*Stanford Exploration Project, Report 115, May 22, 2004, pages 1–444*

# Seismic waves in finely layered VTI media: Poroelasticity, Thomsen parameters, and fluid effects on shear waves

*James G. Berryman*[1]

## ABSTRACT

Layered earth models are well justified by experience, and provide a simple means of studying fairly general behavior of the elastic and poroelastic characteristics of seismic waves in the earth. Thomsen's anisotropy parameters for weak elastic and poroelastic anisotropy are now commonly used in exploration, and can be conveniently expressed in terms of the layer averages of Backus. Since our main interest is usually in the fluids underground, it would be helpful to have a set of general equations relating the Thomsen parameters as directly as possible to the fluid properties. This end can be achieved in a rather straightforward fashion for these layered earth models, and the present paper develops and then discusses these relations. Furthermore, it is found that, although there are five effective shear moduli for any layered VTI medium, one and only one effective shear modulus for the layered system contains all the dependence of pore fluids on the elastic or poroelastic constants that can be observed in vertically polarized shear waves in VTI media. The effects of the pore fluids on this effective shear modulus can be substantial. An increase of shear wave speed on the order of 10% is shown to be possible when circumstances are favorable, which occurs when the medium behaves in an undrained fashion, and the shear modulus fluctuations are large (resulting in strong anisotropy). These effects are expected to be seen at higher frequencies such as sonic and ultrasonic waves for well-logging or laboratory experiments, or at seismic wave frequencies for low permeability regions of reservoirs, prior to hydrofracing. Results presented are strictly for velocity analysis.

## INTRODUCTION

Gassmann's fluid substitution formulas for bulk and shear moduli (Gassmann, 1951) were originally derived for the quasi-static mechanical behavior of fluid saturated rocks. It has been shown recently (Berryman and Wang, 2001) that deviations from Gassmann's results at higher frequencies, especially for shear modes, can be understood when the rock is heterogeneous on the microscale, and in particular when the rock heterogeneity anywhere is locally anisotropic. On the other hand, a well-known way of generating anisotropy in the earth is through fine layering. Then, Backus' averaging (Backus, 1962) of the mechanical behavior of

[1]**email:** berryman@sep.stanford.edu

the layered isotropic media at the microscopic level produces anisotropic mechanical behavior at the macroscopic level. For our present purposes, the Backus averaging concept can also be applied to fluid-saturated porous media, and thereby permits us to study how deviations from Gassmann's predictions could arise in an analytical and rather elementary fashion. We study layers of isotropic elastic/poroelastic materials because this is a simple, explicitly calculable model that nevertheless produces surprising results on the overall poroelastic shear modulus behavior. [If we considered instead layers of anisotropic poroelastic materials, the effects we want to study here concerning fluid-shear interactions would arrive before we begin, because they are often automatically present in anisotropic poroelastic materials as was shown earlier by Gassmann (1951) and others (Schoenberg and Douma, 1988; Sayers, 2002). So we could not show what we have set out to show here concerning the fluid effects by considering such inherently anisotropic models.] By studying both closed-pore and open-pore boundary conditions between layers within the chosen model, we learn in great detail just how violations of Gassmann's predictions can arise in undrained versus drained conditions, or for high versus low frequency waves.

We review some standard results concerning layered VTI media in the first two sections. Then, we discuss singular value composition of the elastic (or poroelastic) stiffness matrix in order to introduce the interpretation of one shear modulus (out of the five shear moduli present) that has been shown recently (Berryman, 2004) to contain all the important behavior related to pore fluid influence on the shear deformation response. These results are then incorporated into our analysis of the Thomsen parameters (originally derived for weak anisotropy, but used here for arbitrary levels of anisotropy). For purposes of analysis, expressions are derived for the quasi-P- and quasi-SV-wave speeds and these results are then discussed from this new point of view. Numerical examples show that the approximate analysis presented is completely consistent with the full theory for layered media. Our conclusions are summarized in the final section of the paper.

## NOTATION AND SOME PRIOR RESULTS

### Notation for VTI media

We begin by introducing some notation needed in the remainder of the paper. For transversely isotropic media with vertical symmetry axis, the relationship between components of stress $\sigma_{kl}$ and strain $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (where $u_j$ is the $j$th component of the displacement vector) is given by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \quad (1)$$

where $a = b + 2m$ (*e.g.*, Musgrave, 1970; Auld, 1973), with $i, j, k, l$ each ranging from 1 to 3 in Cartesian coordinates. The matrix describes isotropic media in the special case when $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$.

The Thomsen (1986) parameters $\epsilon$, $\delta$, and $\gamma$ are related to these stiffnesses by

$$\epsilon \equiv \frac{a - c}{2c}, \tag{2}$$

$$\delta \equiv \frac{(f + l)^2 - (c - l)^2}{2c(c - l)}, \tag{3}$$

$$\gamma \equiv \frac{m - l}{2l}. \tag{4}$$

Certain interpretations are allowed for these parameters when they are small enough. For P-wave propagation in the earth near the vertical, the important anisotropy parameter is $\delta$. For SV-wave propagation near the vertical, the combination $(c/l)(\epsilon - \delta)$ plays essentially the same role as $\delta$ does for P-waves. For SH-waves, the pertinent anisotropy parameter is $\gamma$. All three of the Thomsen parameters vanish for an isotropic medium, and the interpretations mentioned are valid for weakly anisotropic media such that all these parameters are relatively small ($< 1$). However, the definitions are also useful outside the range of these constraints, and we will use the same definitions (and also continue to call them the "Thomsen parameters") even when the smallness condition is violated; there is no fundamental problem doing this as long as it is recognized that the interpretations already mentioned in this paragraph are not necessarily valid any more when the parameters are large. This generalization of the Thomsen parameters will however require us to be careful in our subsequent usage of the parameters, as they cannot always be assumed to be small here as is usual in other treatments. Unless explicitly stated otherwise, the parameters $\epsilon$, $\gamma$, and $\delta$ are *not* small quantities in this paper.

It is also useful to note for later reference that

$$a = c(1 + 2\epsilon), \qquad m = l(1 + 2\gamma), \qquad \text{and} \qquad f \simeq c(1 + \delta) - 2l, \tag{5}$$

where smallness of $\delta$ was in fact assumed in the third expression. In TI media, $c$ and $l$ are directly related to the velocities normal to the layering. Then, $\epsilon$, $\gamma$, and $\delta$ measure the deviations from these normal velocities at other angles. We present the relevant details of the phase velocity analysis later in the paper.

### Gassmann results for isotropic poroelastic media

To understand the significance of the results to follow, we briefly review a well-known result due to Gassmann (1951) [also see Berryman (1999b) for a tutorial]. Gassmann's equation relates the bulk modulus $K^*$ of a saturated, undrained isotropic porous medium to the bulk modulus $K_{dr}$ of the same medium in the drained case:

$$K^* = K_{dr}/(1 - \alpha B), \tag{6}$$

where the parameters $\alpha$ and $B$ [respectively, the Biot-Willis parameter (Biot and Willis, 1957) and Skempton's pore-pressure buildup coefficient (Skempton, 1954)] depend on the porous medium and fluid compliances. For the shear moduli of drained ($\mu_{dr}$) and saturated ($\mu^*$) media, Gassmann's quasi-static theory gives

$$\mu^* = \mu_{dr}. \tag{7}$$

We want to emphasize once more that (7) is a result of the theory, not an assumption. The derivation of (6) and (7) shows that both results are elementary (and coupled) consequences of the theory. Furthermore, the two equations (6) and (7) taken together show that, for isotropic microhomogeneous media, the entire fluid effect on the overall elastic behavior is all contained in the parameter $\lambda^* = K^* - \frac{2}{3}\mu^*$, where $\lambda$ and $\mu$ are the well-known Lamé parameters. This result is crucial for understanding the significance of our later results to oil and gas exploration.

### Backus averaging

Backus (1962) presented an elegant method of producing the effective constants for a thinly layered medium composed of either isotropic or anisotropic elastic layers. This method applies either to spatially periodic layering or to random layering, by which we mean either that the material constants change in a nonperiodic (unpredictable) manner from layer to layer or that the layer thicknesses might also be random. For simplicity, we will assume that the physical properties of the individual layers are constant and isotropic. [For applications to porous earth materials, we implicitly make the typical assumptions of spatial stationarity within these layers as well as scale separation — *i.e.*, the sizes of the pores are much smaller than the thickness of the individual layers in which they reside.] The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying (in depth) coefficients multiply slowly varying stresses or strains.

The derivation has been given many places including Schoenberg and Muir (1989) and Berryman (1999a). Another illuminating derivation has been given recently by Milton (2002). We will not repeat any of the derivations here. The final results will be expressed in terms of averages $\langle Q \rangle$, where the brackets $\langle \cdot \rangle$ surrounding a variable $Q(z)$ indicate the volume average (or, equivalently, the linear average with depth in the vertically layered medium under consideration) of the quantity $Q$. It follows that the anisotropy coefficients in equation (1) are then related to the layer parameters by the following well-known expressions:

$$c = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}, \tag{8}$$

$$f = c \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle, \tag{9}$$

$$l = \left\langle \frac{1}{\mu} \right\rangle^{-1}, \tag{10}$$

$$m = \langle \mu \rangle, \tag{11}$$

$$a = \frac{f^2}{c} + 4m - 4\left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle, \tag{12}$$

and

$$b = a - 2m. \tag{13}$$

When the layering is fully periodic, these results may be attributed to Bruggeman (1937) and Postma (1955), while for more general layered media including random media they should be attributed to Backus (1962). The constraints on the Lamé parameters $\lambda$ and $\mu$ for each individual layer are $0 \le \mu \le \infty$ and $-\frac{2}{3}\mu \le \lambda \le \infty$. Although, for physically stable materials, shear modulus $\mu$ and bulk modulus $K = \lambda + \frac{2}{3}\mu$ must both be nonnegative, these relations mean that $\lambda$ (and also Poisson's ratio $\nu$) may be negative (but nevertheless bounded below, since $\nu \ge -1$, and $\lambda \ge -2\mu/3$). Large fluctuations in $\lambda$ for different layers are therefore entirely possible, in principle, but may or may not be an issue for any given region of the earth.

