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Anisotropic finite-difference traveltimes

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ABSTRACT

Finite-difference traveltime methods are an efficient way to calculate first arrival times at every point in a complex heterogeneous isotropic model. We show that Van Trier and Symes' algorithm can be extended to include anisotropy by replacing the isotropic dispersion relation inherent in their method with an anisotropic one. Unfortunately, the upwind method they use is difficult to implement in the anisotropic case, because it requires high-accuracy solutions of derivatives of complex anisotropic dispersion relations. For our preliminary results we use instead an easy-to-implement fixed-grid finite-difference method. This method works well in homogeneous anisotropic models, but seems to have trouble in heterogeneous ones.

INTRODUCTION

Vidale (1989) first introduced a method for calculating traveltimes using a finite-difference method. His method is efficient, but somewhat cumbersome to implement. Vidale's paper inspired Van Trier and Symes (1990) to introduce a related method that also calculates finite-difference traveltimes but is more streamlined and straightforward.

Neither method allows for anisotropy as published. Vidale's method is based directly on plane wave propagation, and so adding anisotropy to that algorithm is trivial. (In practice it is not so simple, because the oddly-shaped computational fronts in his method make error analysis difficult.)

We will analyze the derivation of Van Trier and Syme's method and show how their method is based on the isotropic dispersion relation. It is then a simple matter to recast their algorithm to use an anisotropic dispersion relation.

For these preliminary results, we use the two-step Lax-Wendroff finite-difference method instead of the upwind method presented in their paper.

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ADDING ANISOTROPY

Theory

The isotropic eikonal equation is

$$\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial z}\right)^2 = s^2, \tag{1}$$

where $\tau(x,z)$ gives the traveltime of some event at each point in space, and $s(x,z)$ gives the slowness (inverse velocity) at each point in space. Van Trier and Symes (1990) start from this equation (equation (1) in their paper) and derive a traveltime extrapolation equation.

What does equation (1) represent? Figure 1 shows a tiny portion of a wavefront at two closely-spaced instants in time. On this scale, the wavefront can be approximated as a plane wave. The plane wave is moving along in some arbitrary direction with a velocity V, so it travels a distance $V\Delta T$ between the two times. At the same time, the plane wave also moves a distance ΔX along the X axis and ΔZ along the Z axis.

The triangle to the lower right of Figure 1 has sides of length ΔX and ΔZ meeting at a right angle, and a height of $V\Delta T$ (if the hypotenuse is considered the base). Three applications of Pythagoras' theorem (one for each of the three right triangles involved) and a little algebra show that

$$\left(V\Delta T\right)^2 = \frac{\Delta X^2 \Delta Z^2}{\left(\Delta X^2 + \Delta Z^2\right)} \tag{2}$$

and thus

$$\left(\frac{\Delta T}{\Delta X}\right)^2 + \left(\frac{\Delta T}{\Delta Z}\right)^2 = \frac{1}{V^2} = s^2. \tag{3}$$

Hence, equation (1) does have a simple geometrical meaning.

How do we make equation (1) anisotropic? From the previous two paragraphs, we can see that $\partial\tau/\partial x$ in equation (1) (or $\Delta T/\Delta X$ in equation (3)) is just the horizontal phase slowness. For a monochromatic plane wave this is k_x/ω. Similarly, $\partial\tau/\partial z = k_z/\omega$. This lets us rewrite equation (1) as a dispersion relation:

$$\left(\frac{k_x}{\omega}\right)^2 + \left(\frac{k_z}{\omega}\right)^2 = s^2. \tag{4}$$

(This is just the Fourier-transformed homogeneous wave equation. Fourier-transformation breaks the wavefield up into plane-wave components.)

Recast as a dispersion relation, $F(u)$ in equation (4) of Van Trier and Symes has a simple interpretation: u is a horizontal phase slowness k_x/ω, and $F(u)$ is a vertical phase slowness k_z/ω. Replacing the isotropic dispersion relation in their paper with

