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Short Note

Wavefield extrapolation in frequency-wavenumber domain for spatially-varying velocity models

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INTRODUCTION

Mixed domain wavefield extrapolation methods can handle, to a large extent, spatial velocity variations by performing part of their computation in the ω-$\mathbf{k}$ domain and part of the computation in the ω-$\mathbf{x}$ domain. The best-known mixed domain methods are Phase Shift Plus Interpolation (PSPI) and extended split-step migration (Gazdag and Sguazzero, 1984; Stoffa et al., 1990; Biondi, 2004). They downward continue the wavefield in the ω-$\mathbf{k}$ domain at each depth step with a series of reference velocities and then interpolate the wavefields in the ω-$\mathbf{x}$ domain possibly after a split-step correction.

There are three main potential sources of error in mixed-domain algorithms: (1) the choice of the reference velocities, (2) the correction to account for the difference between the model velocity and the reference velocity at each spatial location, and (3) the accuracy of the interpolation of the wavefields.

In this paper, we present an alternative to ω-$\mathbf{x}$ – ω-$\mathbf{k}$ downward extrapolation that performs each depth extrapolation completely in the ω-$\mathbf{k}$ domain, yet can handle arbitrary spatial variation of the migration velocities. The proposed algorithm eliminates the need for choosing reference velocities and requires no FFT's at each depth extrapolation level. Making the algorithm efficient, however, is still a research issue.

In the next section, we will present a brief overview of the standard ω-$\mathbf{x}$ – ω-$\mathbf{k}$ methods without getting into any specific implementation details. In the following section, we present our method, and in the last section, we discuss some implementation issues.

OVERVIEW OF MIXED-DOMAIN DOWNWARD EXTRAPOLATION

In this section, we will briefly review, from the mathematical point of view, the ω-$\mathbf{x}$ – ω-$\mathbf{k}$ algorithm. This will serve as the starting point for the presentation of the new extrapolation method in the next section.

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Let $\mathbf{W}^N$ be the wavefield at depth step N in the ω-$\mathbf{k}$ domain and let $\mathbf{W}_l^{N+1}$ be the wavefield extrapolated from depth step N to depth step $N+1$ using reference velocity V_l. That is:

$$\mathbf{W}_l^{N+1} = \mathbf{W}^N e^{ik_{z_l}\Delta z},$$

where Δz is the depth of the Nth layer and k_{z_l} is given by the dispersion relation:

$$k_{z_l} = \sqrt{\frac{\omega^2}{V_l^2} - |\mathbf{k}|^2} \tag{1}$$

with $|\mathbf{k}|$ being the magnitude of the horizontal wavenumber vector. PSPI handles the difference between the true velocity and the reference velocity by interpolating the downward-continued wavefields in the ω-$\mathbf{x}$ domain based on the difference between the reference velocities and the model velocity at each $\mathbf{x}$ position. The interpolated wavefield is therefore given by:

$$\mathbf{w}^{N+1}(j) = \sum_{l=1}^{n_v} \sigma_l \mathbf{w}_l^N(j)$$

where σ_l is an interpolation factor ($\sum_l^{n_v} \sigma_l = 1$), $\mathbf{w}_l^{N+1}(j)$ is the downward-continued wavefield in the ω-$\mathbf{x}$ domain at the spatial location j, and n_v is the number of reference velocities.

Extended split-step adds a correction before the interpolation, the so-called "thin lens term":

$$e^{ik_{ss_l}} \text{ where, } k_{ss_l} = \frac{\omega}{V} - \frac{\omega}{V_l}$$

where V is the true model velocity. Depending on the choice and number of reference velocities, split-step can make significant improvements in accuracy compared to PSPI.

Other methods, such as pseudo-screen and Fourier finite-difference, increase the accuracy of the result by adding high-order spatial derivatives to the computation of the k_{z_l} term (Ristow and Ruhl, 1994; Huang et al., 1999; Xie and Wu, 1999; Biondi, 2004). The more accurate approximation of k_{z_l} relaxes the need for a large number of reference velocities such that with fewer reference velocities similar or even better accuracies can be obtained compared with split-step.