Large fluctuations in $\mu$ are also possible, and the Backus averaging technique is fully capable of handling all such fluctuations properly. But, if these fluctuations are too large, then the weak anisotropy assumption of Thomsen's original work (Thomsen, 1986) will be violated and some care must be taken when writing approximate equations. We do *not* at any point assume weak anisotropy in this paper [except equations (5) and (28)], since the shear behavior we are trying to study will be shown to depend on the presence of strong anisotropy in this sense. We will also find it useful to develop alternatives to some of Thomsen's formulas in order to deal with the strong anisotropy that arises in our analysis.

One very important fact known about the Backus averaging equations (Backus, 1962) is that they reduce to isotropic results with $a = c$, $b = f$, and $l = m$, if the shear modulus is a constant ($= \mu$) — regardless of the behavior of $\lambda$. This fact is also very important for applications involving partial and/or patchy saturation (Mavko *et al.*, 1998; Johnson, 2001). Furthermore, this fact is closely related to the well-known bulk modulus formula of Hill (1963) for isotropic composites having uniform shear modulus, and also to the Hashin-Shtrikman bounds (Hashin and Shtrikman, 1961), which can be used to provide an elementary proof of Hill's equation. Nevertheless, this limit will not be of much interest to us here except as a boundary condition on the results obtained. Furthermore, one of the main purposes of the paper is to show how deviations from these limiting and rather restictive results affect the predictions of the referenced work on partial and patchy saturation.

## THOMSEN PARAMETERS $\epsilon$ AND $\delta$

### Thomsen's $\epsilon$

An important anisotropy parameter for quasi-$SV$-waves (which is our main interest in this paper) is Thomsen's parameter $\epsilon$, defined in equation (2). Formula (12) for $a$ may be rewritten as

$$a = \left\langle \frac{(\lambda+2\mu)^2 - \lambda^2}{\lambda+2\mu} \right\rangle + c \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2, \tag{14}$$

which can be rearranged into the convenient and illuminating form

$$a = \langle \lambda+2\mu \rangle - c \left[ \left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right]. \tag{15}$$

This formula is very instructive because the term in square brackets is in Cauchy-Schwartz form $[\langle q^2 \rangle \langle Q^2 \rangle \geq \langle qQ \rangle^2]$, so this factor is nonnegative. Furthermore, the magnitude of this term depends mainly on the fluctuations in the $\lambda$ Lamé parameter, and is largely independent of $\mu$, since $\mu$ appears only in the weighting factor $1/(\lambda+2\mu)$. Clearly, if $\lambda = constant$, then this bracketed factor vanishes identically, regardless of the behavior of $\mu$. Large fluctuations in $\lambda$ will tend to make this term large. If in addition we consider Thomsen's parameter $\epsilon$ written in a similar fashion as

$$2\epsilon = \left[ \langle \lambda+2\mu \rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - 1 \right] - \left[ \left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right], \tag{16}$$

we find that the term enclosed in the first bracket on the right hand side is again in Cauchy-Schwartz form showing that it always makes a positive contribution unless $\lambda+2\mu = constant$, in which case it vanishes. Similarly, the term enclosed in the second set of brackets is always non-negative, but the minus preceding the second bracket causes this contribution to make a negative contribution to $2\epsilon$ unless $\lambda = constant$, in which case it vanishes. So, in general the sign of $\epsilon$ is indeterminate. The Thomsen parameter $\epsilon$ may have either a positive or a negative sign for a TI medium composed of arbitrary thin isotropic layers. Thomsen (2002) states that $\epsilon > 0$ if $K$ and $\mu$ are positively correlated. But (16) shows that such correlations only produce $\epsilon > 0$ with certainty if they are also supplemented by the stronger condition that $\lambda \simeq const$ [in fact, $\lambda \simeq const$ implies that there is a positive correlation between $K$ and $\mu$, but the reverse does not necessarily hold unless we also assume that the fluctuations in $\mu$ are quite small — an assumption that we do not make here].

Fluctuations of $\lambda$ in the earth have important implications for oil and gas exploration. As we recalled in our earlier discussion, Gassmann's well-known results (Gassmann, 1951) show that, when isotropic porous elastic media are saturated with any fluid, the fluid has no mechanical effect on the shear modulus $\mu$, but — when these results apply — it can have a significant effect on the bulk modulus $K = \lambda + \frac{2}{3}\mu$, and therefore on $\lambda$. Thus, observed (high spatial frequency) variations in layer shear modulus $\mu$ should have no direct information about fluid content, while such variations observed in layer Lamé parameter $\lambda$, especially if they are large variations, may contain important clues about variations in fluid content. So the observed structure of $\epsilon$ in (16) strongly suggests that small positive and all negative values of $\epsilon$ may be important indicators of significant fluctuations in fluid content (Berryman *et al.*, 1999).

**Thomsen's $\delta$**

Thomsen's parameter $\delta$ defined by Eq. (3) is pertinent for near vertical quasi-$P$-waves and can also be rewritten as

$$\delta = -\frac{(c+f)(c-f-2l)}{2c(c-l)}. \tag{17}$$

This parameter is considerably more difficult to analyze than either $\gamma$ or $\epsilon$ for various reasons, some of which we will enumerate shortly. Thomsen (2002) provides some insight into the behavior of $\delta$ by noting that its sign depends only on the variations of the ratio $V_s/V_p$. This can be seen to be true from its definition by noting that

$$c-f-2l = 2cl\left[\left\langle\frac{1}{\mu}\right\rangle\left\langle\frac{\mu}{\lambda+2\mu}\right\rangle - \left\langle\frac{1}{\lambda+2\mu}\right\rangle\right] = -2cl\left\langle\frac{1}{\mu}\cdot\Delta\left(\frac{V_s^2}{V_p^2}\right)\right\rangle, \tag{18}$$

where

$$\Delta\left(\frac{V_s^2}{V_p^2}\right) \equiv \frac{V_s^2}{V_p^2} - \left\langle\frac{V_s^2}{V_p^2}\right\rangle. \tag{19}$$

Because of a controversy surrounding the sign of $\delta$ for finely layered media (*e.g.*, Levin, 1988; Thomsen, 1988; Anno, 1997), Berryman *et al.* (1999) performed a series of Monte Carlo simulations with the purpose of establishing the existence or nonexistence of layered models having positive $\delta$. Those simulation results should be interpreted neither as modeling of natural sedimentation processes nor as an attempt to reconstruct any petrophysical relationships. The main goal was to develop a general picture of the distribution of the sign of $\delta$ using many choices of constituent material properties. This analysis established a similarity in the circumstances between the occurrence of positive $\delta$ and the occurrence of small positive $\epsilon$ (*i.e.*, both occur when Lamé $\lambda$ is fluctuating greatly from layer to layer). The positive values of $\delta$ are in fact most highly correlated with the smaller positive values of $\epsilon$. We should also keep in mind the fact that $\epsilon - \delta \geq 0$ is always true for models with isotropic layers (Postma, 1955; Berryman, 1979) and this fact also plays a role in these comparisons, determining the unoccupied upper left hand corner of a $\delta$ vs. $\epsilon$ plot.

## SINGULAR VALUE DECOMPOSITION FOR STIFFNESS MATRIX

The singular value decomposition (SVD), or equivalently the eigenvalue decomposition, of the real symmetric stiffness matrix appearing in (1) is relatively easy to perform. We can immediately write down four eigenvectors:

$$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \tag{20}$$

and their corresponding eigenvalues, respectively $2l$, $2l$, $2m$, and $a-b=2m$. All four correspond to shear modes of the system. The two remaining eigenvectors must be orthogonal to all four of these and therefore both must have the general form

$$\begin{pmatrix} 1 \\ 1 \\ X \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{21}$$

Applying (21) to the stiffness matrix in (1) shows that the corresponding eigenvalue is

$$\chi = a + b + fX, \tag{22}$$

where the remaining condition that determines both $X$ and $\chi$ is

$$\chi X = 2f + cX. \tag{23}$$

After substitution for $\chi$, we obtain a quadratic equation having the solutions

$$X_{\pm} = \frac{1}{2}\left(-\left[\frac{a+b-c}{f}\right] \pm \sqrt{8 + \left[\frac{a+b-c}{f}\right]^2}\right). \tag{24}$$

The ranges of values for $X_{\pm}$ are $0 \le X_+ \le \infty$ and, since $X_- = -2/X_+$, $-\infty \le X_- \le 0$. The interpretation of the solutions $X_{\pm}$ is simple for the isotropic limit where $X_+ = 1$ and $X_- = -2$, corresponding respectively to pure compression and pure shear modes. So, except for special angles of propagation, these two modes always have mixed character, indicating that pure compression cannot be excited in the system, and must always be coupled to shear. Some types of pure shear modes can still be excited even in the nonisotropic cases, because the other four eigenvectors in (20) are unaffected by this coupling, and they are all pure shear modes. Pure compressional and shear modes are obtained as linear combinations of these two mixed modes according to

$$r\begin{pmatrix} 1 \\ 1 \\ X_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ X_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1+r)\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{25}$$

with $r = -2(X_+ - 1)/[X_+(X_+ + 2)]$ for pure shear, and

$$\begin{pmatrix} 1 \\ 1 \\ X_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ X_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1+s)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{26}$$

with $s = X_+(X_+ - 1)/(X_+ + 2)$ for pure compression.

