be generalized to accommodate coupled waves.

The examples of quartz and PZT2 indicate that we can expect coupling effects in the range of a few percent of velocity change in naturally occurring rock types. I propose measurements in shale and other metamorhpic rocks containing hydrocarbons, to see how large a piezoelectric effect they show. Shales are produced by high pressure and high temperature metamorphosis, similar to the artificial manufacturing process of PZT2. Several rock types containing quartz in one form or another have been found to show piezoelectricity (Nitsan, 1977; Sobolev et al.,1984). However I could not find any literature that records effects in rocks relevant to hydro carbons.

Moreover the literature discribing the measurement of elastic constants in shale lacks many details that would clarify what kind of thermal, elastic or electric boundary conditions were applied. The measured constants might well include electrical stiffening effects, instead of being purely elastic, as previously assumed. The boundary conditions would have a significant effect if piezoelectricity were present.

CONCLUSIONS

Coupled wave propagation in a solid body is possible to describe using a unified formulation inferred from general principles. In this paper coupling between elastic and electromagnetic and thermal quantities is derived as an example; other quantities describing the dynamic state of a solid body, such as magnetic fields, can easily be included. Since fluid flow is excluded, the theory is completely linear. Using coupled conservative and constitutive equations coupled wave equations can be formulated in the time domain. Transforming in the Fourier domain leads to an eigenvalue problem, the generalized Christoffel equation. By solving this equation, wave speeds and particle motions of waves traveling in the medium can be calculated. As we see from the examples of quartz and Lead-Titanate-Zirconate, 3D slowness surface plots show significant difference in uncoupled and coupled wave propagation behavior. The fact, that boundary conditions give a major effect on coupled wave propagation needs careful consideration, and suggests that coupling phenomena can be observed not only, when measuring all coupled quantities, but also when boundary conditions are specified such that one coupled quantity is eliminated. Such a hidden variable still produces a significant coupling phenomena. This suggests that coupling phenomena could prove to be significant in potential geophysical applications such as enhanced oil recovery and detailed studies of hydrocarbonceous target zones.

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REFERENCES

Auld, B.A., 1981, Acoustic fields in waves and solids: John Wiley & Sons.

Balbachan, M.Y., 1982, Changes in electrical resistivity of rocks as a result of mechanical induction of electrical charge: Doklady Akademii Nauk SSSR, **267**, 1, 65-67.

Dellinger, J., 1989, Some anisotropic modeling examples:, SEP **44**, 63-66.

Herrmann, F., 1984, Thermodynamik; Skriptum zur Vorlesung: Universitaet Karlsruhe.

Lanczos, C., 1970, The variational principle of mechanics: Dover Publications Inc.

Landau, L.D., and Lifschitz, E.M., 1984, Mechanik: Akademie Verlag Berlin.

Nitsan, U., 1977, Electromagnetic emission accompanying Fracture of quartz-bearing rocks: Geophysical Research Letters, **4**, 8.

Nye, J.F., Physical properties of crystals: Clarendon Press, 1985.

Sobolev, G.A. , Demin, V.M., Los, V.F., and Maybuk, Y.Y., 1980, Mechanoelectrical emission by ore bodies: Doklady Akademii Nauk SSSR, **252**, 6, 1353-1365.

Sobolev, G.A., Demin V.M., Narod, B.B., and Whaite, 1984, Test of piezoelectric and pulse-radio methods for quartz veins and base-metal sulfides prospecting at Giant Yellowknife Mine, N.W.T. and Sullivan Mine, Kimberley, Canada: Geophysics, **42**, 12, 2178-2185.