The last step in either of these methods is to Fourier transform the interpolated wavefield to the ω-$\mathbf{k}$ space. This wavefield will then be the input to the propagation at the next depth step.

It should be clear that the accuracy of these methods, especially PSPI and extended split-step, is directly related to the accuracy of the wavefield interpolation and the number and choice of the reference velocities.

ω-K DOWNWARD EXTRAPOLATION

The previous section suggests an alternative implementation of mixed-domain migration. To see this more clearly, assume that, at each depth step, we downward continue the wavefield with the true velocities at that depth. In other words, compute $n_v = n_{\mathbf{x}}$ wavefields, each one corresponding to the model velocity at each spatial location. No split-step correction or higher-order approximation of k_z would then be required. The wavefield interpolation in ω-$\mathbf{x}$ domain reduces to a simple selection of the appropriate wavefield, operation that can be expressed as:

$$\mathbf{w}^{N+1}(j) = \sum_{l=1}^{n_v} \mathbf{w}_l^{N+1}(j)\delta_{lj} \tag{2}$$

where $\mathbf{w}_l^{N+1}$ is the row of the array of wavefields extrapolated with the velocity V_l, and δ_{lj} is the Kronecker delta that selects from that row the corresponding $j = l$ component. Notice that, since we are extrapolating as many wavefields as there are spatial positions (traces), $n_v = n_{\mathbf{x}}$. Figure 1 shows a schematic of the velocity selection. In the ω-$\mathbf{k}$ domain, Equation (2)

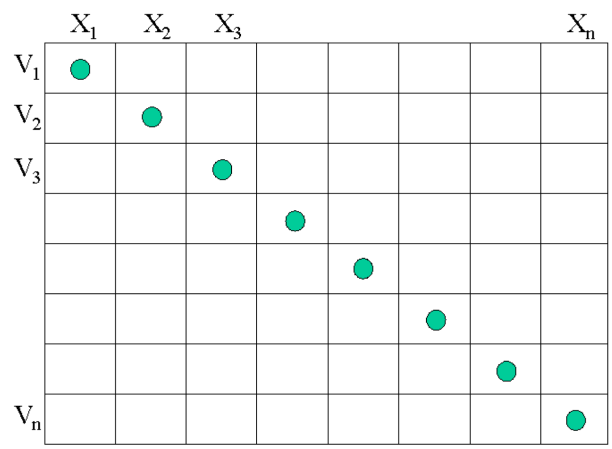

Figure 1: Diagram illustrating velocity selection when there are as many velocities as spatial locations. gabriel1-bin_vels1 [NR]

becomes:

$$\mathbf{W}^{N+1} = \sum_{l=1}^{n_v} \mathbf{W}_l^{N+1} \otimes e^{-ik_x \Delta x_l} \tag{3}$$

where $\Delta x_l = (l-1)\Delta x / n_x$ and we are using a single index to represent the spatial axis in order to simplify the notation. The symbol $\otimes$ represents circular convolution.

Notice that Equation (3) was derived without any approximation. Let's make the computations more explicit in order to gain a better appreciation for what it means:

$$\mathbf{W}^{N+1}(j) = \sum_{l=1}^{n_v} \sum_{m=\langle n_x \rangle} \mathbf{W}_l^{N+1}(m) e^{-ik_x (j-m)\Delta x_l}$$

where $\langle m \rangle$ means that the summation is over the range n_x with modulus n_x. That is,

$$\mathbf{W}^{N+1}(j) = \sum_{l=1}^{n_v} \sum_{m=1}^{n_x} \mathbf{W}^N(m) e^{-ik_{z_l}(m)\Delta z} e^{-ik_x (mod(j-m,n_x))\Delta x_l}.$$