To understand the behavior of $X_+$ in terms of the layer property fluctuations or, alternatively, in terms of the Thomsen parameters, it is first helpful to note that the pertinent functional $F(x) = \frac{1}{2}\left[-x + \sqrt{8+x^2}\right]$ is easily shown to be a monotonic function of its argument $x$. So it is sufficient to study the behavior of the argument $x = (a+b-c)/f$.

**Exact results for isotropic layers**

Combining results from Eqs. (8)–(12), we find after some work on rearranging the terms that

$$\frac{a+b-c}{f} = \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^{-1}\left[\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle + 6\left\langle\frac{m-\mu}{\lambda+2\mu}\right\rangle - 8\left\{\left\langle\frac{\mu^2}{\lambda+2\mu}\right\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - \left\langle\frac{\mu}{\lambda+2\mu}\right\rangle^2\right\}\right], \tag{27}$$

where the correction involving $m-\mu$ in the numerator is the difference of the shear modulus from the layer-averaged shear modulus $m$, and will be the dominant correction when fluctuations in $\mu$ are small. The fact that $\langle(m-\mu)/\mu\rangle = \langle\mu\rangle\langle 1/\mu\rangle - 1 \geq 0$, suggests that this dominant correction to unity (since the leading term is exactly unity) for this expression will be positive if $\lambda$ and $\mu$ are positively correlated throughout all the layers, but the correction could be negative in cases where there is a strong negative correlation between $\lambda$ and $\mu$. On the other hand, the term in curly brackets in (27) is again in Cauchy-Schwartz form (*i.e.*, $\langle q^2\rangle\langle Q^2\rangle - \langle qQ\rangle^2 \geq 0$), and therefore is always non-negative. But, since it is multiplied by $-1$, the contribution to this expression is non-positive. This term is also quadratic in the deviations of $\mu$ from its layer average, and thus is of higher order than the term explicitly involving $m-\mu$. So, if the fluctuations in shear modulus are very large throughout the layered medium, the quadratic terms can dominate — in which case the overall result could be less than unity. Numerical examples developed by applying a code of V. Grechka [used previously in a similar context by Berryman *et al.* (1999)] confirm these analytical results.

Our main conclusion is that the shear modulus fluctuations giving rise to the anisotropy due to stacks of thin isotropic layers are (as expected) the main source of deviations of (27) from unity. But now we can say more, since positive deviations of this parameter from unity are generally associated with smaller magnitude fluctuations of the layer shear modulus, whereas negative deviations from unity must be due to large magnitude fluctuations in these shear moduli.

**Approximate results if Thomsen parameters have small values**

Using the definitions of the Thomsen parameters, we can also rewrite the terms appearing in (27) in order to make connection with this related point of view. Recalling (5) and the fact that $b = a - 2m$, we have

$$\frac{a+b-c}{f} \simeq 1 + \frac{3}{c-2l}(c\delta + 4l\gamma) + \frac{4}{c-2l}\left[c(\epsilon-\delta) - 4l\gamma\right], \tag{28}$$

with some higher order corrections involving powers of $\delta$ and products of $\delta$ with $\epsilon$ and $\gamma$ that we neglected in this equation. We have added and subtracted equally some terms proportional to $\delta$, and others proportional to $\gamma$, in order to emphasize the similarities between the form (28) and that found previously in (27). In particular, the difference $\epsilon - \delta$ is known (Postma, 1955; Berryman, 1979) to be non-negative and its deviations from zero depend on fluctuations in $\mu$ from layer to layer, behavior similar to that of the final term in (27). Since the formula (28) is only approximate and its interpretation requires the use of various other results we derive subsequently for other purposes, for now we will delay further discussion of this to a point later in the paper. [See the discussion of Eq. (62).]

## DISPERSION RELATIONS FOR SEISMIC WAVES

The equations of motion and their solutions for seismic waves in anisotropic media are well known, and have been derived in many places including Berryman (1979) and Thomsen (1986). The dispersion relations for phase velocities are

$$\rho\omega_\pm^2 = \frac{1}{2}\left\{(a+l)k_1^2 + (c+l)k_3^2 \pm \sqrt{[(a-l)k_1^2 - (c-l)k_3^2]^2 + 4(f+l)^2k_1^2k_3^2}\right\}, \tag{29}$$

for quasi-compressional (+) waves and quasi-SV (−) waves (*i.e.*, vertically polarized quasi-shear waves, by which we mean the plane normal to the cross-product of the polarization vector and the propagation vector is vertical) and

$$\rho\omega_s^2 = mk_1^2 + lk_3^2, \tag{30}$$

for horizontally polarized shear waves. In these equations, $\rho$ is the overall density (including fluids when present), $\omega$ is the angular frequency, $k_1$ and $k_3$ are horizontal and vertical wavenumbers (respectively), and the phase velocities are determined simply by $V = \omega/k$ with $k = \sqrt{k_1^2 + k_3^2}$. Elastically, the SH wave depends only on the two parameters $l$ and $m$, which are not dependent in any way on layer Lamé parameter $\lambda$ and, therefore, will play no role in the poroelastic analysis. The densities of any fluids present affect all three wave speeds equally, and cannot therefore contribute to shear wave bi-refringence by itself. Thus, we can safely ignore SH except when we want to check for shear wave splitting — in which case the SH results will be most useful as a baseline for such comparisons.

The dispersion relations for quasi-P- and quasi-SV-waves can be rewritten in a number of instructive ways. One of these that we will choose for reasons that will become apparent shortly is

$$\rho\omega_\pm^2 = \frac{1}{2}\Big[(a+l)k_1^2 + (c+l)k_3^2 \pm \sqrt{[(a+l)k_1^2 + (c+l)k_3^2]^2 - 4[(ak_1^2 + ck_3^2)lk^2 + \{(a-l)(c-l) - (f+l)^2\}k_1^2k_3^2]}\Big]. \tag{31}$$

Written this way, it is obvious that the following two relations hold:

$$\rho\omega_+^2 + \rho\omega_-^2 = (a+l)k_1^2 + (c+l)k_3^2, \tag{32}$$

and

$$\rho\omega_+^2 \cdot \rho\omega_-^2 = (ak_1^2 + ck_3^2)lk^2 + [(a-l)(c-l) - (f+l)^2]k_1^2k_3^2, \tag{33}$$

either of which could have been obtained directly from (29) without the intermediate step of (31).

We are motivated to write the equations in this way in order to try to avoid evaluating the square root in (29) directly. Rather, we would like to arrive at a natural approximation that is quite accurate, but does not involve the square root operation. The desire to do this is not new (Thomsen, 1986), but our goal is different since we *must necessarily* treat strong anisotropy in this paper. From a general understanding of the problem, it is clear that a reasonable way of making use of (32) is to make the identifications

$$\rho\omega_+^2 \equiv ak_1^2 + ck_3^2 - \Delta, \tag{34}$$

and

$$\rho\omega_-^2 \equiv lk^2 + \Delta, \tag{35}$$

with $\Delta$ still to be determined. Then, substituting these expressions into (33), we find that

$$(ak_1^2 + ck_3^2 - lk^2 - \Delta)\Delta = [(a-l)(c-l) - (f+l)^2]k_1^2k_3^2 \tag{36}$$

Solving (36) for $\Delta$ would just give the original results back again. So the point of (36) is not to solve it exactly, but rather to use it as the basis of an approximation scheme. If $\Delta$ is small, then we can presumably neglect it inside the parenthesis on the left hand side of (36) — or we could just keep a small number of terms in an expansion.

The leading term, and the only one we will consider here (but see the Appendix for further discussion), is

$$\Delta = \frac{[(a-l)(c-l) - (f+l)^2]k_1^2k_3^2}{(a-l)k_1^2 + (c-l)k_3^2 - \Delta} \simeq \frac{[(a-l)(c-l) - (f+l)^2]}{(a-l)/k_3^2 + (c-l)/k_1^2}. \tag{37}$$

The numerator of this expression is known to be a positive quantity for layers of isotropic materials (Postma, 1955; Berryman, 1979). Furthermore, it can be rewritten (without approximation) in terms of Thomsen's parameters as

$$[(a-l)(c-l) - (f+l)^2] = 2c(c-l)(\epsilon - \delta). \tag{38}$$

Using the first of the identities noted earlier in (5), we can also rewrite the first elasticity factor in the denominator as $a - l = (c-l)[1 + 2c\epsilon/(c-l)]$. Combining these results in the limit of $k_1^2 \to 0$ (for relatively small horizontal offset), we find that

$$\rho\omega_+^2 \simeq ck^2 + 2c\delta k_1^2, \tag{39}$$

and

$$\rho\omega_-^2 \simeq lk^2 + 2c(\epsilon - \delta)k_1^2, \tag{40}$$

with $\Delta \simeq 2c(\epsilon-\delta)k_1^2$ for very small angles from the vertical. These two equations may be recognized simply as small angle approximations to the weak-anisotropy equations of Thomsen (1986). However, the main thrust of this paper (as we will soon see) requires strong anisotropy and therefore also requires improved approximations, which can be obtained to any desired order with only a little more effort by using (36) instead of the first approximation derived here in (37). Note that Eqs. (39) and (40) were derived without assumptions about the smallness of $\epsilon$ or $\delta$.

