THE DOUBLE SQUARE ROOT EQUATION

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The double square root equation is fundamental to migration. It seems that all current industrial migration techniques, be they finite difference, frequency domain, or Kirchoff summation are approximate implementations of it. It is also a good takeoff point for velocity analysis, both in stratified media and when the velocity is laterally variable. The imaging principle most often used in reflection seismology is that a reflector exists at any place where echoes arrive with zero traveltime. This, of course, implies that the source and receiver have zero offset and are located down at the reflector. Thus, to image the earth it is necessary to mathematically extrapolate both the shots and the receivers into the earth. The equation which does this is called the double square root equation. As will be shown, one square root is for downward extrapolation of the shots, the other is for the geophones. The double square root equation is exact in the sense that there are no error terms dependent on dip angle, offset angle, or departure of the stratification velocity $v(z)$ from a constant. Approximations in the development will suppress multiple reflections and shear waves and will prevent exact treatment of horizontal velocity variation. We begin with the scalar wave equation, which already contains some approximations.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) P = 0 \tag{1}$$

Sinusoidal trial solutions such as

$$\exp\ (-i\omega t + ik_x x + ik_z z) \tag{2}$$

reduce any linear constant coefficient partial differential equation to its *dispersion relation*, which in this case is the equation of a circle

$$k_x^2 + k_z^2 = \frac{\omega^2}{v^2} \tag{3a}$$

In normalized form this is

$$\left(\frac{vk_x}{\omega}\right)^2 + \left(\frac{vk_z}{\omega}\right)^2 = 1 \tag{3b}$$

The Fourier transform variable dual to the z-axis, namely k_z, is the spatial frequency (inverse wavelength) of the wave along the z-axis and ω/v is the spatial frequency along the ray. Simple geometry shows that if θ is the angle from the vertical to the ray then Equation (3b) is equivalent to the trigonometric relation

$$\sin^2\theta + \cos^2\theta = 1 \tag{3c}$$

Our observations are almost always in the (x,t) plane which we may double Fourier transform into the (k_x,ω) plane. Since we never have observations in (x,z) space and that is the space in which we are really interested, we use Equation (3) to solve for k_z.

$$k_z = \pm\left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2} \tag{4}$$

The plus sign is for downgoing waves, and the minus sign is for upgoing waves. [This meaning of the sign is determined by holding phase constant in (2) and observing the relation between z and t at constant x.]

So our raw data which was given in (x,t) space has been converted to (k_x,ω) space by Fourier transform and may now be converted to (k_x, k_z) space by (4). Finally we inverse transform to (x,z) space where the migrated section is seen at zero traveltime t. The method just described is known as the Stolt method. The goal of the remainder of the paper is to fill in some essential details, to clarify the role of shot to geophone midpoint and offset, and to present the equations in such a way as to incorporate velocity stratification and migrated time sections.

First of all consider an upcoming plane wave near the surface of the earth at some fixed x and t. Either common sense or Equation (2) tells us that the depth z dependence of the upcoming pressure field is

$$P = P_0 \, e^{-ik_z z} = P_0 \exp[-i \frac{\omega}{v} \cos(\theta) \; z] \qquad (5)$$

It may at first seem pointless to do so but any sinusoidal function like (5) can as well be specified by a differential equation with the sinusoid as a solution. Thus instead of (5) we could equally write

$$\frac{d}{dz} P = -ik_z P = -i \frac{\omega}{v} \cos(\theta) \; P \qquad (6)$$

Equation (5) is completely incorrect if velocity v is z-dependent but Equation (6) is very accurate. Its solution is

$$P = P_0 \exp \left[-i \; \omega \int_0^z \frac{\cos\theta(z)}{v(z)} \, dz \right] \qquad (7)$$

as may be quickly verified by substitution of (7) into (6). The integral in (7) is evaluating the traveltime from the surface to the depth z. Equation (7) solves both Equation (6), and, if you are willing to neglect the z-derivative of material properties, it also satisfies the full wave Equation (1). The full wave Equation (1) controls both up and downgoing waves and their coupling whereas Equation (6), which is the basis for wave equation migration, controls only upcoming waves.