APPENDIX A:

COUPLED WAVE PROPAGATION IN TERMS OF STRAINS

Elastic Field Equation

The derivation of coupled wave equations in terms of elastic strains differs only slightly from the description in terms of elastic displacements. The constitutive equation in terms of strains is

$$\sigma_1 \;=\; \underset{\sim}{\mathbf{c_{11}}}\;\epsilon_1 \;+\; \underset{\sim}{\mathbf{c_{12}}}\;\epsilon_2 \;+\; \underset{\sim}{\mathbf{c_{13}}}\;\frac{\partial}{\partial t}\epsilon_3 \tag{20}$$

Substituting Equation 20 into 5 gives the following wave equation in terms of elastic strains (omitting the source term):

$$\underset{\sim}{\nabla}^{\mathbf{T}}\;\underset{\sim}{\nabla}\;(\;\underset{\sim}{\mathbf{c_{11}}}\;\epsilon_1 \;+\; \underset{\sim}{\mathbf{c_{12}}}\;\epsilon_2 \;+\; \underset{\sim}{\mathbf{c_{13}}}\;\frac{\partial}{\partial t}\epsilon_3) \;=\; \rho\,\frac{\partial^2}{\partial t^2}\epsilon_1 \tag{21}$$

Electro-magnetic Field Equations

The linear law relating electric stress components to components of elastic strain, electric, and thermal displacements is now

$$\sigma_{\mathbf{2}} \;=\; \underset{\sim}{\mathbf{c_{21}}}\;\epsilon_{\mathbf{1}} \;+\; \underset{\sim}{\mathbf{c_{22}}}\;\epsilon_{\mathbf{2}} \;+\; \underset{\sim}{\mathbf{c_{23}}}\,\frac{\partial}{\partial t}\,\epsilon_{\mathbf{3}} \tag{22}$$

Consequently we end up with the following coupled electromagnetic field equations in terms of elastic strains:

$$-\,\underset{\sim}{\mathbf{curl}}\;\;\underset{\sim}{\mu}^{\mathbf{-1}}\;\;\underset{\sim}{\mathbf{curl}}\;(\underset{\sim}{\mathbf{c_{21}}}\;\epsilon_{\mathbf{1}} \;+\; \underset{\sim}{\mathbf{c_{22}}}\;\epsilon_{\mathbf{2}} \;+\; \underset{\sim}{\mathbf{c_{23}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{3}}) \;=\; \frac{\partial^2}{\partial t^2}\epsilon_{\mathbf{2}} \;+\; \underset{\sim}{\kappa_{\mathbf{e}}}\;\frac{\partial}{\partial t}\sigma_{\mathbf{2}} \tag{23}$$

Entropy Field Equation

The linear law relating changes in elastic strain and electric and thermal displacement to changes in thermal stress is

$$\frac{\partial}{\partial t}\sigma_{\mathbf{3}} \;=\; \underset{\sim}{\mathbf{c_{31}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{1}} \;+\; \underset{\sim}{\mathbf{c_{32}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{2}} \;+\; \underset{\sim}{\mathbf{c_{33}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{3}} \tag{24}$$

Using this relationship we find the coupled wave equation for entropy to be

$$\mathbf{div}\;\{\;\underset{\sim}{\kappa_{\mathbf{h}}}\;\;\mathbf{grad}\;(\underset{\sim}{\mathbf{c_{31}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{1}} \;+\; \underset{\sim}{\mathbf{c_{32}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{2}} + \;\underset{\sim}{\mathbf{c_{33}}}\;\frac{\partial}{\partial t}\epsilon_{\mathbf{3}})\;\}\;=\;\frac{\partial^2}{\partial t^2}\epsilon_{\mathbf{3}} \tag{25}$$

The coupled system in terms of strains

Combining Equations 21, 23 and 25, we end up with a coupled system of partial differential equations. In contrast to Equation 18, the wave equation corresponds to the linear relationship between "generalized stresses" and "generalized strains", as given in Equation 7.

