Stanford Exploration Project, Report 125, January 16, 2007, pages 217–??

Optimized implicit finite-difference migration for TTI media

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ABSTRACT

I develop an implicit finite-difference migration algorithm for tilted transversely isotropic (TTI) media. I approximate the dispersion relation of TTI media with a rational function series, whose coefficients are estimated by least-squares optimization. The dispersion relation of TTI media is not a symmetric function, so an odd rational function series is required in addition to the even one. These coefficients are functions of Thomsen anisotropy parameters. They are calculated and stored in a table before the wavefield extrapolation. Similar to the isotropic and VTI media, in 3D a phase-correction filter is applied after the finite-difference operator to eliminate the numerical error caused by two-way splitting. I generate impulse responses for this algorithm and compare them to those generated using the phase-shift method.

INTRODUCTION

Anisotropy is becoming increasingly important in seismic imaging. A vertical transversely isotropic (VTI) medium is one of the simplest and most practical approximations for anisotropic media. However, the VTI approximation is only valid for simple geologic formations, where the bedding plane is horizontal. In an area where the sediments are steeply dipping, such as anticline structures and thrust sheets, the symmetry axis of the medium is not vertical and the medium cannot be simply approximated as VTI medium. In these area, it is usually better to consider them as tilted transversely isotropic (TTI) media. For VTI media, to image steeply dipping reflector using one-way wave equation, Shan and Biondi (2004) rotate the coordinates. In the new coordinates, the medium becomes TTI media. For both cases, we need to design wavefield-extrapolation operators for TTI media.

Compared to those of isotropic and VTI media, the dispersion relation of TTI media is much more complicated. The dispersion relation of an isotropic medium is very simple, and we have an explicit expression for it. For a VTI medium, under the assumption that the S-wave velocity equals zero, we can still derive an explicit formula for its dispersion relation. The dispersion relation of TTI media is a quartic equation, and we have to solve it numerically. Conventional implicit finite-difference methods rely on the Taylor series approximation of the explicit dispersion relation. It is very hard to derive a Taylor series for the dispersion relation of TTI media. As a result, most wavefield extrapolation algorithms for anisotropic media are based on either explicit finite-difference (Uzcategui, 1995; Zhang et al., 2001a,b; Baumstein and Anderson, 2003; Shan and Biondi, 2005; Ren et al., 2005) or phase-shift plus interpolation method (Rousseau, 1997; Ferguson and Margrave, 1998). For both explicit finite-difference methods and phase-shift plus interpolation (PSPI), the complex dispersion relation does not increase the complexity of the algorithm. However both of them are very expensive; explicit finite-difference methods for TTI media require running 2D convolutions in 3D and PSPI requires extrapolating many reference wavefields.

Implicit finite-difference method has been one of the most attractive methods for isotropic media. It can handle lateral variation of velocity naturally and guarantee stability. Traditional finite-difference methods, such as the 15° equation (Claerbout, 1971) and the 45° equation (Claerbout, 1985), approximate the dispersion relation by the truncation of Taylor series. Lee and Suh (1985) approximate the square-root equation with rational functions, and optimize the coefficient with least-squares. This method improves the accuracy with the same computational cost. Under the weak anisotropy assumption, Ristow and Ruhl (1997) design an implicit scheme for VTI media. Liu et al. (2005) apply a phase-correction operator (Li, 1991) after the finite-difference operator for VTI media and improve the accuracy. Shan (2006) approximates the VTI dispersion relation with rational functions and obtains the coefficients using weighted least-squares optimization.

In this paper, I present an optimized one-way wave equation for TTI media and use a table-driven implicit finite-difference method (Shan, 2006) for laterally varying media. I compared the impulse responses of this algorithm with those of phase-shift methods.