Although the approximations being discussed in this section are of some practical interest in their own right, their elaboration at this point would lead us away from the main theme of the paper. So, to avoid further digression here from the issue of fluid effects on shear modulus, we collect our remaining results concerning these dispersion relation approximations in the Appendix.

## INTERPRETATION OF P AND SV COEFFICIENTS FOR LAYERED MEDIA

### General analysis for VTI media

The correction terms, *i.e.*, those contained in the factor $\Delta$ in (35) for quasi-SV waves in anisotropic media, are proportional to the factor

$$\mathcal{A} \equiv (a-l)(c-l)-(f+l)^2 = 2c(c-l)(\epsilon-\delta), \tag{41}$$

which is sometimes called the *anellipticity parameter*. Similarly, we will call $\Delta$ the *anellipticity correction*. For the case of strong anisotropy that we are considering here, the presence of $\mathcal{A}/(c-l)$ in (40) just introduces ellipticity into the move out, but the higher order corrections that we neglected can introduce deviations from ellipticity — hence anellipticity.

Clearly, from (40) for quasi-SV-waves [and in layered media at this order of approximation], the anellipticity parameter holds all the information about the presence or absence of fluids that is not already contained in the density factor $\rho$. So it will be worth our time to study this factor in more detail. First note that, after rearrangement, we have the general identity

$$\mathcal{A} = (f+l)(a+c-2f-4l)+(a-f-2l)(c-f-2l), \tag{42}$$

which is true for all transversely isotropic media.

In some earlier work (Berryman, 2003), the author has shown that it is convenient to introduce two special-purpose effective shear moduli $\mu_1^*$ and $\mu_3^*$ associated with $a$ and $c$, namely,

$$\mu_1^* \equiv a-m-f \qquad \text{and} \qquad 2\mu_3^* \equiv c-f. \tag{43}$$

Furthermore, it was shown that the combination defined by

$$G_{eff} = (\mu_1^*+2\mu_3^*)/3 \tag{44}$$

plays a particular role in the theory, as it is only this effective shear modulus for the anisotropic system that can also contain information about fluid content. It turns out that (42) can be rewritten in terms of this effective shear modulus if we first introduce two more parameters:

$$\mathcal{K} = f + l + \left[ \frac{1}{a - f - 2l} + \frac{1}{c - f - 2l} \right]^{-1} \tag{45}$$

and

$$\mathcal{G} = \left[ 3G_{eff} + m - 4l \right] / 3. \tag{46}$$

Then, (42) can be simply rewritten as

$$\mathcal{A} = 3\mathcal{K}\mathcal{G}. \tag{47}$$

This result is analogous to, but distinct from, a product formula relating the effective shear modulus $G_{eff}$ and the bulk modulus

$$K = f + \left[ \frac{1}{a - m - f} + \frac{1}{c - f} \right]^{-1} \tag{48}$$

to the eigenvalues of the elastic matrix according to

$$\chi_+ \chi_- = 6KG_{eff}. \tag{49}$$

Eq. (49) can be motivated by noting that, in the isotropic limit, the eigenvalues are $3K$ and $2\mu$.

[Side notes concerning layered materials: In the isotropic limit, when $\mu \rightarrow constant$, we have $K \rightarrow f + 2\mu/3$, while $\mathcal{K} \rightarrow f + \mu$. So these two parameters are not the same, but they do have strong similarities in their behavior. In contrast, $G_{eff} \rightarrow \mu$, while $\mathcal{G} \rightarrow 0$ in the same limit. It is also possible to show for layered materials that in general $l \leq \mathcal{K} - f \leq m$, with the lower limit being optimum, *i.e.*, attainable.]

Also, since Thomsen's $\delta$ plays an important role in (39), it is helpful to note that (17) can also be rewritten as

$$c\delta = -(c - f - 2l) \left[ 1 - \frac{c - f - 2l}{2(c - l)} \right], \tag{50}$$

which shows that, at least for weakly anisotropic media (in which case the deviation from unity inside the brackets is neglected), $c\delta$ is very nearly a direct measure of the quantity $c - f - 2l$.

### Analysis for isotropic layers

The analysis presented in the previous subsection is general for all VTI elastic media. But we can say more by assuming now that the anisotropy arises due to layers of isotropic elastic (or possibly poroelastic) media. Then, using (8)-(12), we have the following relations

$$f + 2l = c \left\langle \frac{\lambda + 2l}{\lambda + 2\mu} \right\rangle, \tag{51}$$

$$c - f - 2l = 2c\left\langle \frac{\mu - l}{\lambda + 2\mu} \right\rangle, \tag{52}$$

and

$$a - f - 2l = 2c\left\{\left\langle \frac{2m - \mu - l}{\lambda + 2\mu} \right\rangle - 2\left[\left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle\left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle^2\right]\right\}. \tag{53}$$

Eq. (51) is an easy consequence of the Backus averaging formulas. Then, (52) shows that $c$ differs from $f + 2l$ only by a term that measures the difference in the weighted average of $\mu$ and $l$. Eq. (53) shows that $a$ differs from $f + 2l$ in a more complicated fashion that depends on the difference in the weighted average of $(2m - l)$ and $\mu$, as well as a term that is higher order in the fluctuations of the layer $\mu$ values. Combining these results, we have

$$G_{eff} = m - \frac{4c}{3}\left[\left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle\left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle^2\right], \tag{54}$$

showing that all the interesting behavior (including strong $\mu$ fluctuations in the layers together with $\lambda$ dependence) is collected in $G_{eff}$. Since the product of (52) and (53) is clearly of higher order in the fluctuations of the layer shear moduli, it is not hard to see that, to leading order when these fluctuation effects are small,

$$\mathcal{A} \simeq (c - l)(3G_{eff} + m - 4l). \tag{55}$$

To give a quick estimate, note that if all the layers have the same value of Poisson's ratio, then the ratio $r = \lambda/\mu$ is constant. Then, it is easy to show that $G_{eff} = m - 4(m - l)/3(2 + r)$. Since $-2/3 \le r \le \infty$, the effective shear modulus for this class of models lies in the range $l \le G_{eff} \le m$. From this fact, we can conclude that the important coefficient in (40) is given to a good approximation by

$$2c(\epsilon - \delta) \simeq 3G_{eff} + m - 4l, \tag{56}$$

and ranges from $2l\gamma$ to $8l\gamma$.

To study the fluid effects, the drained Lamé parameter $\lambda$ in each layer should be replaced under undrained conditions by

$$\lambda^* = K^* - 2\mu/3, \tag{57}$$

where $K^*$ was defined by (6). Then, for small fluctuations in $\mu$, Eq. (56) shows that the leading order terms due to these shear modulus variations contributing to $\epsilon - \delta$ actually do not depend on the fluids at all (since $m - l$ does not depend on them). With no fluid in the pores, there is a contribution to the shear wave speed for SV in layered media, just due to the fluctuations in the shear moduli. One part of the contribution is always independent of any fluids that might be present, but the magnitude of this contribution (which is always positive) is small whenever the difference $m - l$ is also small. If $m - l$ is large, then the magnitude of the additional increase due to liquids in the pores can be very substantial as we will see in the following examples.

So the effects of liquids on $G_{eff}$ will generally be weak when the fluctuations in $\mu$ are weak, and strong when they are strong.

Furthermore, when the product $\alpha B \neq 0$, we first choose to define

$$ratio_{\alpha B} = \frac{m - G_{eff}}{m - l}. \tag{58}$$

so that, for all possible layered models, we have $0 \leq ratio_{\alpha B} \leq 1$. Then, we consider plotting the quantity $1 - ratio_{\alpha B}/ratio_0$ versus $\gamma$ (which we treat as a simple quantitative measure of the fluctuations in the layer shear moduli). To generate a class of 900 models for each of three choices of $\alpha$ (treated as a single constant for all layers in each individual model) in order to illustrate the behavior of these quantities, I made use of a code of V. Grechka [used previously in a joint publication (Berryman *et al.*, 1999)]. This code chooses layer parameters randomly from within the following (arguable, but generally reasonable) range of values: $1.5 \leq V_p \leq 5.0$ km/s, $0.1 \leq V_s/V_p \leq 0.8$, and $1.8 \leq \rho \leq 2.8 \times 10^3$ kg/m$^3$. The results are displayed in Figure 1 for $\alpha = 0.5$, 0.8, and 0.9. We find empirically that (for $B = 1$) the values never exceed $\alpha$ for any set of choices for the layer model parameters. This apparent fact (as determined by these computer experiments) does not appear to be easy to prove from the general formula. But one simple though nontrivial calculation we can do is based again on an assumption that the bulk moduli in the layers are always proportional to the shear modulus, so $K = s\mu$, for some fixed value of of the proportinality factor $s > 0$. Then, for a given model, we find that

$$1 - \frac{ratio_{\alpha B}}{ratio_0} = \frac{\alpha B}{1 + 4(1 - \alpha B)/3s} \leq \alpha B, \tag{59}$$

in agreement with the empiricial result from the synthetic data shown in Figure 1.

To check the corresponding result for P-waves, we need to estimate $\delta$. Making use of (50), we have

$$c\delta = -2c\left\langle \frac{\mu - l}{\lambda + 2\mu} \right\rangle \left[ 1 - l^{-1} \left\langle \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \right\rangle^{-1} \left\langle \frac{\mu - l}{\lambda + 2\mu} \right\rangle \right]. \tag{60}$$

Working to the same order as we did for the final expression in (56), we can neglect the second term in the square brackets of (60). What remains shows that pore fluids would have an effect on this result. The result is

$$c^*\delta^* \simeq -2c^* \left\langle \frac{\mu - l}{\lambda^* + 2\mu} \right\rangle. \tag{61}$$

If desired, a similar replacement can also be made for $G_{eff}$ in (44) using the fact that $2(\mu_3^* - l) = c - f - 2l$. Eq. (61) shows that, since $c^*$ and $\delta^*$ both depend on the $\lambda^*$'s (although in opposite ways, since one increases while the other decreases as $\lambda^*$ increases), the product of these factors will have some dependence on fluids. The degree to which fluctuations in $\lambda^*$ and $\mu$ are correlated, or anticorrelated, as they vary from layer to layer will also affect these results in predictable ways.