$$\begin{array}{rlllll}
-\,\underset{\sim}{\mathbf{K}}^{\mathbf{T}}\;\;\underset{\sim}{\mathbf{K}}\;\;\rho^{-1} & (\underset{\sim}{\mathbf{c_{11}}}\;\;\epsilon_{\mathbf{1}} & +\;\underset{\sim}{\mathbf{c_{12}}}\;\;\epsilon_{\mathbf{2}} & -\;i\;\omega\;\;\underset{\sim}{\mathbf{c_{13}}}\;\;\epsilon_{\mathbf{3}}) & = & \omega^2\;\epsilon_{\mathbf{1}} \\
\underset{\sim}{\mathbf{R}}\;\;\underset{\sim}{\mu}^{\mathbf{-1}}\;\;\underset{\sim}{\mathbf{R}}\;\;-\;i\;\omega\;\;\underset{\sim}{\kappa_{\mathbf{e}}} & (\underset{\sim}{\mathbf{c_{12}}}\;\;\epsilon_{\mathbf{1}} & +\;\underset{\sim}{\mathbf{c_{22}}}\;\;\epsilon_{\mathbf{2}} & -i\omega\,\underset{\sim}{\mathbf{c_{23}}}\,\epsilon_{\mathbf{3}}) & = & \omega^2\;\epsilon_{\mathbf{2}} \\
-\mathbf{Div}\;\;\underset{\sim}{\kappa_{\mathbf{h}}}\;\;\mathbf{Grad} & (-i\;\omega\;\;\underset{\sim}{\mathbf{c_{31}}}\;\;\epsilon_{\mathbf{1}} & -\;i\;\omega\;\;\underset{\sim}{\mathbf{c_{32}}}\;\;\epsilon_{\mathbf{2}} & -i\;\omega\;\;\underset{\sim}{\mathbf{c_{33}}}\;\;\epsilon_{\mathbf{3}}) & = & \omega^2\;\epsilon_{\mathbf{3}}
\end{array} \tag{26}$$

However in this case, the spatial derivative operator in matrix form K has to be in uncompacted notation

$$\underset{\sim}{\mathbf{K}} \;=\; i\begin{pmatrix} k_x & \frac{k_y}{2} & \frac{k_z}{2} & \frac{k_y}{2} & 0 & 0 & \frac{k_z}{2} & 0 & 0 \\ 0 & \frac{k_x}{2} & 0 & \frac{k_x}{2} & k_y & \frac{k_z}{2} & 0 & \frac{k_z}{2} & 0 \\ 0 & 0 & \frac{k_x}{2} & 0 & 0 & \frac{k_y}{2} & \frac{k_x}{2} & \frac{k_y}{2} & z \end{pmatrix} \tag{27}$$

in order to cast the problem in eigenvalue form

$$\underset{\sim}{\mathbf{C}}\ \epsilon \ = \ v^2\ \epsilon, \tag{28}$$

which is the generalized Christoffel equation in elastic strain notation. The eigenvalues ($v^2 \ = \ \frac{\omega^2}{\mathbf{k}^2}$) of this equation are again propagation velocities of plane waves in the medium. The eigenvectors are directly related to the particle motion. With this formulation, in contrast to the displacement formulation, we have to carry out an additional computational step in order to obtain physically meaningful displacements, given the strains for a wave traveling in a particular direction.

Solving for particle motions

Equation 26 is a wave equation that combines elastic strains, electric displacements, and entropy. It describes how, in a given medium, a traveling wavetype is typically neither purely elastic, nor purely electromagnetic, nor purely thermal. Instead, one wave includes all such attributes. However it is useful to term them quasi-elastic when most of the energy is transferred by strain changes, or quasi-electric when most of the energy propagates in form of electromagnetic waves.

Solving the eigenvalue equation 28, we obtain as eigenvalues the propagation velocities for the elastic waves, the electromagnetic and the thermal wave. Although $\epsilon_{\mathbf{1}}$ is a 9 component tensor there are only three non-zero eigenvalues, corresponding to three propagating elastic wavetypes. The electromagnetic part will typically have two non-zero eigenvalues, corresponding to two propagating wavetypes. The eigenvalue for the thermal wave will be complex. Consequently we can also calculate propagation velocities and particle motion for the electromagnetic wavetypes and the heat wave.

