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Putting Schoenberg–Muir to the test

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ABSTRACT

The Schoenberg-Muir averaging technique is a powerful method for calculating the bulk properties of layered and fractured elastic media. The theory is exact only for infinite layers and infinitely low frequency waves. It is already known that the approximations are accurate for infinite flat layers if the wavelength of the slowest elastic wave is long enough to contain a representative sample of the layers at any position in the medium. In this paper we run models testing the infinite flat layers assumption. We find that if the layers are much longer than they are thick this approximation is a good one and Schoenberg-Muir averaging is applicable.

INTRODUCTION

Schoenberg and Muir (1989), present an efficient technique for finding the *homogeneous* bulk equivalent of a *heterogeneous* layered medium. This S&M technique is finding wider and wider use in predicting not only the bulk behavior of layered media, but the effects of adding sets of cracks to uncracked background media as well. Because this technique is so convenient, it is tempting to apply it without verifying that the underlying approximations are applicable. In a strict sense, the technique is on firm theoretical footing only when it is used to predict the behavior of a flat, infinite stack of layers (possibly with some of the layers representing "cracks") under a static traction.

In practice, it is known that if the waves propagating through the stack have sufficiently long wavelengths the zero-frequency approximation can be very accurate. The problem of what "sufficiently long wavelength" means in practice has been investigated by Nichols (1988). He concluded that if the medium were approximately uniform in the distribution of types of layers on the scale of a wavelength, the wave would not "feel" the layers. What about the "flat infinite layer" assumption? This assumption is commonly broken when the S&M method is used to model truncated dipping layers (such as those in Figure 1), cracks cross-cutting layers, or non-parallel

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sets of cracks.

The purpose of this paper is to test how well the method works when applied to truncated dipping layers. For our models we use an annulus of layered media embedded in its S&M homogeneous equivalent. The source is in the center of the annulus. We compare the wavefields after propagation through the annulus with the wavefields in the corresponding homogeneous model.

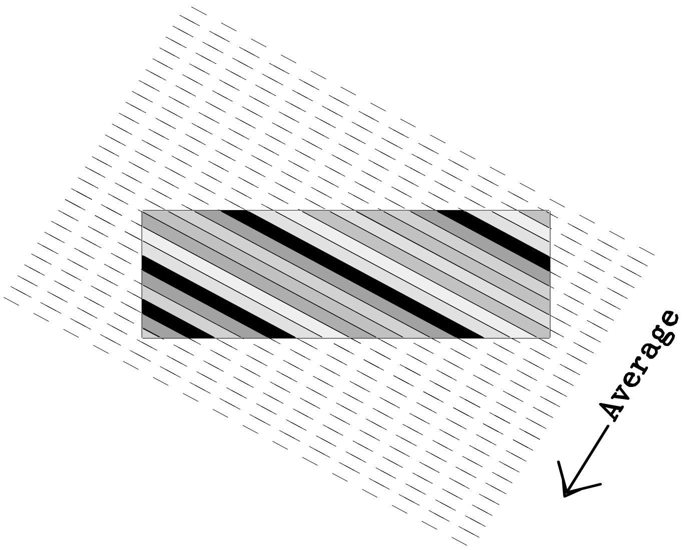

Figure 1: Schoenberg-Muir averaging for canted layers. S&M averaging assumes stationarity in the direction of the arrow, and infinite layers in the perpendicular direction. The truncated layers in the shaded block violate these assumptions. joe1-layers [NR]

SCHOENBERG-MUIR THEORY

Suppose we have a flat stack of infinite elastic layers, and apply some static traction across the top and bottom surfaces. Schoenberg and Muir's layer averaging method tells us how to find the equivalent homogeneous medium that exactly mimics our original stack. (This is really a glorified elastic version of the standard physics problem of finding out the effective spring constant when two different springs are connected end-to-end.) Although the Schoenberg-Muir method is exact for static tractions, it should also approximately apply to elastic waves if their wavelength is sufficiently large. There are two reasons why long wavelengths are required. First, the time scale of the wave passing by must be sufficiently slow that the medium has time to adjust in a quasi-static way. Second, each wavelength must "feel" a good representative sample of the set of layers.

