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Wave-equation angle-domain Hessian

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ABSTRACT

A regularization in the reflection angle dimension (and, more generally in the reflection and azimuth angles) is necessary to stabilize the wave-equation inversion problem. The angle-domain Hessian can be computed from the subsurface-offset Hessian by an offset-to-angle transformation. This transformation can be done in the image space following the Sava and Fomel (2003) approach. To perform the inversion, the angle-domain Hessian matrix can be used explicitly, or implicitly as a chain of the offset-to-angle operator and the subsurface offset Hessian matrix.

INTRODUCTION

Seismic imaging using non-unitary migration operators (Claerbout, 1992) often produce images with reflectors correctly positioned but biased amplitudes (Nemeth et al., 1999; Duquet and Marfurt, 1999; Ronen and Liner, 2000; Chavent and Plessix, 1999). One way to solve this problem is to use the inversion formalism introduced by Tarantola (1987) to solve geophysical imaging problems, where the image can be obtained by weighting the migrated image with the inverse of the Hessian matrix. However, when the dimensions of the problem get large, the explicit calculation of the Hessian matrix and its inverse becomes unfeasible.

Valenciano and Biondi (2004) proposed computing the Hessian in a target-oriented fashion to reduce the size of the problem. The zero-offset inverse image can be estimated as the solution of a non-stationary least-squares filtering problem, by means of a conjugate gradient algorithm (Valenciano et al., 2005b,a). This approach, renders unnecessary an explicit computation of inverse of the Hessian matrix.

In this paper, we define the wave-equation angle-domain Hessian from the subsurface offset wave-equation Hessian via an angle-to-offset transformation following the Sava and Fomel (2003) approach. To perform the inversion, the angle-domain Hessian matrix can be used explicitly or, implicitly as a chain of the offset-to-angle operator and the subsurface offset Hessian matrix.

The definition of the wave-equation angle-domain Hessian allows the angle-domain regularization required to stabilize the wave equation inversion problem (Prucha et al., 2000; Kuehl and Sacchi, 2001). It also allows to obtain a prestack inverse image, adding the possibility of

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doing amplitude vs. angle (AVA) analysis for reservoir characterization.

LINEAR LEAST-SQUARES INVERSION

Tarantola (1987) formalizes the geophysical inverse problem by giving a theoretical approach to compensate for experimental deficiency (e.g., acquisition geometry, complex overburden), while being consistent with the acquired data. His approach can be summarized as follows: given a linear modeling operator $\mathbf{L}$, compute synthetic data $\mathbf{d}$, using,

$$\mathbf{d} = \mathbf{L}\mathbf{m}, \tag{1}$$

where $\mathbf{m}$ is a reflectivity model. Given the recorded data $\mathbf{d}_{obs}$, a quadratic cost function,

$$S(\mathbf{m}) = \|\mathbf{d} - \mathbf{d}_{obs}\|^2 = \|\mathbf{L}\mathbf{m} - \mathbf{d}_{obs}\|^2, \tag{2}$$

is formed. The reflectivity model $\hat{\mathbf{m}}$ that minimizes $S(\mathbf{m})$ is given by

$$\hat{\mathbf{m}} = (\mathbf{L}'\mathbf{L})^{-1}\mathbf{L}'\mathbf{d}_{obs} \tag{3}$$

$$\hat{\mathbf{m}} = \mathbf{H}^{-1}\mathbf{m}_{mig}, \tag{4}$$

where $\mathbf{L}'$ (migration operator) is the adjoint of the linear modeling operator $\mathbf{L}$, $\mathbf{m}_{mig}$ is the migration image, and $\mathbf{H} = \mathbf{L}'\mathbf{L}$ is the Hessian of $S(\mathbf{m})$.

The main difficulty with this approach is the explicit calculation of the Hessian inverse. In practice, it is more feasible to compute the least-squares inverse image as the solution of the linear system of equations,

$$\mathbf{H}\hat{\mathbf{m}} = \mathbf{m}_{mig}, \tag{5}$$

by using an iterative conjugate gradient algorithm.

The inversion inherent in equation 5 needs regularization. Prucha et al. (2000) and Kuehl and Sacchi (2001) propose smoothing the image in the offset ray parameter dimension, which is equivalent to the same procedure in the reflection angle dimension. This idea can be generalize to include the azimuth dimension.

