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Wavefield extrapolation in phase-ray coordinates

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ABSTRACT

A ray theoretic formulation is developed that allows rays to be traced directly from existing solutions to the Helmholtz equation. These rays, termed phase-rays, are defined by the direction normal to surfaces of constant wavefield phase. Phase-rays exhibit a number of attractive characteristics, including triplication-free ray-fields, an ability to shoot rays forward or backward, and an ability to shoot infill rays for ensuring adequate ray density. Because of these traits, we use phase-rays as a coordinate basis on which to extrapolate wavefields using the generalized coordinate system approach. Examples of wavefields successfully extrapolated in phase-ray coordinates are presented, and the merits and drawbacks of this approach, relative to conventionally traced ray coordinates, are discussed.

INTRODUCTION

Ray theory is routinely applied to generate characteristics to solutions of the Helmholtz equation. The usual ray theoretic approach introduces an ansatz representation of the wavefield solution into the Helmholtz equation to yield coupled eikonal and transport equations. Computation of eikonal equation solutions is usually facilitated by the introduction of a high-frequency approximation that removes the amplitude dependence from the full eikonal equation. Consequently, conventionally computed rays are independent of frequency and often triplicate due to their broad-band nature.

In contrast, whenever independent solutions to the Helmholtz equation exist, full ray theoretic formulae may be used to trace rays (Foreman, 1989). These rays are termed phase-rays herein owing to a ray direction that is always orthogonal to surfaces of constant phase. One of their beneficial traits is that, unlike conventionally traced rays, computed ray-fields are caustic-free. A second advantage is that ray position is the only required initial condition for ray-tracing. However, one obstacle seems to restrict the use of the phase-ray formulation in seismic imaging problems: wavefield solutions to the Helmholtz equation must be known in advance of ray-field computation.

One situation where phase-rays (and conventional rays) are of use is in wavefield extrapolation in generalized (i.e. non-Cartesian) coordinates (Sava and Fomel, 2003). A natural set of coordinates for wavefield extrapolation is the general ray family represented by a ray

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direction and shooting angle(s). Wavefield extrapolation in ray coordinates thus uses a ray-field as the coordinate system on which to extrapolate wavefields. This approach transforms the physics of one-way wave propagation from the usual Cartesian grid so that it is valid in a ray coordinate system. Wavefield extrapolation operators are then applied more accurately because extrapolation can occur at lower angles to the ray direction than usually occurs with Cartesian coordinates. The result is then mapped from ray coordinates to Cartesian. Although ray domain extrapolated wavefields are generally more accurate, issues still remain when using conventionally-traced rays; in particular, how to robustly deal with infinite amplitudes at locations of coordinate system triplication.

Motivated by this issue, this paper examines the use of phase-rays as a coordinate system for wavefield extrapolation. The main advantage of phase-ray coordinates is that they are caustic-free, and thereby avoid complications arising from triplicating coordinates. The paper begins with a general discussion of ray theory and an approach for calculating phase-rays from solutions to the Helmholtz equation. Phase-ray examples from a salt body model are then presented, and are followed by results illustrating wavefield extrapolation in phase-ray coordinates for a Gaussian velocity perturbation model. Finally, a method is proposed for propagating broadband wavefields using frequency-dependent phase-ray coordinates.

THEORY

The theory outlined in this section closely follows that of Foreman's exact ray theory (Foreman, 1989), but is summarized here for completeness. Ray theory may be used to compute the characteristics to the time-independent, homogeneous Helmholtz equation,

$$\nabla^2 \Psi + k^2 \Psi = 0, \tag{1}$$

where Ψ is the desired wavefield solution, and k is the wavenumber. In most ray theoretic developments, the wavefield is represented by an ansatz solution, $\Psi = Ae^{i\phi}$, where $A(\mathbf{r})$ and $\phi(\mathbf{r})$ are the amplitude and phase functions, respectively. Substituting this representation into the Helmholtz equation yields the usual eikonal and transport equations,

$$\begin{aligned} K^2 = \mathbf{K}\cdot\mathbf{K} = \nabla\phi\cdot\nabla\phi &= k^2 + \frac{\nabla^2 A}{A}, \\ 2\nabla A\cdot\nabla\phi + A\nabla^2\phi &= 0, \end{aligned} \tag{2}$$

where $\mathbf{K}$ is the phase gradient vector (see figure 1). Solutions to equations (2) are the ray paths and the amplitudes along these ray paths, respectively. In isotropic media the gradient of the phase function, $\mathbf{K} = \nabla\phi$, is orthogonal to surfaces of constant phase and represents the instantaneous direction of propagation. Explicitly, this may be written,

$$\mathbf{K} = \nabla\phi = |K|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}, \tag{3}$$

where $\mathrm{d}s$ is an element of length, and $\mathrm{d}\mathbf{r}$ is a vector element of the ray-path. A ray-path equation is developed by taking the gradient of the phase gradient magnitude (i.e. ∇K),