As an example of S&M layer-averaging, Figure 2 shows two snapshots of qS waves propagating away from a point source. The model on the left is *not* anisotropic, despite the prominent shear-wave triplication visible. It consists of 512 layers; each individual layer is chosen randomly from the set of 11 possible isotropic media listed in the table in Appendix 1. (Note that this is rather an extreme test; the P velocity, S velocity, and density all vary by a factor of 2 between the extremal layers. We used 11 layers so that there was little chance the same elastic constants would be picked several times in a row, resulting in a "thick" layer.) The model on the right uses the homogeneous S&M medium equivalent to the stack of layers on the left. The shear waves are high enough in frequency to "feel" the layers in the left model to some extent (especially the occasional double-thick ones), so there is some scattered energy, but otherwise the wavefields in the layered and homogeneous models are nearly identical. The two results cannot be subtracted because they show *displacement* instead of *energy*, which depends strongly on the elastic constants of the medium.

Why did we choose this particular model? The S&M calculus is quite general, and we could have chosen layers with a high degree of anisotropy. Instead we elected to use isotropic layers. There are many people who believe that anisotropy is a consequence of heterogeneity, without any intrinsic quality. We wished to demonstrate that triplication is a normal, geometric consequence of layering, and not a special artifact of a homogeneous anisotropic medium.

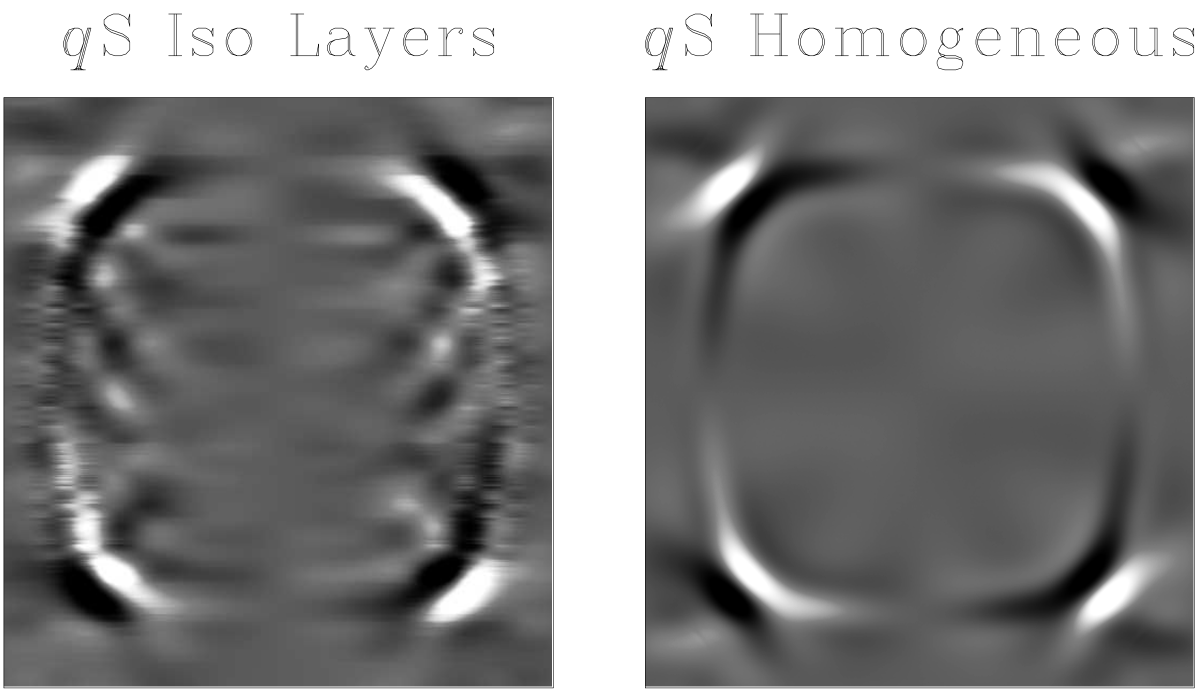

Figure 2: Left: Wave propagation in a layered medium. There are 512 layers from top to bottom, chosen randomly from a set of 11. Right: The homogeneous S&M medium equivalent to the stack of layers on the left. For both models, the source is a vertical force at the center. The X component of displacement is shown. The wavelength of the qS waves is about 75 gridpoints, which is not quite enough for a good average if the waves are traveling nearly parallel to the layers. joe1-carlos [NR]

Running rings around Schoenberg-Muir

S&M layer averaging works for waves propagating in a stack of flat layers at any angle (although as we can see in Figure 2, the zero-frequency approximation can't be pushed as far for waves propagating *along* the layers as it can for waves propagating *across* them).