Since we envision our surface data being representable in (ω, k_x) space it seems appropriate to express (6) with these parameters from (4) instead of the shorthand $\cos\theta$.

$$\frac{d}{dz} P = -i \frac{\omega}{v} \left[1 - \left(\frac{v k_x}{\omega} \right)^2 \right]^{1/2} P \qquad (8)$$

If we have a continuum of geophones stretched out along the x-axis then we can have a new philosophical view of (8). Instead of thinking of it as a downward extrapolation equation for a wave field we can think of it as a downward extrapolation equation for our geophones. It tells us what data would have been recorded if we had buried geophones. Considering both shots and geophones to be continuously distributed along the x-axis we will have data not as a function of x but data as a function of both s and g. These data could be Fourier transformed over space to become a function of k_s and k_g. Just as the downward extrapolation of geophones can be achieved by replacing k_x by k_g, so can the extrapolation of shots be accomplished by replacing k_x by k_s. Moreover, the simultaneous downward extrapolation can be achieved by adding the phase shifts giving the *double square root equation*

$$\frac{d}{dz} P = -i \frac{\omega}{v} \left\{ \left[1 - \left(\frac{vk_g}{\omega} \right)^2 \right]^{1/2} + \left[1 - \left(\frac{vk_s}{\omega} \right)^2 \right]^{1/2} \right\} P \tag{9}$$

where the first square root is the cosine of the arrival angle of the ray and the second square root is the cosine of the takeoff angle from the source. Sometimes I become confused about the signs of the square roots. For example, since there is a downgoing wave at the shots and an upcoming wave at the geophones, perhaps the second square roots should have the opposite sign? No, the signs in (9) are correct. Since traveltime to a deep reflector decreases [recall discussion after (4)] as the geophones are moved downward, the same sign on the square root will ensure that it also decreases as the shots are moved downward.

Next we will convert the shot-geophone double square root Equation (9) to the midpoint-offset double square root equation. We formally define a coordinate transformation from (s,g) coordinates to the coordinates of midpoint y and half-offset h

$$y = \frac{g + s}{2} \tag{10a}$$

$$h = \frac{g - s}{2} \tag{10b}$$

Clearly, data as a function of s and g which could be two-dimensionally Fourier transformed to k_s and k_g could instead be transformed from (s,g) space to (y,h) space with (10) and then Fourier transformed to (k_y,k_h) space. The question is then, what form would the double square root Equation (9) take in terms of the spatial frequencies (k_y,k_h)? Let us define the seismic data field in either coordinate system as

$$P(s,g) = P'(y,h) \tag{11}$$

This introduces a new mathematical function P' with the same physical meaning as P but, like a computer subroutine or function call, there is a different subscript look-up procedure if you enter with (y,h) than if you enter with (s,g). Applying the chain rule for partial differentiation to (11) we get

$$\frac{\partial P}{\partial s} = \frac{\partial y}{\partial s}\frac{\partial P'}{\partial y} + \frac{\partial h}{\partial s}\frac{\partial P'}{\partial h} \tag{12a}$$

$$\frac{\partial P}{\partial g} = \frac{\partial y}{\partial g}\frac{\partial P'}{\partial y} + \frac{\partial h}{\partial g}\frac{\partial P'}{\partial h} \tag{12b}$$

and utilizing (10) in (12), we get

$$\frac{\partial P}{\partial s} = \frac{1}{2}\left(\frac{\partial P'}{\partial y} - \frac{\partial P'}{\partial h}\right) \tag{13a}$$

$$\frac{\partial P}{\partial g} = \frac{1}{2}\left(\frac{\partial P'}{\partial y} + \frac{\partial P'}{\partial h}\right) \tag{13b}$$

In Fourier transform space where $\partial/\partial x$ transforms to ik_x, Equation (13), upon canceling the i and canceling $P = P'$, becomes

$$k_s = \frac{1}{2}(k_y - k_h) \tag{14a}$$

$$k_g = \frac{1}{2}(k_y + k_h) \tag{14b}$$

Substituting (14) into (9) we achieve the main purpose of this paper, to get the double square root migration equation in midpoint-offset coordinates

$$\frac{d}{dz}P = -i\frac{\omega}{v}\left\{\left[1 - \left(\frac{vk_y+vk_h}{2\omega}\right)^2\right]^{1/2} + \left[1 - \left(\frac{vk_y-vk_h}{2\omega}\right)^2\right]^{1/2}\right\}P \tag{15}$$