$$K^2 = \mathbf{K}\cdot\mathbf{K},$$

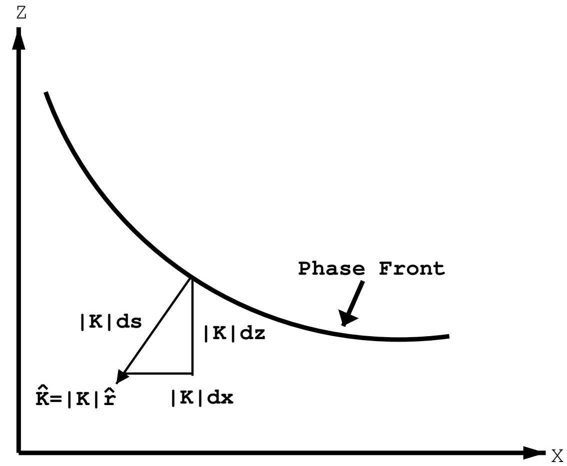

Figure 1: Schematic of phase-front gradient quantities. **K** is the phase gradient vector, $|K|\mathrm{d}s$ is the gradient magnitude along step $\mathrm{d}s$, and $|K|\mathrm{d}x$ and $|K|\mathrm{d}z$ are the projections of $|K|\mathrm{d}s$ along the x and z coordinates, respectively. jeff1-Rays [NR]

$$\begin{aligned} 2K\nabla K &= 2(\mathbf{K}\cdot\nabla)\mathbf{K} + 2\mathbf{K}\times(\nabla\times\mathbf{K}), \\ \nabla K &= \left[(\mathbf{K}/K\cdot\nabla\right]\mathbf{K}, \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}\cdot\nabla\right)K = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{K}, \end{aligned} \tag{4}$$

where $\nabla\times\mathbf{K} = \nabla\times\nabla\phi = 0$ has been employed. Using equation (3) this may be written explicitly,

$$\nabla K = \frac{\mathrm{d}}{\mathrm{d}s}\left(K\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}\right) \tag{5}$$

Coupling between the eikonal and transport equations is evident through the dependence of the eikonal equation on amplitude function, A. In many cases a high-frequency approximation (i.e. $\frac{\nabla^2 A}{A}\approx 0$) is used to decouple these equations. Use of this approximation yields the usual form of the ray path equation,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}\right) = \nabla\left(\frac{1}{c}\right), \tag{6}$$

where c is the velocity. Use of this approximation also eliminates the frequency dependence of ray trajectories (see Appendix A). One manner of reintroducing frequency-dependent ray trajectories is discussed in a latter section.

Phase-ray formulation

When a solution, Ψ, to the Helmholtz equation is known, obtaining a ray trajectory using equation (5) is relatively trivial. The expression for the wavefield gradient, $\nabla\Psi = \nabla(Ae^{i\phi})$, divided by the wavefield, Ψ, is,

$$\frac{\nabla\Psi}{\Psi} = \frac{\nabla A}{A} + i\mathbf{K}. \tag{7}$$

An expression for the wavefield gradient vector, **K**, is obtained by retaining the imaginary component of equation (7) and using the expression for **K** in equation (3),

$$K\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r} = \mathrm{Im}\left(\frac{\nabla\Psi}{\Psi}\right) \tag{8}$$

Equation (8) may be rewritten explicitly as a system of two decoupled ordinary differential equations,

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} x \\ z \end{bmatrix} = \frac{1}{\sqrt{(\frac{\partial \Psi}{\partial x})^2 + (\frac{\partial \Psi}{\partial z})^2}} \mathrm{Im}\left(\frac{1}{\Psi}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} \end{bmatrix}\Psi\right). \tag{9}$$

The solution for ray-path, **r**, is computed through an initial evaluation of the right hand side of equations (9), and an iterative forward step by a constant interval using the precomputed quantity to determine the proper apportioning of the step along each coordinate. Note that because these differential equations are first-order, only one initial condition (position) is required for ray computation, and rays may be started from any location in the wavefield solution. Finally, although the two-dimensional formulation is presented here, the extension to three dimensions is trivial.

PHASE-RAY EXAMPLES

Examples of traced phase-rays are presented in this section using a salt body velocity field as a didactic model. The background velocity of the model, shown in figure 2e, is a typical Gulf of Mexico v(z) velocity gradient. The superposed salt body is characterized by higher wave speeds (4700 m/s) and a fairly rugose bottom of salt interface.

Figure 2 presents five phase-rays computed from four different monochromatic wavefields. The wavefields were generated for a shot point located at 11700 m using a split-step Fourier operator (Stoffa et al., 1990) in a Cartesian coordinate system. Each ray begins at the same point in all panels. The rays to the extreme left and right in each panel show little variability in their spatial location; however, the three remaining rays are attracted to regions of greater wavefield amplitude and their spatial locations vary with a range up to 2000 m. Accordingly, because rays originate at the same spot, observed phase-ray movement is caused by changes in wavefield solution and indicates frequency-dependent behavior.