OPTIMIZED ONE-WAY WAVE EQUATION OPERATOR FOR TTI

In a VTI medium, the phase velocity of qP- and qSV-waves in Thomsen's notation can be expressed as (Tsvankin, 1996):

$$\frac{V^2(\theta)}{V_{P0}^2} = 1 + \varepsilon \sin^2(\theta) - \frac{f}{2} \pm \frac{f}{2}\sqrt{\left(1 + \frac{2\varepsilon\sin^2(\theta)}{f}\right)^2 - \frac{2(\varepsilon-\delta)\sin^2(2\theta)}{f}}, \tag{1}$$

where θ is the phase angle of the propagating wave, and $f = 1 - (V_{S0}/V_{P0})^2$. V_{P0} and V_{S0} are the qP- and qSV- wave velocities in the vertical direction, respectively. ε and δ are anisotropy parameters defined by Thomsen (1986):

$$\varepsilon = \frac{C_{11} - C_{33}}{2C_{33}}, \delta = \frac{(C_{11} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})},$$

where C_{ij} are elastic moduli. In equation (1), $V(\theta)$ is qP-wave phase-velocity when the sign in front of the square root is positive, and the qSV-wave phase velocity for a negative sign.

Rotating the symmetry axis from vertical to a tilted angle φ, we obtain the phase velocity of a tilted TI medium whose symmetry axis forms an angle φ with the vertical direction:

$$\frac{V^2(\theta,\varphi)}{V_{P0}^2} = 1 + \varepsilon \sin^2(\theta-\varphi) - \frac{f}{2} \pm \frac{f}{2}\sqrt{\left(1 + \frac{2\varepsilon\sin^2(\theta-\varphi)}{f}\right)^2 - \frac{2(\varepsilon-\delta)\sin^2 2(\theta-\varphi)}{f}}. \tag{2}$$

Here, in contrast to equation (1), ε and δ are now defined in a direction tilted by the angle φ from the vertical direction. V_{P0} is the qP-wave velocity in the direction parallel to the symmetry axis.

The phase angle θ is related to the wavenumbers k_x and k_z by:

$$\sin\theta = \frac{V(\theta,\varphi)k_x}{\omega}, \qquad \cos\theta = \frac{V(\theta,\varphi)k_z}{\omega}, \tag{3}$$

where ω is the temporal frequency. Let $S_x = k_x/\frac{\omega}{V_{P0}}$, and $S_z = k_z/\frac{\omega}{V_{P0}}$. We can obtain a dispersion relation equation from equations (2) and (3):

$$d_4 S_z^4 + d_3 S_z^3 + d_2 S_z^2 + d_1 S_z + d_0 = 0, \tag{4}$$

where the coefficients d_0, d_1, d_2, d_3, and d_4 are as follows:

$$\begin{aligned}
d_0 &= (2+2\varepsilon\cos^2\varphi - f)S_x^2 - 1 - \left[(1-f)(1+2\varepsilon\cos^2\varphi) + \tfrac{f}{2}(\varepsilon-\delta)\sin^2 2\varphi\right]S_x^4,\\
d_1 &= [2\varepsilon(1-f)\sin 2\varphi - f(\varepsilon-\delta)\sin 4\varphi]S_x^3 - 2\varepsilon\sin 2\varphi S_x,\\
d_2 &= [f(\varepsilon-\delta)\sin^2 2\varphi - 2(1-f)(1+\varepsilon) - 2f(\varepsilon-\delta)\cos^2 2\varphi]S_x^2 + (2+2\varepsilon\sin^2\varphi - f),\\
d_3 &= [f(\varepsilon-\delta)\sin 4\varphi + 2\varepsilon(1-f)\sin 2\varphi]S_x,\\
d_4 &= f - 1 + 2\varepsilon(f-1)\sin^2\varphi - \tfrac{f}{2}(\varepsilon-\delta)\sin^2 2\varphi.
\end{aligned}$$

Equation (4) is a quartic equation and there is no explicit expression for its solution.