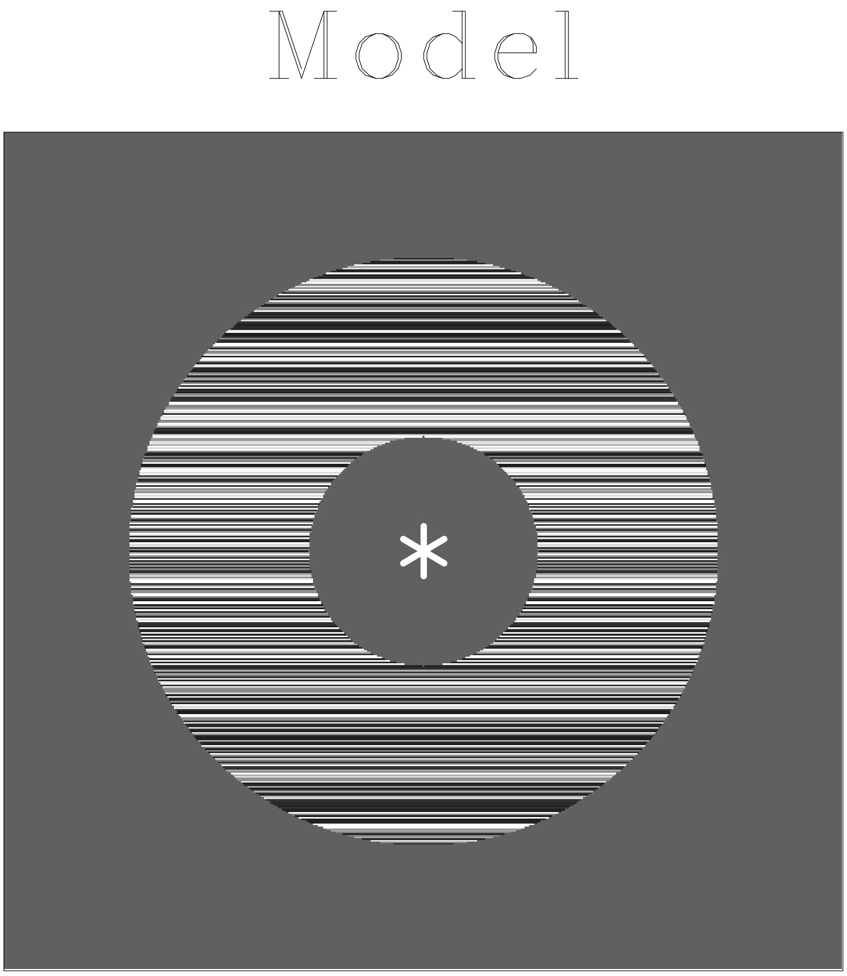

Figure 3: A typical model. The intensity on this plot shows the value of C_{13} at each gridpoint. The model is 512^2 gridpoints. The ring is 110 gridpoints across from inner to outer edge. The source is a vertical point force at the "$*$". joe1-model [NR]

What about a stack of dipping layers? Schoenberg-Muir theory says nothing about the boundary conditions on the ends of the layers; the math assumes infinite flat layers so that the troublesome ends are infinitely far away and can be ignored. Oblique stacks of layers, like a canted row of dominos, violate this assumption because the waves must encounter the *edges* of the layers to enter the stack. (See Figure 1.) Does this violation of the underlying assumptions cause noticeable effects? Figure 3 shows a 512^2 gridpoint model consisting of a annulus-shaped set of layers embedded in its homogeneous S&M equivalent medium. (The layers are chosen the same way as in Figure 2.) S&M theory tells us that if the wavelength of the source is sufficiently long, and the edge effects of the layers are ignored, the waves should emerge from the stack looking the same as they would if the medium were homogeneous everywhere. Figure 4 shows the actual results; the qP wave is low enough in spatial frequency to work nearly perfectly for all angles of propagation. The main difference is that the amplitude of the qP wave in the layered model is about 10% lower. The edge effects do not seem to be significant in this model.