Let $\tilde{m}_j = mod(j-m,n_x)$ and exchange the order of summation:

$$\mathbf{W}^{N+1}(j) = \sum_{m=1}^{n_x} \mathbf{W}^N(m) \sum_{l=1}^{n_v} e^{-i[k_{z_l}(m)\Delta z + k_x(\tilde{m}_j)]\Delta x_l}.$$

This equation shows that in order to compute the jth component of the extrapolated wavefield in the ω-$\mathbf{k}$ domain, we need to take the dot product of the wavefield at the previous depth step with a vector that contains all the velocity and interpolation information. That is,

$$\mathbf{W}^{N+1}(j) = \mathbf{W}^{N} \cdot \mathbf{f}_j \tag{4}$$

where $\mathbf{f}_j$ is the vector given by

$$\mathbf{f}_j = \sum_{l=1}^{n_v} e^{-i[k_{z_l}(m)\Delta z + k_x(\tilde{m}_j)]\Delta x_l}. \tag{5}$$

PRACTICAL IMPLEMENTATION

The algorithm described by equations 4 and 5 is theoretically attractive because it shows in one equation that downward extrapolation can be done entirely in the ω-$\mathbf{k}$ domain, even for arbitrary spatial velocity variations. From the practical point of view, however, the algorithm is proportional, at each depth step and each frequency, to the cube of the spatial dimensions. Clearly, the cost is associated with the unreasonable demand that we extrapolated, at each depth step, as many wavefields as spatial positions. This is not really necessary, as we will see below.

Computation of the vertical wavenumber

Equation (5) contains all the velocity information in the data and can be precomputed, at least in part. Notice that, although we described the algorithm for $n_v = n\mathbf{x}$, that is not necessary because the range of velocities is limited and independent of the spatial dimensions of the data (although the velocities themselves vary spatially).

Start by binning the velocities in small bins, for example at 10 m/s (which would imply a maximum velocity error of 5 m/s, well below the likely error in the estimation of the velocities themselves) such that the vertical wavenumber k_{z_l} (that is, the dispersion relation), needs to be computed only a few hundred times and can thus be stored as a function of the horizontal wavenumber and the velocity. From the standpoint of the theoretical algorithm, all that changes is the selection process to choose the trace from the extrapolated wavefield that corresponds to the binned velocity at each spatial location. That is, instead of the selection being simply a multiplication by a Kronecker delta to choose $l = j$ as it was before, it is now a multiplication with a Kronecker delta, to select $l = p(j)$, that is, the wavefield that was migrated with the binned velocity corresponding to the bin of $V(j)$. Equation (2) can then be rewritten as:

$$\mathbf{w}^{N+1} = \sum_{l=1}^{n_v} \mathbf{w}_l^{N+1} \sum_{p} \delta_{pl}.$$

The equation for the wavefield extrapolation then becomes:

$$\mathbf{W}^{N+1}(j) = \sum_{m=1}^{n_x} \mathbf{W}^N(m) \sum_{l=1}^{n_v} \left(e^{-ik_{z_l}(m)\Delta z} \sum_{p} e^{-ik_x(\tilde{m}_j)\Delta x_p} \right). \tag{6}$$

Notice that summation over l involves summing over all the binned velocities whereas summation over p involves selecting the different wavefield components that correspond to a given velocity. That is, p ranges over the spatial locations whose binned velocity is V_l for each l. Figure 2 shows the velocity selection. This time, since we don't have a wavefield migrated with each velocity, at each spatial location, it is likely that several locations correspond to the same wavefield, since they correspond to the same velocity bin. There is, obviously, just one possible velocity at each spatial location, but many spatial locations may share the same velocity. Also, it is possible for a particular velocity not to be required at a specific depth step.