Generally, the Padé approximation suggests that if the function $S_z(S_r) \in C^{n+m}$, then $S_z(S_r)$ can be approximated by a rational function $R_{n,m}(S_r)$:

$$R_{n,m}(S_r) = \frac{P_n(S_r)}{Q_m(S_r)}, \tag{5}$$

where

$$P_n(S_r) = \sum_{i=0}^{n} a_i S_r^i$$

and

$$Q_m(x) = \sum_{i=0}^{m} b_i S_r^i$$

are polynomials of degree n and m, respectively. The coefficients a_i and b_i can be obtained either analytically by Taylor-series analysis or numerically by least-squares fitting.

For an isotropic or VTI medium, S_z is an even function of S_x. We can approximate the dispersion relation with even rational functions, such as S_x^2, S_x^4. For TTI media, S_z is not an symmetric function of S_x. Therefore, in addition to even rational functions, we need odd rational functions to approximate the dispersion relation, such as S_x, S_x^3. The fourth order approximation for the dispersion relation of TTI media is as follows:

$$S_z(S_x) \approx S_{z0} + \frac{a_1 S_x^2 + c_1 S_x}{1+b_1 S_x^2} + \frac{a_2 S_x^2 + c_2 S_x}{1+b_2 S_x^2}, \tag{6}$$

where $S_{z0} = S_z(0)$ and the coefficients $c_1, b_1, a_1, c_2, b_2, a_2$ can be estimated by least-squares methods. They are functions of the anisotropy parameters ε, δ and the tilting angle ϕ. When these parameters vary laterally, the coefficients $c_1, b_1, a_1, c_2, b_2, a_2$ also vary laterally. It is too expensive to run least-squares estimation for each grid point during the wavefield extrapolation. They can be calculated and stored in a table before the wavefield extrapolation. During the wavefield extrapolation, given the anisotropy parameters ε, δ, and the tilting angle ϕ, we search for these coefficients from the table and put them into the finite-difference algorithm. Given the coefficients found from the table, the finite difference algorithm in TTI media is similar to the isotropic media.

FINITE-DIFFERENCE SCHEME

In the approximated dispersion relation (6), replacing S_z and S_x by the partial differential operators $i\frac{\partial}{\partial z}/\frac{\omega}{V_{P0}}$ and $i\frac{\partial}{\partial x}/\frac{\omega}{V_{P0}}$, we obtain a partial differential equation as follows:

$$\frac{\partial}{\partial z}P = i\frac{\omega}{V_{P0}}\left(S_{z0} + \frac{-a_1\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2} + ic_1\frac{V_{P0}}{\omega}\frac{\partial}{\partial x}}{1 - b_1\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2}} + \frac{-a_2\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2} + ic_2\frac{V_{P0}}{\omega}\frac{\partial}{\partial x}}{1 - b_2\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2}}\right)P. \tag{7}$$

Equation (7) can be solved by cascading as follows:

$$\frac{\partial}{\partial z}P = i\frac{\omega}{V_{P0}}S_{z0}P, \tag{8}$$

$$\frac{\partial}{\partial z}P = i\frac{\omega}{V_{P0}}\left(\frac{-a_1\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2} + ic_1\frac{V_{P0}}{\omega}\frac{\partial}{\partial x}}{1 - b_1\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2}}\right)P, \tag{9}$$

$$\frac{\partial}{\partial z}P = i\frac{\omega}{V_{P0}}\left(\frac{-a_2\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2} + ic_2\frac{V_{P0}}{\omega}\frac{\partial}{\partial x}}{1 - b_2\frac{V_{P0}^2}{\omega^2}\frac{\partial^2}{\partial x^2}}\right)P. \tag{10}$$