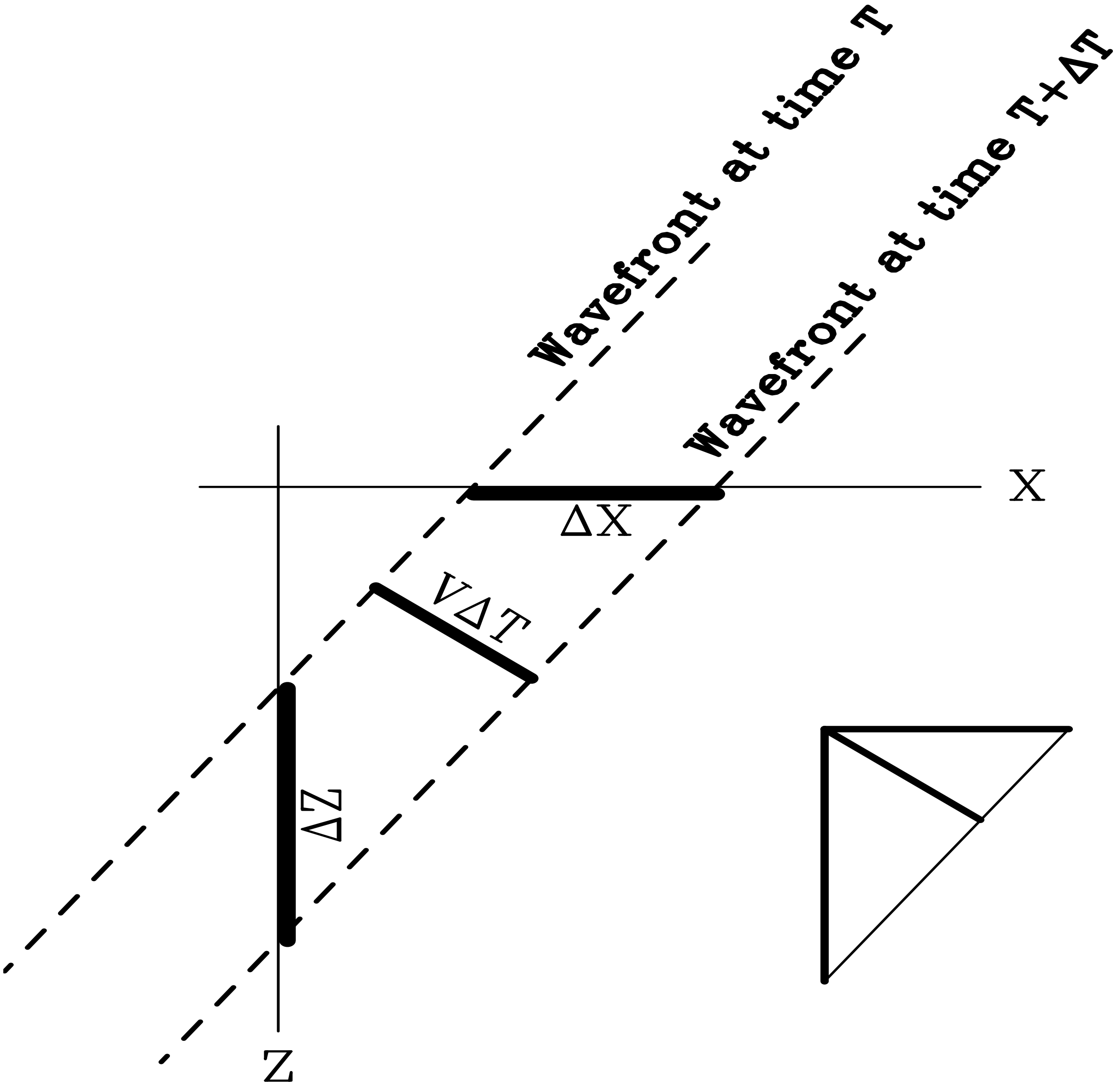

Figure 1: Geometrical properties of plane-wave propagation. joe2-planar [NR]

an appropriate anisotropic one gives an anisotropic traveltime extrapolation equation. Unfortunately, solving for k_z as a function of k_x can be difficult for some anisotropic media. In the general case of arbitrary three-dimensional anisotropy it can require solving a sixth order polynomial.

Implementation

The isotropic $F(u)$ in Van Trier and Symes is $\sqrt{s^2 - u^2}$ in cartesian coordinates and $\sqrt{s^2 - (u/r)^2}$ in polar ones. For our examples we shall restrict ourselves to transverse isotropy with a vertical axis of symmetry, one of the simplest types of anisotropy.