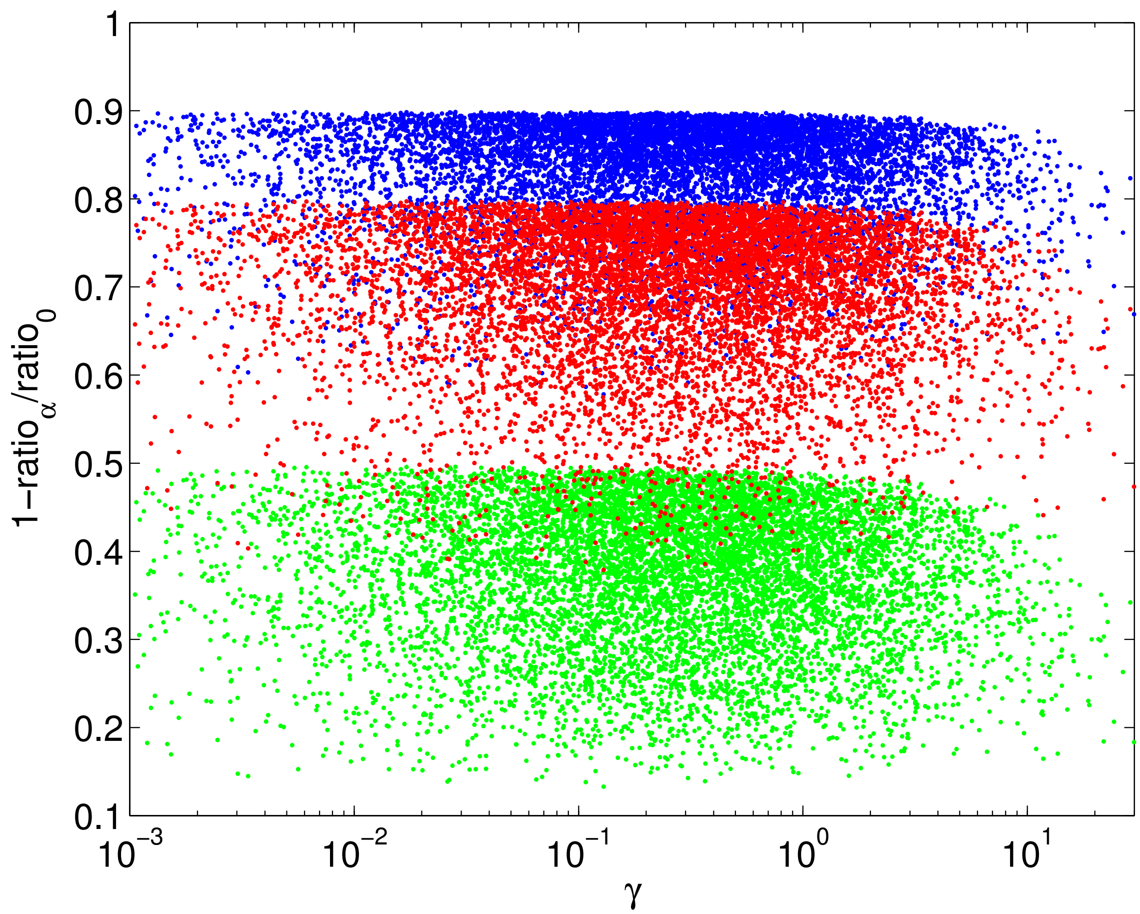


Figure 1: .
Blue dots are for $\alpha = 0.9$, red for $\alpha = 0.8$, and green for $\alpha = 0.5$. Note, that in each case, all the points for a particular choice of $\alpha$ are bounded above precisely by the value of $\alpha$. (A general proof of this empirical observation is currently lacking.) ]Scatter plot illustrating how $G_{eff}$ varies over a physically sensible range of layered isotropic media (see text for details) with 2700 distinct models and $B = 1$ [see Eq. (58) in the text for the definition of $ratio_\alpha$]. Blue dots are for $\alpha = 0.9$, red for $\alpha = 0.8$, and green for $\alpha = 0.5$. Note, that in each case, all the points for a particular choice of $\alpha$ are bounded above precisely by the value of $\alpha$. (A general proof of this empirical observation is currently lacking.) jim1-scatter [NR]

### Interpretation of the results

Now we have derived all the results needed to interpret Eq. (28) and show how it is related to (27). First, we note some of the main terms missing from (28) are those due to approximations made to $\delta$ and the denominators of (27), which have been approximated as $f \simeq c - 2l$ instead of $f \simeq c(1+\delta) - 2l$. Then, from (56), it is easy to see that the final term in (28) vanishes to lowest order, and that the remainder is given exactly by the shear modulus fluctuation terms in brackets in (53) — in complete agreement with the final terms of (27). Then, from (60), it follows that the leading contribution to the factor $c\delta + 4l\gamma$ is

$$c\delta + 4l\gamma \simeq 2c\left\langle \frac{m-\mu}{\lambda+2\mu} \right\rangle, \tag{62}$$

in complete agreement with the second term on the right hand side of (27).

In the case of very strong fluctuations in the layer shear moduli, then (53) and (60) both show that pore fluids effects are magnified due to the fluctuations in layer shear moduli and, therefore, contribute more to the anisotropy correction factors $2c^*(\epsilon^* - \delta^*)$ and $2c^*\delta^*$ for undrained porous media. So these effects will be more easily observed in seismic, sonic, or ultrasonic data under these circumstances. When these effects are present, the vertically polarized quasi-shear mode will show the highest magnitude effect, the horizontally polarized shear mode will show no effect, and the quasi-compressional mode will show an effect of intermediate magnitude. It is known that these effects, when present, are always strongest at 45°, and are diminished when the angle of propagation is either 0° or 90° relative to the layering direction. We will test these analytical predictions with numerical examples in the next section.

To summarize our main result here: The most significant contributions of the liquid dependence to shear waves comes into the wave dispersion formulas through coefficient $a$ (or equivalently $\epsilon$). Equations (53) and (54) show that

$$a = 2f - c + m + 3G_{eff}. \tag{63}$$

For small fluctuations in $\mu$, coefficients $a$ and $c$ have comparable magnitude dependence on the fluid effects, but of opposite sign. For large fluctuations, the effects on $a$ are much larger (quadratic) than those on $c$ (linear). Propagation at normal incidence will never show much effect due to the liquids, while propagation at angles closer to 45° can show large enhancements in both quasi-P and quasi-SV waves (when shear fluctuations are large), but still no effect on SH waves.

## COMPUTED EXAMPLES

From previous work (Berryman, 2003), we know that large fluctuations in the layer shear moduli are required before significant deviations from Gassmann's quasi-static constant result, thereby showing that the shear modulus dependence on fluid properties can become noticeable. To generate a model that demonstrates these results, again I made use of the same code of V.

Grechka as described when presenting Figure 1. But this time I arbitrarily picked just one of the models that seemed to be most interesting for the present purposes. The parameters of this model are displayed in TABLE 1. The results for the various elastic coefficients and Thomsen parameters are displayed in TABLE 2. The results of the calculations for $V_p$ and $V_{sv}$ are shown in Figures 1 and 2.

The model calculations were simplified in one way: the value of the Biot-Willis parameter was chosen to be a uniform value of $\alpha = 0.8$ in all layers. We could have actually computed a value of $\alpha$ from the other layer parameters, but to do so would require another assumption about the porosity values in each layer. Doing this seemed an exercise of little value because we are just trying to show in a simple way that the formulas given here really do produce the types of results predicted analytically, and also to get a feeling for the magnitude of the effects. Furthermore, if $\alpha$ is a constant, then it is only the product $\alpha B$ that matters. Whatever choice of constant $\alpha \leq 1$ is made, it mainly determines the maximum value of the product $\alpha B$ for $B$ in the range $[0,1]$. So, for a parameter study, it is only important not to choose too a small value of $\alpha$, which is why the choice $\alpha = 0.8$ was made. This means that the maximum amplification of the bulk modulus due to fluid effects can be as high as a factor of 5 $[= 1/(1-\alpha)]$ for the present examples.