Equation (8) can be solved by a phase-shift in the space domain. Let $P_i^n = P(\omega, n\Delta z, i\Delta x)$, where Δx and Δz are the grid size of finite-difference scheme. In equation (9), replacing the partial differential operators by the finite-difference operators as follows:

$$\frac{\partial}{\partial x}P(\omega, n\Delta z, i\Delta x) \approx \delta_x P_i^n = \frac{P_{i+1}^n - P_{i-1}^n}{2\Delta x}$$

and

$$\frac{\partial^2}{\partial x^2}P(\omega, n\Delta z, i\Delta x) \approx \delta_x^2 P_i^n = \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{\Delta x^2},$$

we can derive the following finite difference equation:

$$\left(1 + (\frac{ia_1\Delta z}{2}\frac{V_{P0}}{\omega} - b_1\frac{V_{P0}^2}{\omega^2})\delta_x^2 + \frac{c_1\Delta z}{2}\delta_x\right)P_i^{n+1} = \left(1 + (-\frac{ia_1\Delta z}{2}\frac{V_{P0}}{\omega} + b_1\frac{V_{P0}^2}{\omega^2})\delta_x^2 - \frac{c_1\Delta z}{2}\delta_x\right)P_i^n \tag{11}$$

Fourier analysis shows that the finite-difference scheme (11) is stable. Its computational cost is almost same as that of the finite-difference scheme for isotropic media. Equation (10) can be solved similarly.

IMPULSE RESPONSES

Figure 1 shows the impulse responses of the optimized implicit finite-difference method. The medium is homogeneous, in which the vertical velocity of the medium is 2000 m/s, the anisotropy parameter ε is 0.4, the anisotropy parameter δ is 0.2 and the tilting angle is 30°. The travel time of the impulse responses are 0.4s, 0.6s and 0.8s, respectively. The impulse location is at $x = 4000$m. For comparison, I also present the impulse responses of phase-shift (Figure 2) and the impulse responses of plane-wave migration in tilted coordinates (Figure 3). Notice in Figure 2, although we use the phase-shift method, we can not achieve 90° in the right section. The reason is that the waves on the right side overturn when the propagation direction is close to the horizontal and the phase-shift method is still a one-way equation based method. Figure 3 shows the impulse responses generated by plane-wave migration in tilted coordinates (Shan and Biondi, 2004), which extrapolate the wavefield accurately even when the waves overturn. Comparing the left section of the impulse responses of Figure 1-3, I find that the impulse responses of optimized implicit finite-difference is very close to the other two. In the right section in Figure 1, optimized implicit finite-difference loses accuracy when the waves propagate almost horizontally or they overturn, since it is still a one-way based method. The impulse responses in Figure 1 have heart-shaped noise, which are typical evanescent energy in implicit finite-difference.

CONCLUSION

I present optimized implicit finite-difference method for wavefield extrapolation in TTI media. It is stable and can handle laterally variation easily. The impulse responses show that it is accurate for waves that do not overturn.

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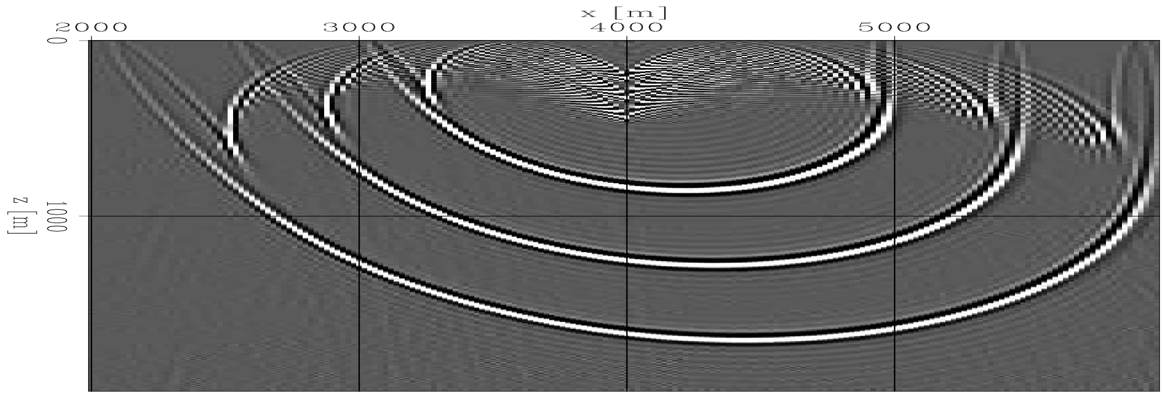