In order to calculate the elastic displacements we have to extract the three non-zero eigenvalues (v_1, v_2, v_3) and eigenvectors $(\mathbf{V_1}, ..., \mathbf{W_3})$ from the coupled system. We can combine them as follows:

$$\underset{\sim}{\mathbf{A}} = (\,\mathbf{V_1} \quad \mathbf{V_2} \quad \mathbf{V_3}\,) \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} (\,\mathbf{W_1} \quad \mathbf{W_2} \quad \mathbf{W_3}\,) \tag{29}$$

to produce the purely elastic wave equation which uniquely describes the propagation of strains with the correct (coupled) velocity. This purely elastic equation can be turned into a generalized eigenvalue problem for the elastic displacements u, as follows:

$$\underset{\sim}{\mathbf{K}}\ \underset{\sim}{\mathbf{A}}\ \underset{\sim}{\mathbf{K}}^{\mathbf{T}}\ \mathbf{u} \ = \ \underset{\sim}{\mathbf{K}}\ \underset{\sim}{\mathbf{K}}^{\mathbf{T}}\ \mathbf{u}. \tag{30}$$

The eigenvalues and eigenvectors represent now propagation velocity and particle motion respectively and are physically meaningful.

APPENDIX B: SYMBOLS

The following definitions translate the symbols used in the notation into commonly used terms:

- $\sigma_{\mathbf{1}}$ = elastic stress (second rank tensor)
- $\sigma_{\mathbf{2}}$ = electric stress ("electric field", first rank tensor, vector)
- $\sigma_{\mathbf{3}}$ = heat stress ("Temperature", zeroth rank tensor, scalar)
- $\epsilon_{\mathbf{1}}$ = elastic strain (second rank tensor)
- $\epsilon_{\mathbf{2}}$ = electric strain ("electric displacement", first rank tensor, vector)
- $\epsilon_{\mathbf{3}}$ = heat strain ("Entropy", zeroth rank tensor, scalar)
- $\underset{\sim}{\mathbf{C}}$ = generalized stiffness matrix
- $\underset{\sim}{\mathbf{K}}$ = symmetric spatial derivative matrix

The matrix notation for $\underset{\sim}{\mathbf{K}}$ is

$$\underset{\sim}{\mathbf{K}} \;=\; i \begin{pmatrix} k_x & 0 & 0 & 0 & k_z & k_y \\ 0 & k_y & 0 & k_z & 0 & k_x \\ 0 & 0 & k_z & k_y & k_x & 0 \end{pmatrix} \tag{31}$$

The matrix formulation for **curl** is

$$\underset{\sim}{\mathbf{R}} \;=\; i \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \tag{32}$$

Grad and **Div** are denoted as

$$\mathbf{Grad} \;=\; i \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \tag{33}$$

$$\mathbf{Div} \;=\; i \begin{pmatrix} k_x & k_y & k_z \end{pmatrix} \tag{34}$$

APPENDIX C

DIMENSIONALITY - PIEZOELECTRICIY

To avoid confusion about dimensionality of quantities, I list the ones important for piezoelectricity. One can easily verify the correctness of the wave equations and the consistency of other relations. All quantities are given in SI units. Some useful relations to check out the consistency:

- $C = A\ s$
- $Joule = V\ A\ s\ = N\ m$
- $N\ = \frac{kg\ m}{s^2}$
- $\sigma_{\mathbf{1}}$ in $[\frac{N}{m^2}]$
- $\sigma_{\mathbf{2}}$ in $[\frac{V}{m}]$
- $\epsilon_{\mathbf{1}}$ $dimensionless$
- $\epsilon_{\mathbf{2}}$ in $[\frac{C}{m^2}]$
- $\utilde{\mathbf{c_{11}}}$ in $[\frac{N}{m^2}]$
- $\utilde{\mathbf{c_{22}}}$ in $[\frac{V\ m}{C}]$
- $\utilde{\mathbf{c_{21}}}$ in $[\frac{V}{m}]$ (piezoelectric stress constants times inverse electrical permittivity)
- $\utilde{\mathbf{c_{12}}} = \utilde{\mathbf{c}}^{\mathbf{T}}_{\mathbf{21}}$
- eps_0 in $[\frac{C}{Vm}]$
- $\utilde{\kappa_{\mathbf{e}}}$ in $[\frac{A}{Vm}]$
- $\utilde{\mu}$ in $[\frac{Vs}{Am}]$
- ω in $[\frac{1}{s}]$
- $\utilde{\mathbf{K}}$ in $[\frac{1}{m}]$
- $\utilde{\mathbf{R}}$ in $[\frac{1}{m}]$