Raising the frequency

Perhaps the edge effects become more pronounced at higher frequencies? Figure 5 shows the same model as in Figure 4, but uses a source with twice the dominant frequency. Although the results look pretty good, there is a significant velocity error, especially for the qS waves. Subtracting the two wavefields in this case is not very informative because of the resulting phase differences. The overall shapes of the wavefronts are still correct, though.

The qP wavelength in this example is about 65 gridpoints. By Nichols' rules of thumb (1988), this would seem to be more than enough of a distance to average over, since there are only 11 kinds of layers, each one gridpoint thick. Probably this difference in results is because we used a much more extreme variation between layers in our examples.

We know the S&M approximations break down when the layers are too thick compared to the wavelengths of the waves passing through, as in Figure 5. Muir has conjectured that perhaps S&M gives an answer that is correct in expectation in such cases. Figure 6 tests this hypothesis. To make it, we ran 20 models similar to that in Figure 5, but with independently chosen sets of random layers. Because of the symmetry of the problem, we averaged in each result and each result with the Z axis reversed, to effectively double the number of models at no additional cost. The results are much cleaner, but there is still a slight phase shift that did not average out. This is especially noticeable for the higher-frequency qS wave. (This suggests that it might be useful to work on extending S&M theory to non-zero frequencies.)

DISCUSSION

Some conclusions

Schoenberg-Muir averaging offers a clean, straightforward method of calculating bulk properties of layered and fractured media. It is tempting to apply the method to arbitrary stacks of layers, oblique layers, and fractures, even though these cases are not really covered by the theory.

We have shown that oblique layers, at least, seem to work pretty well anyway.