The least squares solution of equation 5 is obtained using the fitting goals,

$$\begin{aligned} \mathbf{H}(\mathbf{x},\Theta;\mathbf{x}',\Theta')\hat{\mathbf{m}}(\mathbf{x},\Theta) - \mathbf{m}_{mig}(\mathbf{x},\Theta) &\approx 0, \\ \mathbf{D}(\Theta)\hat{\mathbf{m}}(\mathbf{x},\Theta) &\approx 0, \end{aligned} \tag{6}$$

where $\Theta = (\theta, \alpha)$ are the reflection and the azimuth angles, and $\mathbf{D}(\Theta)$ is a smoothing operator in the reflection and azimuth angle dimensions.

The next sections show how to include the subsurface offset dimension in the Hessian computation and how to go from subsurface offset to reflection and azimuth angle dimensions following the Sava and Fomel (2003) approach.

EXPANDING HESSIAN DIMENSIONALITY

Valenciano et al. (2005b) define the zero subsurface-offset Hessian by using the adjoint of the zero subsurface-offset migration as the modeling operator **L**. Then the zero-offset inverse image can be estimated as the solution of a non-stationary least-squares filtering problem, by means of a conjugate gradient algorithm (Valenciano et al., 2005b,a). But, from the results reported by Prucha et al. (2000), Kuehl and Sacchi (2001), and Valenciano et al. (2005a), regularization in the reflection angle dimension is necessary to stabilize the wave-equation inversion problem.

Subsurface-offset Hessian

The prestack migration image (subsurface offset domain) for a group of shots positioned at $\mathbf{x}_s = (x_s, y_s, 0)$ and a group of receivers positioned at $\mathbf{x}_r = (x_r, y_r, 0)$ can be given by the adjoint of a linear operator **L** acting on the data-space $\mathbf{d}(\mathbf{x}_s, \mathbf{x}_r; \omega)$ as

$$\begin{aligned} \mathbf{m}(\mathbf{x},\mathbf{h}) &= \mathbf{L}'\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega) \\ &= \sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r} \mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\sum_{\mathbf{h}}{}'\sum_{\mathbf{x}}{}'\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega), \end{aligned} \tag{7}$$

where $\mathbf{G}(\mathbf{x}, \mathbf{x}_s; \omega)$ and $\mathbf{G}(\mathbf{x}, \mathbf{x}_r; \omega)$ are the Green functions from shot position $\mathbf{x}_s$ and receiver position $\mathbf{x}_r$ to a model space point $\mathbf{x} = (x, y, z)$, and $\mathbf{h} = (h_x, h_y, h_z)$ is the subsurface offset. The symbols $\sum_{\mathbf{h}}'$ and $\sum_{\mathbf{x}}'$ are spray (adjoint of the sum) operators in the subsurface offset and model space dimensions, respectively.

The synthetic data can be modeled (as the adjoint of equation 7) by the linear operator **L** acting on the model space $\mathbf{m}(\mathbf{x}, \mathbf{h})$ with $\mathbf{x} = (x, y, z)$ and $\mathbf{h} = (h_x, h_y)$

$$\begin{aligned} \mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega) &= \mathbf{L}\mathbf{m}(\mathbf{x},\mathbf{h}) \\ &= \sum_{\mathbf{x}}\sum_{\mathbf{h}} \mathbf{G}(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\sum_{\mathbf{x}_r}{}'\sum_{\mathbf{x}_s}{}'\sum_{\omega}{}'\mathbf{m}(\mathbf{x},\mathbf{h}), \end{aligned} \tag{8}$$

where the symbols $\sum_{\mathbf{x}_r}'$, $\sum_{\mathbf{x}_s}'$, and $\sum_{\omega}'$ are spray operators in the shot, receiver, and frequency dimensions, respectively.

In equations 7 and 8 the Green functions are computed by means of the one-way wave equation (Ehinger et al., 1996) and the extrapolation is performed using the adequate paraxial wave equations (flux conservation) (Bamberger et al., 1988).