For example, we can verify the dimensions of the system of equations 18, omitting thermal effects, by substituting from the list as follows:

$$\begin{pmatrix} \frac{m^2}{kg} & 0 \\ 0 & \frac{A}{V\ m\ s} \end{pmatrix} \begin{pmatrix} \frac{N}{m^3} & \frac{V}{m} \\ \frac{V}{m^2} & \frac{V\ m}{C} \end{pmatrix} \begin{pmatrix} - \\ \frac{C}{m^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{s^2 m} & \frac{C}{m^3\ s^2} \\ \frac{m^2}{C\ s^2} & \frac{1}{s^2} \end{pmatrix} \begin{pmatrix} - \\ \frac{C}{m^2} \end{pmatrix} = \frac{1}{s^2} \begin{pmatrix} - \\ \frac{C}{m^2} \end{pmatrix} \tag{35}$$

If the whole system is divided by $|\mathbf{k}|^2$ then the eigenvalues have the dimension $[\frac{m}{s}]$, which is the propagation velocity.

APPENDIX D

The parameters I used to calculate the slowness surfaces are as follows:

QUARTZ

trigonal, symmetry class 32

density

$$\rho \;=\; 2651\frac{kg}{m^3}$$

stiffness coefficients

$$\underset{\sim}{\mathbf{C}}[\frac{10^{10}N}{m^2}] = \begin{pmatrix} 8.674 & 0.699 & 1.191 & -1.791 & . & . \\ & 8.674 & 1.191 & 1.791 & . & . \\ & & 10.72 & . & . & . \\ & & & 5.794 & . & . \\ & & & & 5.794 & -1.791 \\ & & & & & 3.988 \end{pmatrix}$$

piezoelectric stress coefficients

$$\underset{\sim}{\mathbf{d}}\ [\frac{C}{m^2}] = \begin{pmatrix} 0.171 & -0.171 & . & -0.0436 & . & . \\ . & . & . & . & 0.0436 & -0.171 \\ . & . & . & . & . & . \end{pmatrix}$$

electrical permittivity coefficients

$$\underset{\sim}{\mathbf{eps}}\ [10^{-12}\frac{C}{Vm}] = \begin{pmatrix} 39.8438 & . & . \\ & 39.8438 & . \\ & & 40.7293 \end{pmatrix}$$

magnetic permeabilty coefficients

$$\underset{\sim}{\mu}\ [10^{-6}\frac{Vs}{Am}] = \begin{pmatrix} 1.2566 & . & . \\ & 1.2566 & . \\ & & 1.256 \end{pmatrix}$$

LEAD TITANATE-ZIRCONATE

uniaxial, hexagonal, symmetry class 6mm

density

$$\rho \;=\; 7600\frac{kg}{m^3}$$

stiffness coefficients

$$\underset{\sim}{\mathbf{C}}[\frac{10^{10}N}{m^2}] = \begin{pmatrix} 13.5 & 6.79 & 6.81 & . & . & . \\ & 13.5 & 6.81 & . & . & . \\ & & 11.3 & . & . & . \\ & & & 2.22 & . & . \\ & & & & 2.22 & . \\ & & & & & 3.355 \end{pmatrix}$$

piezoelectric stress coefficients

$$\underset{\sim}{\mathbf{d}}\ [\frac{C}{m^2}] = \begin{pmatrix} . & . & . & . & 9.8 & . \\ . & . & . & 9.8 & . & . \\ -1.9 & -1.9 & 9.0 & . & . & . \end{pmatrix}$$

electrical permittivity coefficients

$$\underset{\sim}{\mathbf{eps}}\ [10^{-9}\frac{C}{Vm}] = \begin{pmatrix} 4.46 & . & . \\ & 4.46 & . \\ & & 2.301 \end{pmatrix}$$

magnetic permeabilty coefficients

$$\underset{\sim}{\mu}\ [10^{-6}\frac{Vs}{Am}] = \begin{pmatrix} 1.2566 & . & . \\ & 1.2566 & . \\ & & 1.256 \end{pmatrix}$$