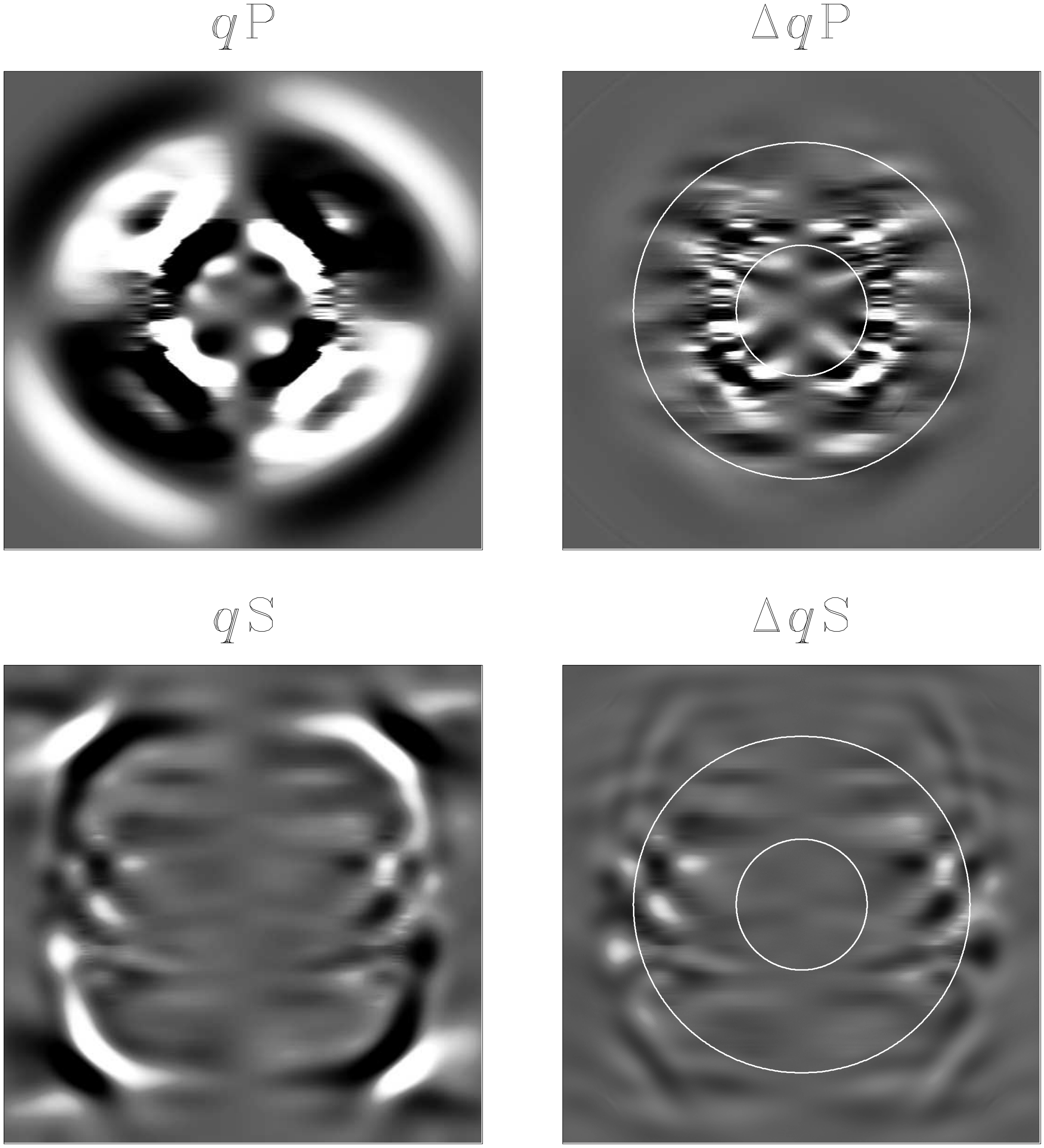

Figure 4: Left: Snapshots of waves propagating in the annulus model shown in Figure 3. The model is 512^2 gridpoints, the annulus is 110 gridpoints wide, the qP waves have a wavelength of about 135 gridpoints, and the qS waves have a wavelength of about 75 gridpoints. Right: The difference between waves in the S&M equivalent homogeneous model and the annulus model on the left. The two white circles mark the edges of the heterogeneous part of the model, where the differing underlying elastic constants and density between the two models makes subtracting displacements meaningless. $\boxed{\text{joe1-blob}}$ [NR]