$F(u)$ in this case for "simple" cartesian coordinates is

$$F_{\mathrm{P}}(u) = (-\sqrt{A} + B)/2$$

for qP waves and

$$F_{\mathrm{S}}(u) = (\sqrt{A} + B)/2$$

for qS waves, where

$$A = \frac{-4(\mathrm{C}_{11}\mathrm{C}_{44}u^4 - (\mathrm{C}_{11} + \mathrm{C}_{44})u^2 + 1)}{\mathrm{C}_{33}\mathrm{C}_{44}} + \left(\frac{-(\mathrm{C}_{13}^2 u^2) + \mathrm{C}_{11}\mathrm{C}_{33}u^2 - 2\mathrm{C}_{13}\mathrm{C}_{44}u^2 - (\mathrm{C}_{33} + \mathrm{C}_{44})}{\mathrm{C}_{33}\mathrm{C}_{44}}\right)^2 \tag{5}$$

and

$$B = \frac{(\mathrm{C}_{13}^2 u^2) - \mathrm{C}_{11}\mathrm{C}_{33}u^2 + 2\mathrm{C}_{13}\mathrm{C}_{44}u^2 + (\mathrm{C}_{33} + \mathrm{C}_{44})}{\mathrm{C}_{33}\mathrm{C}_{44}}.$$

In principle, there is no theoretical obstacle to extrapolation in polar coordinates, but in that case $F(u)$ becomes much worse than equation (5). For that reason, we have remained with the simpler cartesian extrapolation method for our preliminary examples here. (Of course we could have followed Vidale's lead and used expanding square calculation fronts. However, this would have put "corners" of the calculation fronts right where we wanted to model shear triplications. We decided to stick with simple extrapolation in z for now, so we know the strange wavefront shapes are from the anisotropy, not the finite-difference method.)

Again, in principle, there should be no objection to using the upwind finite-difference method that worked so well in Van Trier and Symes. In practice the upwind method has proven surprisingly recalcitrant. One possible reason is that the quantity $\bar{u}$ required by the upwind method (it is defined by $F'(\bar{u}) = 0$) is typically quite complicated, both in behavior and to calculate. (In the isotropic case it is ridiculously easy: $\bar{u} = 0$.)

For the examples in this paper we use the two-step Lax-Wendroff method. This is a fixed-stencil second-order method (Press et al., 1988). Theoretically this second-order method should be more accurate than the first-order upwind method, but it is difficult to predict the behavior of various finite-difference methods in practice. Symes suggests that any fixed-stencil method should be unstable for this application (Symes, 1990).

EXAMPLES

A homogeneous isotropic example

Figure 2 shows three examples of finite-difference traveltimes in homogeneous media. For the moment limit your attention to the top plot, which shows an isotropic medium. The wavefronts in this example are nearly circular, but they are not as accurate as the example in Figure 3 of Van Trier and Symes (page 49 of this report).

There are several reasons for this. We used extrapolation in z instead of r, which is not as accurate for near-circular wavefronts. It especially has trouble at the top of the model, where the extrapolation is (almost) at right angles to the wave propagation direction. Calculating the initial boundary conditions is harder when extrapolating in z, too. Our model shows a "bump" directly under the source. This is due to the initial discontinuity in $u = \partial\tau/\partial x$ at the source. The initial error at $z = 0$ is propagated along when we integrate $F(u) = \partial\tau/\partial z$ over z to get the final result $\tau(x, z)$, causing the vertical line of "bumps" under the source.

Two homogeneous anisotropic examples

How does the method work for homogeneous anisotropic models? The middle plot in Figure 2 shows a somewhat anisotropic model. The isotropic plot on top shows the SV wave for $C_{11} = 1.$, $C_{33} = 1.$, $C_{55} = .3$, and $C_{13} = .4$. For the middle plot, we set $C_{13} = .3$. The theoretical and calculated curves for this model overlay one another about as well as they did for the isotropic example.

The bottom plot in Figure 2 is more interesting. Here we have set $C_{13} = 0.$, resulting in an extremely anisotropic model. The homogeneous qSV wave has a multi-valued traveltime. The traveltime extrapolation method cannot be multivalued, and instead "cuts off" the triplication in an interesting way.

A heterogeneous example

Figure 3 shows a heterogeneous model example. The left and top parts of the model are isotropic (the same elastic constants as the isotropic example in Figure 2), while the block to the right is anisotropic with $C_{11} = .75$, $C_{33} = .75$, $C_{55} = .225$, and $C_{13} = 0.$. (This is just a slower version of the "strongly anisotropic" medium in Figure 2.)