The quadratic cost function is

$$\begin{aligned} S(\mathbf{m}) &= \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r} \|\mathbf{d}-\mathbf{d}_{obs}\|^2 \qquad (9) \\ &= \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r} [\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}]'\,[\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}], \end{aligned}$$

while its first derivative, with respect to the model parameters $\mathbf{m}(\mathbf{x},\mathbf{h})$, is

$$\frac{\partial S(\mathbf{m})}{\partial \mathbf{m}(\mathbf{x},\mathbf{h})} = \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\{\mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)[\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}] + [\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}]'\mathbf{G}(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\}, \quad (10)$$

and its second derivative with respect to the model parameters $\mathbf{m}(\mathbf{x},\mathbf{h})$ and $\mathbf{m}(\mathbf{x}',\mathbf{h}')$ is the subsurface offset Hessian:

$$\mathbf{H}(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}') = \frac{\partial^2 S(\mathbf{m})}{\partial \mathbf{m}(\mathbf{x},\mathbf{h})\partial \mathbf{m}(\mathbf{x}',\mathbf{h}')} \quad (11)$$

$$\mathbf{H}(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}') = \sum_{\omega}\sum_{\mathbf{x}_s}\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{G}(\mathbf{x}'+\mathbf{h}',\mathbf{x}_s;\omega)\sum_{\mathbf{x}_r}\mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}'-\mathbf{h}',\mathbf{x}_r;\omega).$$

The next subsection shows how to go from subsurface offset to reflection and azimuth angle dimensions following the Sava and Fomel (2003) approach.

Angle-domain Hessian

Sava and Fomel (2003) define an image space transformation from subsurface offset to reflection and azimuth angle as:

$$\mathbf{m}(\mathbf{x},\Theta) = \mathbf{T}'(\Theta,\mathbf{h})\mathbf{m}(\mathbf{x},\mathbf{h}), \quad (12)$$

where $\Theta = (\theta,\alpha)$ are the reflection and the azimuth angles, and $\mathbf{T}'(\Theta,\mathbf{h})$ is the adjoint of the angle-to-offset transformation operator (slant stack).

Substituting the prestack migration image (subsurface offset domain) in equation 7 into equation 12 we obtain the expression for the prestack migration image in the angle-domain that follows:

$$\begin{aligned}\mathbf{m}(\mathbf{x},\Theta) &= \mathbf{T}'(\Theta,\mathbf{h})\mathbf{L}'\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega) \quad (13)\\ &= \mathbf{T}'(\Theta,\mathbf{h})\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\sum_{\mathbf{h}}{}'\sum_{\mathbf{x}}{}'\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega).\end{aligned}$$

The synthetic data can be modeled (as the adjoint of equation 14) by the chain of linear operator $\mathbf{L}$ and the angle-to-offset transformation operator acting on the model space,

$$\begin{aligned}\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega) &= \mathbf{L}\mathbf{T}(\Theta,\mathbf{h})\mathbf{m}(\mathbf{x},\Theta)\\ &= \sum_{\mathbf{x}}\sum_{\mathbf{h}}\mathbf{G}(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\sum_{\mathbf{x}_r}{}'\sum_{\mathbf{x}_s}{}'\sum_{\omega}{}'\mathbf{T}(\Theta,\mathbf{h})\mathbf{m}(\mathbf{x},\Theta), \quad (14)\end{aligned}$$

The quadratic cost function is

$$\begin{aligned}S(\mathbf{m}) &= \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\|\mathbf{d}-\mathbf{d}_{obs}\|^2\\ &= \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}[\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}]'[\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}], \quad (15)\end{aligned}$$

while its first derivative with respect to the model parameters $\mathbf{m}(\mathbf{x},\Theta)$ is

$$\frac{\partial S(\mathbf{m})}{\partial \mathbf{m}(\mathbf{x},\Theta)} = \frac{1}{2}\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\{\mathbf{T}'(\Theta,\mathbf{h})\mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\,[\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}] + [\mathbf{d}(\mathbf{x}_s,\mathbf{x}_r;\omega)-\mathbf{d}_{obs}]'\,\mathbf{G}(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{T}(\Theta,\mathbf{h})\} \quad (16)$$