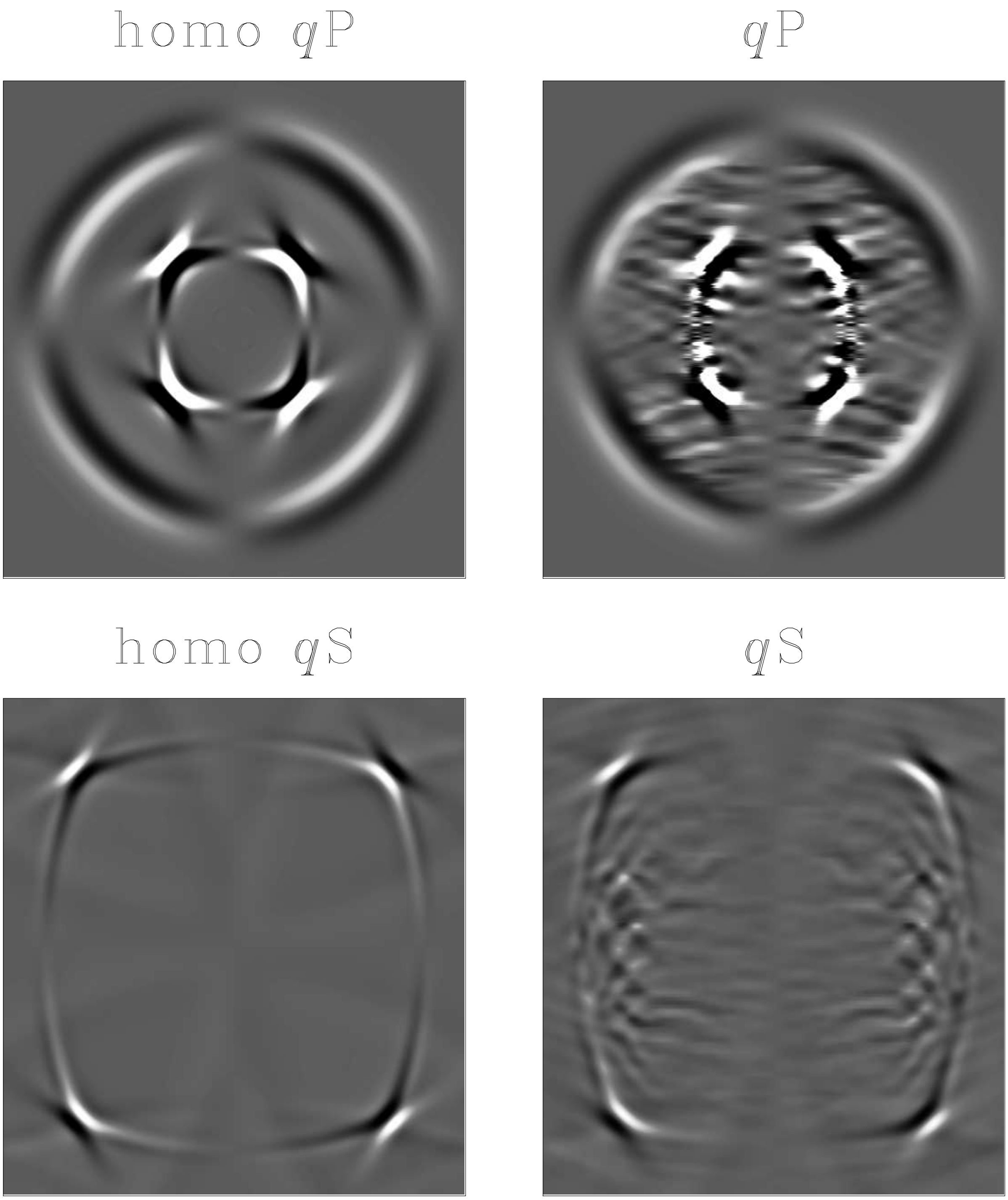

Figure 5: Left: Wave propagation in the homogeneous S&M medium equivalent to the stack of layers on the right. Right: Wave propagation in the same layered medium as in Figure 4, but with a higher frequency source. The qP waves have a wavelength of about 65 gridpoints, the qS about 35. joe1-twelve [NR]