Figure 1: Impulse responses of the optimized implicit finite-difference. guojian1-ttiimfd [ER]

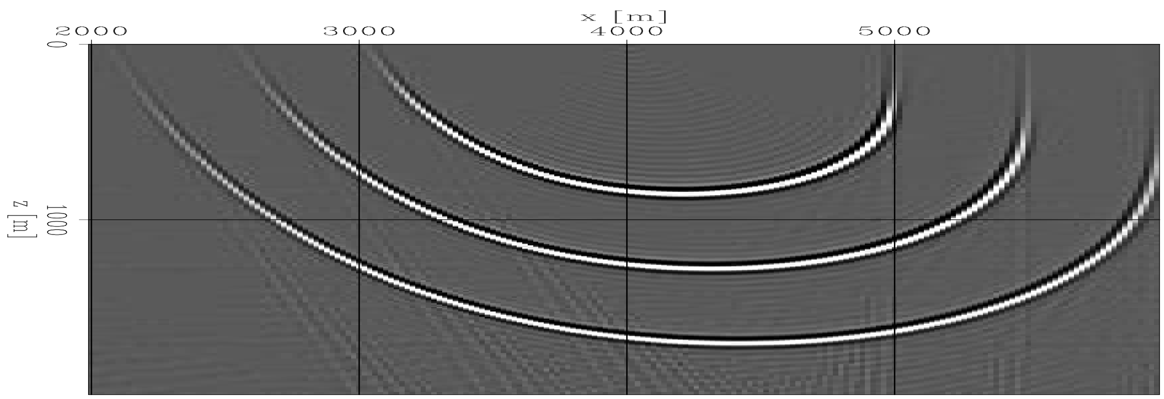

Figure 2: Impulse responses of the phase-shift method. guojian1-ttiphase [ER]

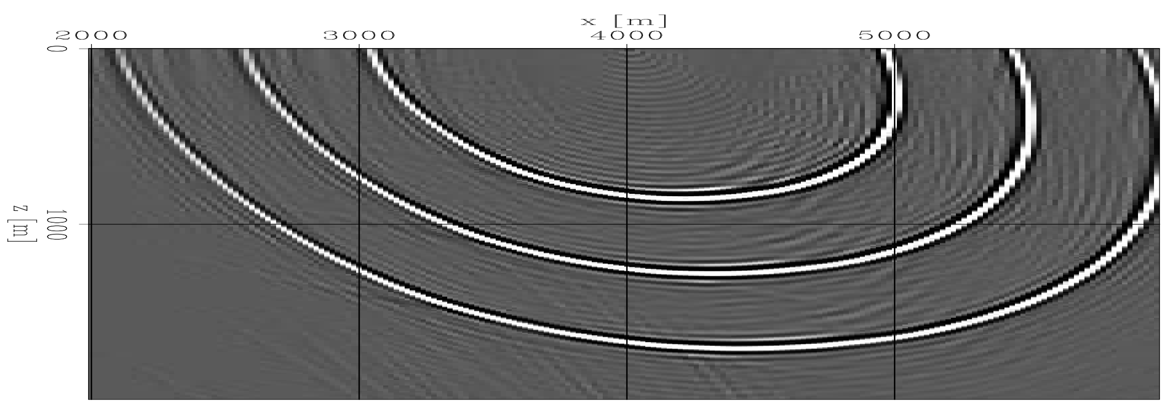

Figure 3: Impulse responses of plane-wave migration in tilted coordinates guojian1-im2d5pttilt [ER]