TABLE 1. Layer parameters for the three materials in a simple layered medium used to produce the examples in Figures 2 and 3. For this model, $\gamma = 7.882$ (indicating strong anisotropy).

| *Constituent* | $K$ (GPa) | $\mu$ (GPa) | $z$ (m/m) |
|---|---|---|---|
| 1 | 9.4541 | 0.0965 | 0.477 |
| 2 | 14.7926 | 4.0290 | 0.276 |
| 3 | 43.5854 | 8.7785 | 0.247 |

TABLE 2. The VTI elastic coefficients and Thomsen parameters for the materials (see Table 1) used in the computed examples of Figures 2 and 3.

| *Elastic Parameters and Density* | *Case* $B = 0$ | *Case* $B = \frac{1}{2}$ | *Case* $B = 1$ |
|---|---|---|---|
| $a$ (GPa) | 33.8345 | 50.3523 | 132.7003 |
| $c$ (GPa) | 33.1948 | 50.4715 | 134.2036 |
| $f$ (GPa) | 22.2062 | 38.5857 | 120.7006 |
| $l$ (GPa) | 4.0138 | 4.0138 | 4.0138 |
| $m$ (GPa) | 6.7777 | 6.7777 | 6.7777 |
| $G_{eff}$ (GPa) | 5.2797 | 5.8841 | 6.2417 |
| $\delta$ | -0.0847 | -0.0733 | -0.0399 |
| $\epsilon - \delta$ | 0.0943 | 0.0745 | 0.0343 |
| $\gamma$ | 0.3443 | 0.3443 | 0.3443 |
| $\rho$ (kg/m$^3$) | 2120.0 | 2310.0 | 2320.0 |

We took the porosity to be $\phi = 0.2$, and the overall density to be $\rho = (1-\phi)\rho_s + \phi S \rho_l$, where $\rho_s = 2650.0$ kg/m$^3$, $S$ is liquid saturation ($0 \leq S \leq 1$), and $\rho_l = 1000.0$ kg/m$^3$. Then, three cases were considered: (1) Gas saturation $S = 0$ and $B = 0$, which is also the drained case, assuming that the effect of the saturating gas on the moduli is negligible. (2) Partial liquid saturation $S = 0.95$ and $B = \frac{1}{2}$ [which is intended to model a case of partial liquid saturation], intermediate between the other two cases. For smaller values of liquid saturation, the effect of the liquid might not be noticeable, since the gas-liquid mixture when homogeneously mixed will act much like the pure gas in compression, although the density effect will still be present. When the liquid fills most of the pore-space, and the gas occupies less than about 3% of the entire volume of the rock, the gas starts to become disconnected, and we expect the effect of the liquid to start becoming more noticeable, and therefore we choose $B = \frac{1}{2}$ to be representative of this case. And, finally, (3) full liquid saturation $S = 1$ and $B = 1$, which is also the fully undrained case. We assume for the purposes of this example that a fully saturating liquid has the maximum possible stiffening effect on the locally microhomogeneous, isotropic, poroelastic medium.

The results shown in Figures 2 and 3 are in complete qualitative and quantitative agreement with the analytical predictions described, as expected.

## DISCUSSION AND CONCLUSIONS

The primary question we address in this paper is this: Does the effective shear wave speed of a long-wavelength quasi-SV wave in a finely layered VTI material depend on the fluid in the porous layers, even though Gassmann's results (Gassmann, 1951) say that — without doubt — the shear modulus in each individual layer is mechanically independent of the fluid? Perhaps surprisingly, we show that the answer to the question is positive. The quasi-SV wave always does depend on the fluid mechanics, unless the shear modulus of all the layers is exactly a uniform constant. Furthermore, the magnitude of this effect is largest when the layer shear modulus fluctuations are large.

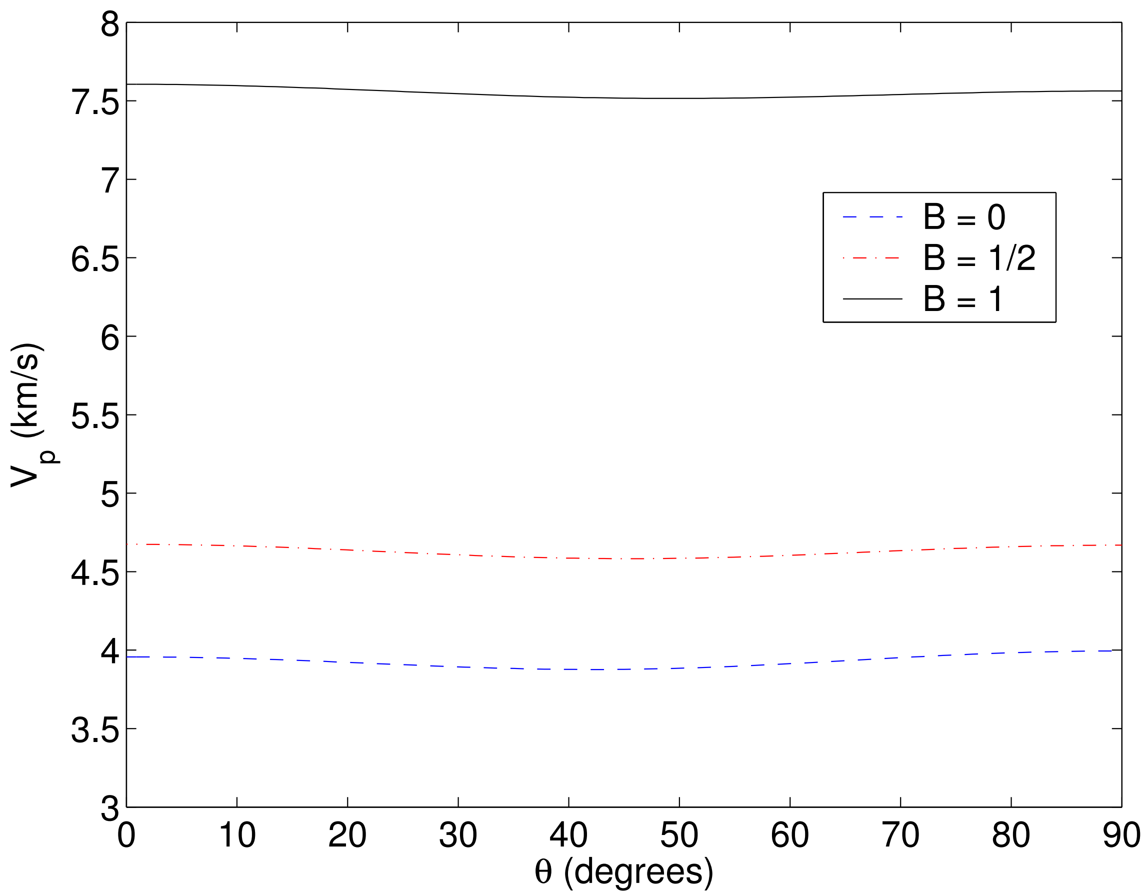


Figure 2: Compressional wave speed $V_p$ as a function of angle $\theta$ from the vertical. Two curves shown correspond to choices of Skempton's coefficient $B = 0$ for the drained case (dashed line) and $B = 1$ for the undrained case (solid line). The case $B = \frac{1}{2}$ (dot-dash line) is used to model partial saturation conditions as described in the text. The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers. jim1-vpnew [NR]

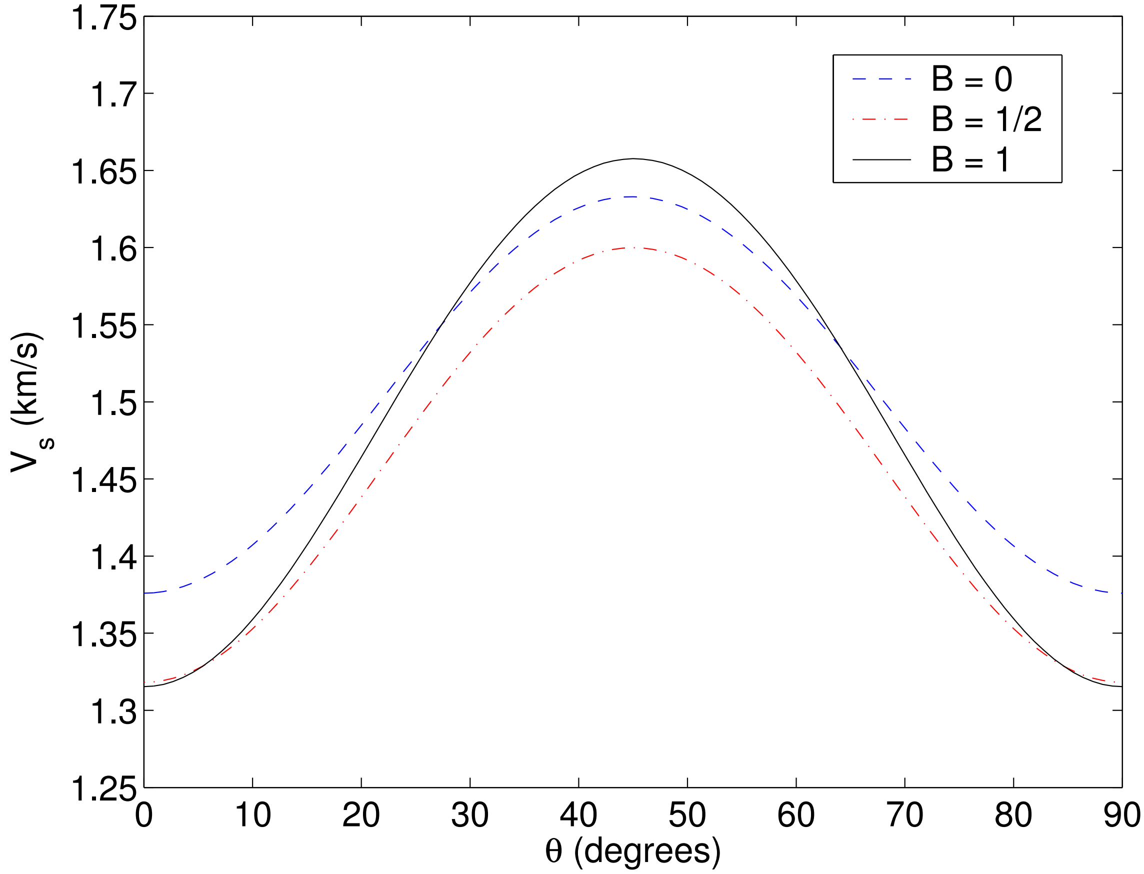


Figure 3: Vertically polarized shear wave speed $V_{sv}$ as a function of angle $\theta$ from the vertical. Two curves shown correspond to choices of Skempton's coefficient $B = 0$ for the drained case (dashed line) and $B = 1$ for the undrained case (solid). The case $B = \frac{1}{2}$ (dot-dash line) is used to model partial saturation conditions as described in the text. The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers. jim1-vsnew [NR]

In addition, our analysis leads us to consider some different ways of expanding the formulas for the dispersion relationships for the quasi-P and quasi-SV modes. These secondary results may also have some practical benefits and are illustrated in the Appendix.