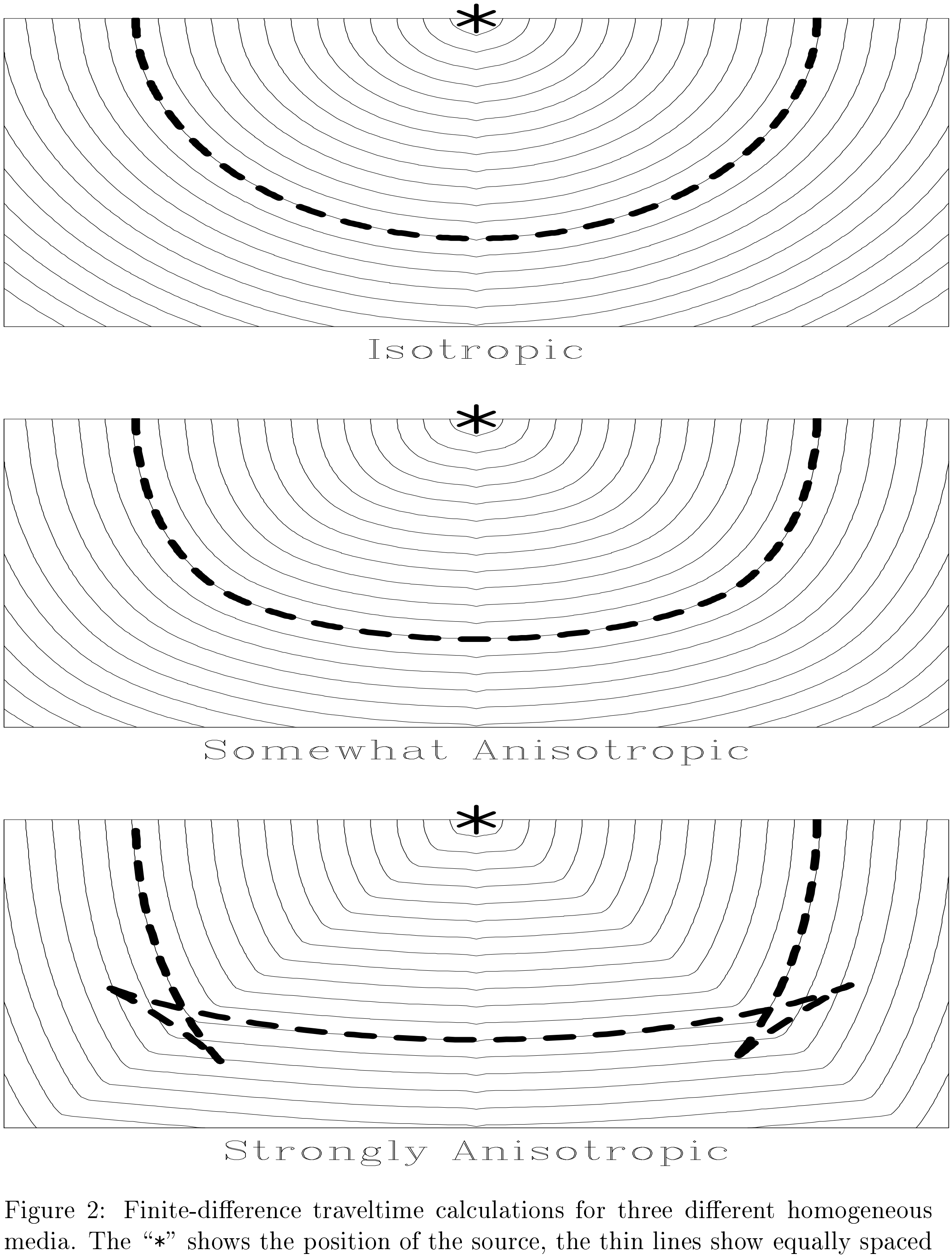

Figure 2: Finite-difference traveltime calculations for three different homogeneous media. The "*" shows the position of the source, the thin lines show equally spaced traveltime contours, and the thick dashed curve shows a theoretical contour for comparison. joe2-homo [NR]