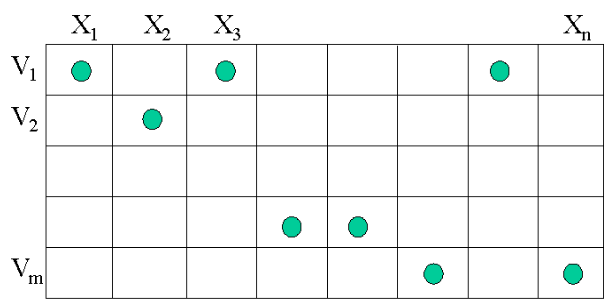

Figure 2: Diagram illustrating velocity selection when there are fewer velocities than spatial locations. gabriel1-bin_vels2 [NR]

Computation of the horizontal wavenumbers

The horizontal wavenumber term $k_x(\tilde{m}_j)$ can also be stored as a function of the circular-convolution index $\tilde{m}$ since it does not depend on the data itself.

Subsampling in ω-k Space

The previous subsections showed that the velocity part of the computation can be precomputed and therefore the cost of the algorithm becomes essentially quadratic in the spatial dimensions. For prestack 3D migration that is still too expensive. Notice however, that the algorithm doesn't have any significant approximation, since the velocities can be binned as finely as required by their intrinsic accuracy without significantly affecting the cost.

The question is whether we can reduce the computation time significantly by introducing reasonable approximations in the computation of each trace of the wavefield. It should be clear from Equation (6) that the cost of the algorithm comes from having to consider every single trace of the wavefield in the computation of every wavefield trace. We could, for instance, compute only every other wavefield trace in each of the axis of the ω-$\mathbf{k}$ space. For 3D prestack migration that alone would reduce the computation cost to one sixteenth. The extrapolated wavefield would then be interpolated at each depth step. Or, we can compute only say one in four traces in the cmp inline wavenumber axis and every trace in the cross-line offset wavenumber axis. This may be better since the cmp inline wavenumber axis is likely to be over-sampled whereas the xline offset wavenumber axis is not. Similarly, we may only consider traces of the wavefield in a given neighborhood for the computation of a given trace of the wavefield. If, for example, for the computation of each wavefield trace we use only the half traces closest to the trace being computed along each axis, again, for 3D prestack migration, that would imply a reduction of computation to only one sixteenth of the total computation. If we combine the two forms of computation savings we end up with an algorithm that may begin to be competitive with the mixed-domain algorithms, but that is simpler and more accurate in handling arbitrary lateral velocity variations.

Subsampling in the ω-$\mathbf{k}$ domain is akin to reducing the lateral extent of the wavefield in the ω-$\mathbf{x}$ domain. Whether this is acceptable and to what degree in each of the spatial axis is an unresolved issue at this point in our research. On physical grounds we can argue that the wavefield expands as it propagates so perhaps the approximation is valid at small depths but deteriorates at larger depths. Nothing prevents the subsampling to be a function of depth, making it an interesting issue to investigate further. Limiting the number of wavefield components that are actually used to the computation of another component may be acceptable in most cases since the wavefield is expected to be coherent in the ω-$\mathbf{k}$ domain. However, in specific, important cases, the wavefield may be irregular in the presence of sharp velocity discontinuities. In those cases it is not clear to what extent the approximation deteriorates.

Other issues

One advantage, that may be difficult to quantify, but that may none the less be significant is the simplicity of the algorithm. By doing away with the computation of FFT's, we are also implicitly simplifying the data access, which may be significant advantage for large datasets.

CONCLUSION AND FUTURE WORK

We have shown a theoretically attractive formulation of downward extrapolation in ω-$\mathbf{k}$ domain capable of handling arbitrary spatial velocity variations. The initial algorithm can be easily modified to make its cost quadratic in the spatial dimensions. This may be appropriate for 2D data or for post-stack 3D data but is certainly not good enough for prestack 3D data. By sacrificing some accuracy, yet to be quantified, we may be able to arrive to a competitive algorithm that is much easier to implement and whose accuracy can be improved at run time.

We need to research the issues of wavefield subsampling and the extent to which wavefield traces influence one another.

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