Although there are five effective shear moduli for any layered VTI medium, the main result of the paper is that there is just one effective shear modulus for the layered system that contains *all* the dependence of elastic or poroelastic constants on pore fluids — all that can be observed in vertically polarized shear waves in VTI media. The relevant modulus $G_{eff}$ is related to uniaxial shear strain and the relevant axis of symmetry is the vertical one, normal to the bedding planes. The pore-fluid effects on this effective shear modulus can be substantial when the medium behaves in an undrained fashion, as might be expected at higher frequencies such as sonic and ultrasonic for well-logging or laboratory experiments, or at seismic frequencies for lower permeability regions of reservoirs. These predictions are clearly illustrated by the example in Figure 2.

The stiffness coefficients $a$, $b$, $c$, and $f$, all contain contributions from fluid effects for undrained layers. However, only stiffness $a$ and Thomsen parameter $\epsilon$ contain terms quadratic in layer shear modulus fluctuations, and these contributions are the ones creating the most significant effects on shear waves for strong anisotropy.

## ACKNOWLEDGMENTS

I thank P. A. Berge and V. Y. Grechka for their insight and collaboration during our earlier related studies in this research area. I thank S. R. Pride and L. Thomsen for constructive criticism that helped improve the final form of the paper.

## REFERENCES

Alkhalifah, T., and Tsvankin, I., 1995, Velocity analysis for transversely isotropic media: *Geophysics*, **60**, 1550–1566.

Anno, P., 1997, Equivalent anisotropy for finely-layered media: Colorado School of Mines, Center for Wave Phenomena, Project Review, pp. 307–312.

Auld, B. A., 1973, *Acoustic Fields and Waves in Solids*, Vols. 1 and 2, John Wiley & Sons, New York.

Backus, G. E., 1962, Long-wave elastic anisotropy produced by horizontal layering: *J. Geophys. Res.*, **67**, 4427–4440.

Berryman, J. G., 1979, Long-wave elastic anisotropy in transversely isotropic media: *Geophysics*, **44**, 896–917.

Berryman, J. G., 1999a, Transversely isotropic elasticity and poroelasticity arising from thin isotropic layers: in *Theoretical and Computational Acoustics '97*, *Proceedings of the Third*

*International Conference on Theoretical and Computational Acoustics, July 14–18, 1997, Newark, New Jersey*, edited by Y.-C. Teng, E.-C. Shang, Y.-H. Pao, M. H. Schultz, and A. D. Pierce (World Scientific, Singapore), pp. 457–474.

Berryman, J. G., 1999b, Origin of Gassmann's equations: *Geophysics*, **64**, 1627–1629.

Berryman, J. G., 2004, Poroelastic shear modulus dependence on pore-fluid properties arising in a model of thin isotropic layers: *Geophys. J. Int.*, **157**, 415–425.

Berryman, J. G., Grechka, V. Y., and Berge, P. A., 1999, Analysis of Thomsen parameters for finely layered VTI media: *Geophysical Prospecting*, **47**, 959–978.

Berryman, J. G., and Wang, H. F., 2001, Dispersion in poroelastic systems: *Phys. Rev. E*, **64**, 011303.

Biot, M. A., and Willis, D. G., 1957, The elastic coefficients of the theory of consolidation: *J. Appl. Mech.*, **24**, 594–601.

Bruggeman, D. A. G., 1937, Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen III. Die elastischen Konstanten der quasiisotropen Mischkörper aus isotropen Substanzen: *Annalen der Physik*, **29**, 160–178.

Gassmann, F., 1951, Über die Elastizität poröser Medien: *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, **96**, 1–23.

Hashin, Z., and Shtrikman, S., 1961, Note on a variational approach to the theory of composite elastic materials: *J. Franklin Inst.*, **271**, 336–341.

Hill, R., 1963, Elastic properties of reinforced solids: Some theoretical results: *J. Mech. Phys. Solids*, **11**, 357–372.

Johnson, D. L., 2001, Theory of frequency dependent acoustics in patchy-saturated porous media: *J. Acoust. Soc. Am.*, **110**, 682–694.

Levin, F. K., 1988, Discussion on "Weak elastic anisotropy," by L. Thomsen: *Geophysics*, **53**, 558-559.

Mavko, G., Mukerji, T., and Dvorkin, J., 1998, *The Rock Physics Handbook*, Cambridge, New York, 1998, pp. 206–207.

Milton, G. W., 2002, *The Theory of Composites*, Cambridge University Press, Cambridge, UK, pp. 163–165.

Musgrave, M. J. P., 1970, *Crystal Acoustics: Introduction to the Study of Elastic Waves and Vibrations in Crystals*, Holden-Day, San Francisco, California.

Postma, G. W., 1955, Wave propagation in a stratified medium: *Geophysics*, **20**, 780–806.

Rüger, A., 2002, *Reflection Coefficients and Azimuthal AVO Analysis in Anisotropic Media*, SEG, Tulsa, OK, p. 42.

Sayers, C. M., 2002, Fluid-dependent shear-wave splitting in fractured media: *Geophys. Prospecting*, **50**, 393–401.

Schoenberg, M., and Douma, J., 1988, Elastic wave propagation in media with parallel fractures and aligned cracks: *Geophys. Prospecting*, **36**, 571–590.

Schoenberg, M., and Muir, F., 1989, A calculus for finely layered anisotropic media: *Geophysics*, **54**, 581–589.

Skempton, A. W., 1954, The pore-pressure coefficients $A$ and $B$: *Geotechnique*, **4**, 143–147.

Thomsen, L. 1986, Weak elastic anisotropy: *Geophysics*, **51**, 1954–1966.

Thomsen, L., 1988, Reply by author to discussion on "Weak elastic anisotropy": *Geophysics*, **53**, 559–560.

Thomsen, L., 2002, *Understanding Seismic Anisotropy in Exploration and Exploitation*, Distinguished Instructor Series, No. 5, SEG, Tulsa, OK.

Tsvankin, I., and Thomsen, L., 1994, Nonhyperbolic reflection moveout in anisotropic media: *Geophysics*, **59**, 1290–1304.

## APPENDIX A

## APPROXIMATE PHASE VELOCITIES

Probably the most common way to write the linearized equations for the phase velocities in VTI media (Thomsen, 1986; 2002; Rüger, 2002) is

$$V_p(\theta) \simeq V_{p0}\left[1 + \delta \sin^2\theta\cos^2\theta + \epsilon\sin^4\theta\right], \tag{A-1}$$

$$V_{sv}(\theta) \simeq V_{s0}\left[1 + \frac{V_{p0}^2}{V_{s0}^2}(\epsilon - \delta)\sin^2\theta\cos^2\theta\right], \tag{A-2}$$

and

$$V_{sh}(\theta) \simeq V_{s0}\left[1 + \gamma\sin^2\theta\right]. \tag{A-3}$$

The approximations in all three cases are made based on assumed smallness of the parameters $\delta$, $\epsilon$, $\gamma$, which in fact may or may not hold for any particular layered medium. However, the work in this paper demands better approximations than these, because the assumptions of weak anisotropy are always violated in the cases of most interest, *i.e.*, when the SV-wave velocity actually does depend in a significant way on fluid content. Thomsen (2002) (in his Appendix III) quotes another form of the dispersion relation for $V_p(\theta)$ that is more useful for our purposes (modified here to correct an obvious typo in the leading term):

$$V_p^2(\theta) \simeq V_{p0}^2 + 2V_{nmo}^2(\delta\sin^2\theta + \eta\sin^4\theta), \tag{A-4}$$

where $V_{p0}^2 = c/\rho$, $V_{nmo} \simeq V_{p0}(1+\delta)$, and $\eta = (\epsilon-\delta)/(1+2\delta)$ is the combination of parameters introduced by Alkhalifah and Tsvankin (1996). Although (A-4) is still an approximation, it is much closer in form to the dispersion relation quoted here in (34). The form (A-4) still assumes smallness of the anisotropy parameters, but the usual square root approximation has not been made yet, so the correspondence with (34) is easier to scan. If we neglect higher order contributions to $\Delta$ and thereby make the approximation that

$$\Delta \simeq 2k^2 c(\epsilon-\delta)\sin^2\theta\cos^2\theta, \tag{A-5}$$

then (34) becomes

$$V_p^2(\theta) \simeq V_{p0}^2 + 2V_{p0}^2\delta\sin^2\theta + 2V_{p0}^2(1+2\delta)\eta\sin^4\theta. \tag{A-6}$$

If in addition we also make the small anisotropy approximations $V_{p0}^2\delta \simeq V_{nmo}^2\delta$ and $V_{p0}^2(1+2\delta) \simeq V_{nmo}^2$ in (A-6), then the result recovers (A-4).