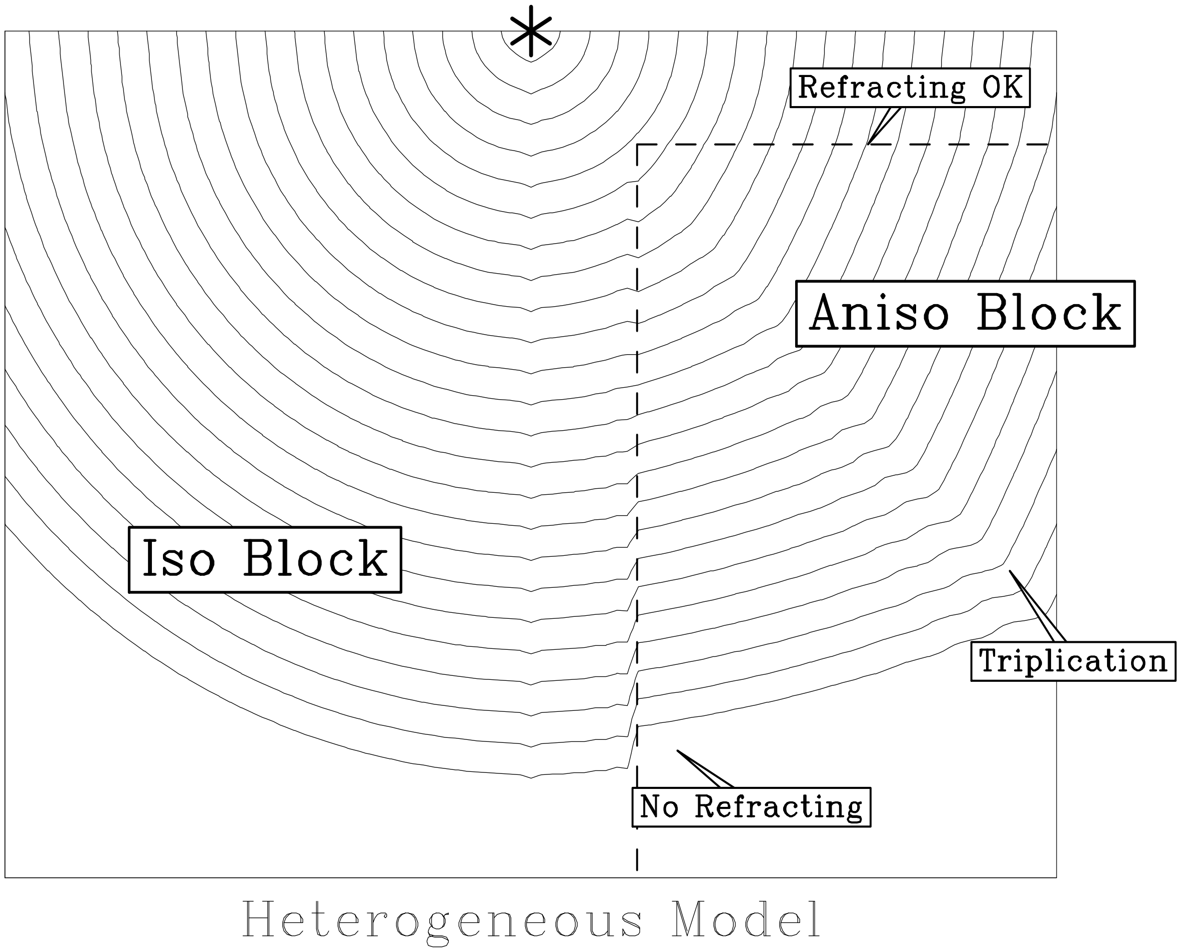

Figure 3: A heterogeneous example. The top and left part of the model are isotropic. The block on the right is very anisotropic. joe2-hetero [NR]

Along the top edge of the anisotropic block in Figure 3 (marked by the horizontal dashed line), the wavefronts correctly refract into the anisotropic block. Inside the anisotropic block, the waves correctly start to triplicate. Along the left edge of the anisotropic block, however, something strange happens. The wave inside the block starts out slightly faster than the direct wave in the isotropic part of the model. Towards the bottom of the block the direct isotropic arrival becomes faster. This is not strange; it is a correct result of the anisotropy in the model. What is disturbing is that the contoured traveltimes do not show the fastest arrivals, but the "direct" arrivals. There should be waves refracting across the vertical boundary; instead the wavefronts on either side have become decoupled.

DISCUSSION

Currently it is not clear what this result means. Symes (1990) has suggested that fixed-stencil methods like the one used here (in contrast to upwind methods like the one in Van Trier and Symes) cannot track first arrivals across velocity discontinuities properly. Perhaps that is what is happening here, but Symes suggests that such methods should also necessarily be unstable. This method does appear to be stable, it just doesn't always follow the *first* arrival. If Symes is right, switching to upwind finite-differences will probably resolve the problem evidenced in Figure 3.

Referring back to the bottom plot in Figure 2, we see the traveltime contours also did not follow the leading branch of the shear wave triplication in the homogeneous case. Would upwind finite-differences have behaved the same way? It isn't clear. Triplicating shear waves don't even really *have* a well-defined "first arrival"—precursive energy reaches out all the way to the P wave.

In any case, it appears that calculating anisotropic finite-difference traveltimes is possible, if not quite as straightforward to implement as one might hope.

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