and its second derivative with respect to the model parameters $\mathbf{m}(\mathbf{x},\Theta)$ and $\mathbf{m}(\mathbf{x}',\Theta')$ is the angle-domain Hessian

$$\begin{aligned}
\mathbf{H}(\mathbf{x},\Theta;\mathbf{x}',\Theta') &= \frac{\partial^2 S(\mathbf{m})}{\partial \mathbf{m}(\mathbf{x},\Theta)\partial \mathbf{m}(\mathbf{x}',\Theta')} \\
&= \sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\mathbf{T}'(\Theta,\mathbf{h})\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{G}(\mathbf{x}'+\mathbf{h}',\mathbf{x}_s;\omega) \\
&\times\ \mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}'-\mathbf{h}',\mathbf{x}_r;\omega)\mathbf{T}(\Theta',\mathbf{h}') \\
&= \mathbf{T}'(\Theta,\mathbf{h})\sum_{\omega}\sum_{\mathbf{x}_s}\sum_{\mathbf{x}_r}\mathbf{G}'(\mathbf{x}+\mathbf{h},\mathbf{x}_s;\omega)\mathbf{G}(\mathbf{x}'+\mathbf{h}',\mathbf{x}_s;\omega) \\
&\times\ \mathbf{G}'(\mathbf{x}-\mathbf{h},\mathbf{x}_r;\omega)\mathbf{G}(\mathbf{x}'-\mathbf{h}',\mathbf{x}_r;\omega)\mathbf{T}(\Theta',\mathbf{h}') \\
&= \mathbf{T}'(\Theta,\mathbf{h})\mathbf{H}(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}')\mathbf{T}(\Theta',\mathbf{h}'). \qquad (17)
\end{aligned}$$

Explicit vs. implicit Hessian matrix computation

Equation 17 expresses the angle-domain Hessian as a chain of the offset-to-angle operator and the subsurface offset Hessian matrix. This implies that to implement the angle-domain wave-equation inversion using a conjugate gradient algorithm there is no need to explicitly compute the angle Hessian matrix. But the possible drawback is that, for each iteration, the offset-to-angle transformation needs to be performed.

A different strategy might be to explicitly compute the angle-domain Hessian matrix. This can be done by a simple manipulation the terms in equation 17

$$\begin{aligned}
\mathbf{H}(\mathbf{x},\Theta;\mathbf{x}',\Theta') &= \mathbf{T}'(\Theta,\mathbf{h})\mathbf{H}(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}')\mathbf{T}(\Theta',\mathbf{h}') \\
&= \mathbf{T}'(\Theta,\mathbf{h})\left(\mathbf{T}'(\Theta',\mathbf{h}')\mathbf{H}'(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}')\right)'. \qquad (18)
\end{aligned}$$

Due to the symmetry of the Hessian matrix equation 18 turns into:

$$\mathbf{H}(\mathbf{x},\Theta;\mathbf{x}',\Theta') = \mathbf{T}'(\Theta,\mathbf{h})\left(\mathbf{T}'(\Theta',\mathbf{h}')\mathbf{H}(\mathbf{x},\mathbf{h};\mathbf{x}',\mathbf{h}')\right)'. \qquad (19)$$

In practice, equation 19 takes the subsurface offset Hessian matrix and applies an offset-to-angle transformation, then transposes the resulting matrix and reapplies the same offset-to-angle transformation.

This explicit angle Hessian matrix computation could be an expensive operation, but it has the advantage of only needing to be performed once. In contrast to the application of a chain of the offset-to-angle operator and the subsurface offset Hessian matrix (implicit approach) which needs to be performed at each conjugate gradient iteration. Each approach has its advantages and disadvantages, thus the specific application will dictate which path to follow.

CONCLUSIONS

The wave-equation angle-domain Hessian can be computed from the subsurface offset wave-equation Hessian via an angle-to-offset transformation following the approach presented by Sava and Fomel (2003). This result allow us to implement an angle-domain regularization that stabilizes the the wave equation inversion problem.

In order to perform the wave-equation angle-domain inversion, the angle-domain Hessian matrix can be used explicitly or, implicitly as a chain of the offset-to-angle operator and the subsurface offset Hessian matrix. Since each approach has its advantages and disadvantages the specific application will dictate which path to follow.

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