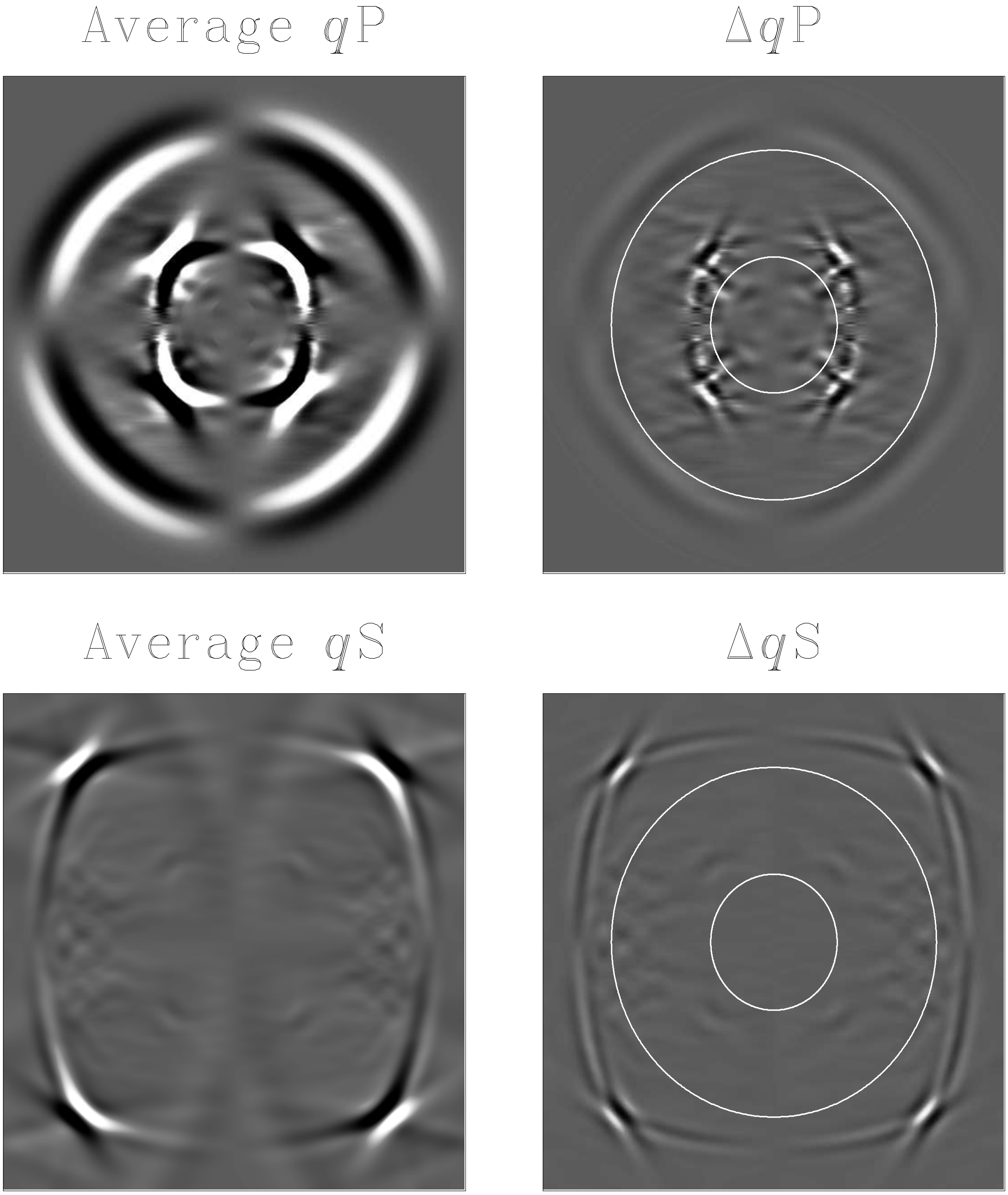

Figure 6: Left: The results of averaging 40 wavefields like the one in Figure 5. Right: The difference between the average wavefield on the left and the wavefield in the homogeneous S&M equivalent model. joe1-sum [NR]

Work remains to be done to quantify how long the layers need to be relative to their width before the "infinite layer" approximation breaks down. (In the limit of "layers" consisting of single gridpoints, we know that the result of averaging a set of isotropic media must be isotropic.) Our results also indicate that the sensitivity of the S&M approximations to layer thickness depends on the contrast between the layers. More work also remains to quantify this effect.

A possible modeling application of S&M

A constant irritation in computer modeling is that continuous and smooth boundaries between regions have no obvious counterpart in a space-sampled world. S&M may provide a simple solution. Assign an area of influence to each point. Where a region boundary crosses such an area, chose as elastic constants for that point those of the equivalent medium that honors the direction of the boundary, and the relative proportions into which the area is divided.

Where do we go from here?

At first sight the notion of an equivalent model that is exact only at zero frequency is not very satisfying. Our branch of Seismology deals with dynamic and kinematic problems, not with static ones. Fortunately it appears that S&M is both statically and kinematically correct. As far as we know this is not proven, but there are reasonable hand-waving arguments in its behalf. By "kinematically correct" we mean that in the far field, the center (defined in some reasonable way) of the wavelet should travel with the velocity predicted by S&M. A suggested line of argument is via one of the Abel/Tauber theorems that tie behavior at infinity in the Time Domain to behavior around the origin in the Frequency Domain (and vice versa).

The extension of S&M to Dynamic correctness looks difficult but doable. There are two ideas worth pursuing. The first is to recognize that in a randomly layered medium, energy is continuously converted from coherent to scattered. Since we are interested in the behavior of the coherent energy, this suggests equivalencing the heterogeneous, scattering medium with a homogeneous, visco-elastic one. In both cases energy is lost out of the coherent system, and the analogy is sufficient for many of our purposes. The second idea is to employ an ergodic principle, and instead of averaging over layers as the S&M paper did, average over realizations as we did in Figure 6.

APPENDIX

Layer	C_{11}	C_{13}	C_{33}	C_{44}	ρ
1	0.48	0.32	0.48	0.15	0.75
2	0.50	0.39	0.50	0.10	0.70
3	0.68	0.31	0.68	0.36	0.90
4	0.88	0.36	0.88	0.51	0.80
5	1.12	0.72	1.12	0.40	1.25
6	1.27	1.03	1.27	0.24	1.00
7	1.33	0.66	1.33	0.66	1.10
8	1.43	0.84	1.43	0.58	1.05
9	1.48	0.68	1.48	0.80	0.95
10	1.65	0.82	1.65	0.82	1.15
11	1.71	1.06	1.71	0.65	1.30
S&M Average	1.07	0.55	0.94	0.32	0.99

Table 1: Elastic constants for the 11 isotropic layers used in the layered models in this paper.

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