Our main goal in this Appendix is to make a direct comparison between the exact formulas (34) and (35), the approximate formulas resulting from (34) and (35) when $\Delta$ is replaced by its first approximation (37), and either standard equations (A-1) and (A-2) or approximation (A-4) and some yet to be determined companion equation for quasi-SV waves. The easiest and most consistent way to arrive at an appropriate approximate form for $V_{sv}^2$ is to use the exact relations (34) and (35) to determine what effective value of $\Delta_{eff}$ has been used in (A-4) and then use it again in (35). We find

$$\Delta_{eff}(\theta) \simeq 2k^2\rho V_{po}^2(\epsilon-\delta)\sin^2\theta\cos^2\theta - 2k^2\rho V_{p0}^2\delta^2(2+\delta)\sin^2\theta. \tag{A-7}$$

However, this formula has the undesirable characteristic that it does not vanish as it should for $\theta = 90°$. The offending terms are second order in $\delta$ and therefore are usually neglected for weak anisotropy. But the weak anisotropy assumptions implicit in (A-4) are not valid in the present context, so this is nevertheless a problem for us here. Making proper allowance for this, we can arrive at a corrected $\Delta$ that has the desired behavior and still agrees with the prior results under weak anisotropy conditions:

$$\Delta_{corr}(\theta) \simeq 2k^2\rho V_{nmo}^2\eta\sin^2\theta\cos^2\theta, \tag{A-8}$$

and, therefore, that a good choice for $V_{sv}^2$ to the same level of approximation is

$$V_{sv}^2(\theta) = V_{so}^2 + \Delta_{corr}(\theta)/k^2\rho. \tag{A-9}$$

But these modifications have led us back to the approximation (34) and therefore provide nothing new. So instead of comparisons to (A-4) and (A-9), we will choose to make our comparisons to (A-1) and (A-2). In particular, these two equations amount to using

$$\Delta_{eff} = 2k^2\rho V_{p0}^2(\epsilon-\delta)\sin^2\theta\cos^2\theta, \tag{A-10}$$

except for some higher order corrections in $\Delta_{eff}$ which would always be small if the anisotropy were really always weak. Although Eqs. (A-8) and (A-10) are apparently the same, this fact

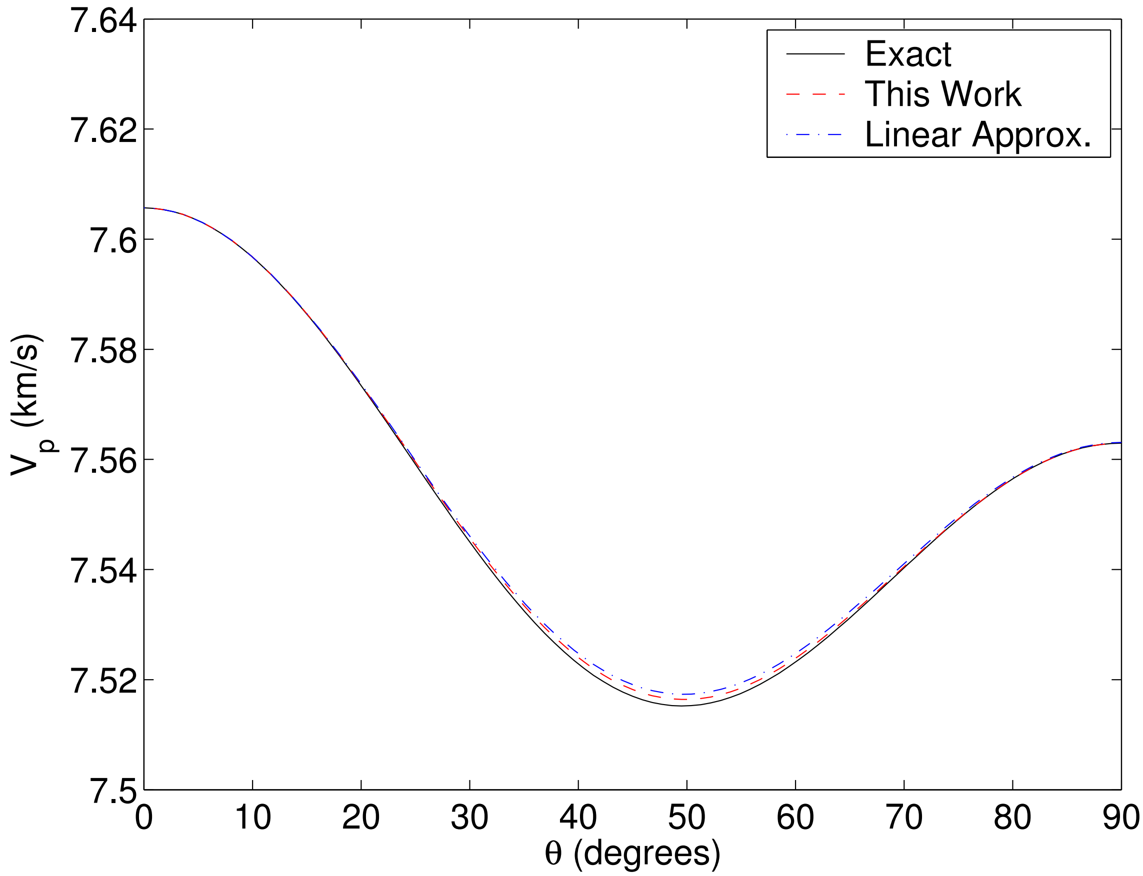


Figure A-1: Compressional wave velocities as computed exactly from the dispersion relation (34), by (34) using approximation (37) for $\Delta$, and by the linear approximation (64). This layered model is the same as in Figures 2 and 3 for the case $B = 1$. jim1-compvels [NR]

is a bit misleading since $\Delta$ and its corrections arise in the final results in different ways because of differing square root approximations and different assumptions about the presence or absence of strong anisotropy.

Numerical comparisons of these three sets of results for quasi-P and quasi-SV waves are summarized in Figures 4 and 5 for one strong anisotropy example. The comparison is obviously not a fair one for the weak anisotropy equations since they are being used beyond their acknowledged (and expected) range of validity. The main point of the exercise is to see that the approximations made here give reasonable approximations to the exact results for the full range of possible incidence angles for strong anisotropy conditions, while the standard results do not fair as well. All the methods agree quite well for compressional waves in this model. The evaluation of (35) using (37) to approximate $\Delta$ gives a clear improvement over (A-2) for the quasi-SV wave velocity in a range of intermediate angles. Overall, the weak anisotropy formulas (A-1) and (A-2) give better results for strong anisotropy in this case than might have been expected.

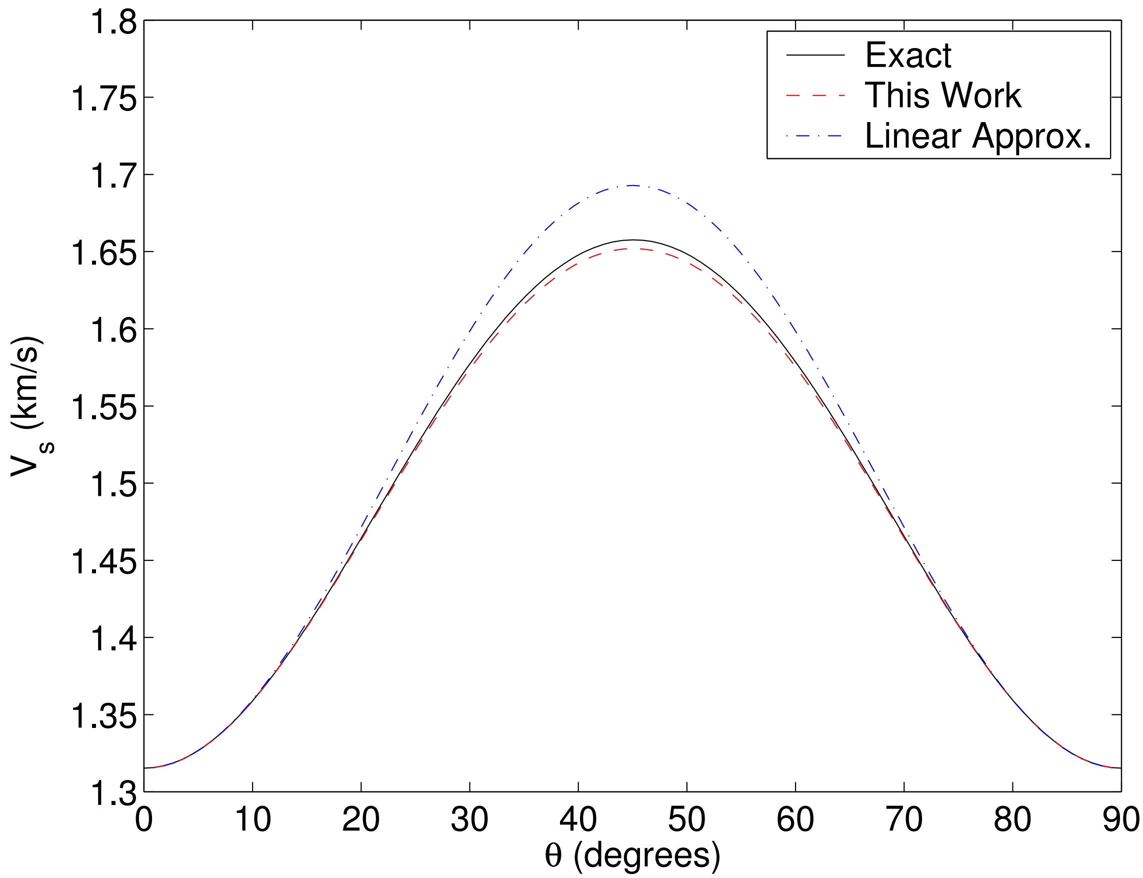


Figure A-2: Shear wave velocities as computed exactly from the dispersion relation (35), by (35) using approximation (37) for $\Delta$, and by the linear approximation (66). This layered model is the same as in Figures 2 and 3 for the case $B = 1$. jim1-shearvels [NR]