Equation (15) is the takeoff point for many kinds of common-midpoint seismogram analyses. Some convenient definitions to simplify its appearance are

$$G = \frac{vk_g}{\omega} \tag{16a}$$

$$S = \frac{vk_s}{\omega} \tag{16b}$$

$$Y = \frac{vk_y}{2\omega} \tag{16c}$$

$$H = \frac{vk_h}{2\omega} \tag{16d}$$

As noted earlier, the definitions of S and G are the sines of the takeoff angle and arrival angle of a ray. When these sines are at their limits of ± 1 they refer to the steepest possible slopes in (s,t) or (g,t) space. Similarly, if $H = 0$, then Y is bounded by ± 1 [since $Y + (H=0) = G = \pm 1$]. Thus, the quantities S, G, Y, H are all sine-like and refer to angles from vertical in the spaces of shot, geophone, midpoint and offset. With these definitions (15) becomes slightly less cluttered

$$\frac{d}{dz} P = -i \frac{\omega}{v} \left\{ \left[1 - (Y+H)^2 \right]^{1/2} + \left[1 - (Y-H)^2 \right]^{1/2} \right\} P \tag{17}$$

Simpleminded derivations [such as (8)] fail to consider offset. In such derivations H is simply absent. Setting $H = 0$ in (17) allows the two square roots to become identical. It then reduces to (8) except for a discrepancy factor of 2 in velocity. Equation (17) is actually correct; the simpleminded derivations usually insert the factor of 2 as a *post facto* correction to convert two-way traveltime to one-way time. One-way time is said to be appropriate because of the "explosive reflectors" imaging concept. The most widespread use of (17) is on moveout corrected common midpoint stacks. In this application H is almost always absent. One could rationalize setting $H = 0$ by noting that $k_h = 0$ is the zero spatial frequency component in the half offset direction. This zero frequency component is just the integration over offset *without moveout correction*. In old jargon this was called *vertical stack*. Thus the greatest contribution would be at the tangency zone at zero offset. In other words, $H = 0$ is roughly like $h = 0$ and a common depth point stack is supposed to push everything into zero offset. Anyway, supposing that $H = 0$, the two square roots in (17) become identical and we get an equation like the single square root Equation (8).

A final matter is to convert Equation (17) to a form where depth z does not explicitly appear and its role is replaced by a variable τ which represents the two-way vertical traveltime. We have

$$\tau = 2 \int_0^z \frac{dz}{v(z)}$$

$$\frac{d\tau}{dz} = \frac{2}{v} \tag{18}$$

Dividing both sides of (17) by (18) we get the double square root equation for the migrated time section

$$\frac{d}{d\tau} P = -i \frac{\omega}{2} \left\{ \left[1 - (Y+H)^2 \right]^{1/2} + \left[1 - (Y-H) \right]^2 \right\}^{1/2} P \tag{19}$$

This equation is integrated down into the earth from the surface $\tau = 0$. The migrated time section at time τ is found at any τ at zero offset h and zero traveltime t.

It is not our present objective to investigate implementations of (19). It may look as if (19) is in the frequency domain for time and for all space coordinates. Migration could be done that way. But inverse Fourier transformation could be performed on any or all coordinates and the implementation could be done in a finite difference form. Fundamental advantages of the finite difference methods are that they extend to incorporate multiple removal and that horizontal space variation in velocity and absorption are more readily manageable. A practical advantage of finite difference over Fourier methods is the absence of periodicity. A practical advantage of Fourier methods is the ability to work at frequencies approaching the Nyquist. This is especially an advantage on the horizontal space axis. In different situations it may be advantageous to work in (ω,y,h) space, (ω,y,H) space, (t,y,H) space or some other composite space.

The Kirchoff methods may be interpreted in terms of (19) as follows: On the right side of (19) we see a product of an operator with a data field P. This is a product in the k_y spatial frequency domain. Such a product could be expressed as a convolution in the midpoint y domain. Likewise, the solution to (19) is of the form of Equation (5) or (7) where there is a product of an exponential with the surface data P_0. The The exponential function of ω and k_y could be transformed to the time and midpoint domain. Not surprisingly, the resulting function very closely resembles a hyperbola in (y,t) space. A convolution of this hyperboloid along the midpoint axis of the data amounts to the "hyperbola summation" method of migration.