Phase-rays computed according to equations (9) may be traced in reverse, from observation to source point, by using a negative step interval. Figure 3 illustrates this situation with the same model as in figure 2. Initial ray locations are points at regular intervals on a semicircular arc of radius 5000 m. Calculated phase-rays do not overlap and the ray-field is caustic-free. Phase-ray density, though, is frequency-dependent, with significant coverage gaps of variable size appearing in all four panels. This suggests that an additional condition is required to ensure that, when needed, ray density is more uniform. One solution is to shoot a new ray between two successive rays wherever intra-ray distance exceeds some threshold value.

In summary, these results illustrate a number of advantageous characteristics of phase-rays: i) phase-ray ray-fields are triplication-free; ii) ray tracing from areas of low wavefield amplitude (e.g. shadow zones) to the source point is possible; and iii) sufficient phase-ray density may be ensured by an additional shooting of phase-rays wherever intra-ray spacing is too large. These three traits provide the main impetus for using phase-ray coordinates as a generalized coordinate system for wavefield extrapolation (Sava and Fomel, 2003).

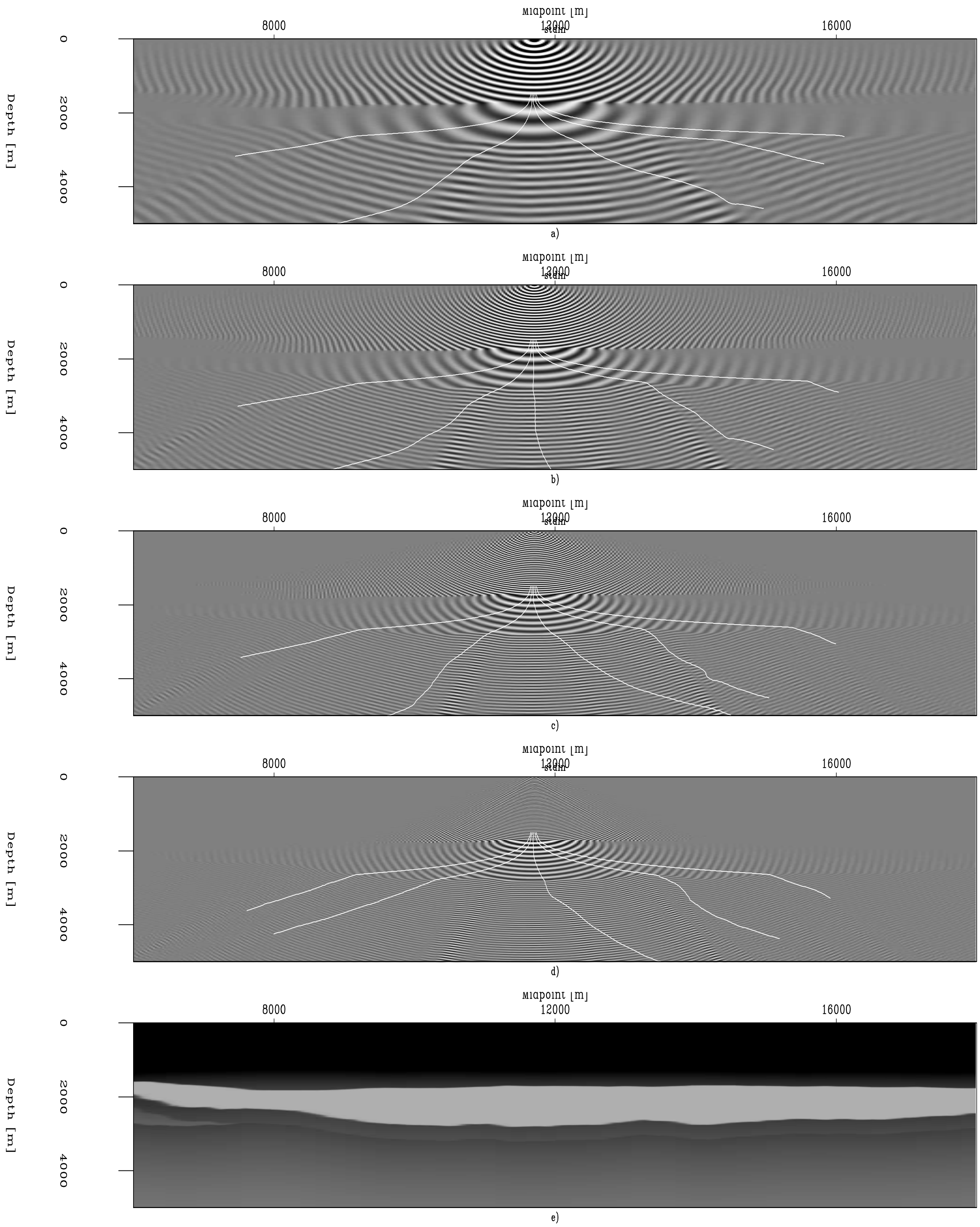

Figure 2: Five phase-rays traced through monochromatic wavefields. Wavefields were generated by a split-step Fourier operator in Cartesian coordinates for a shot point at 11700 m using the velocity model illustrated in e). a) 5 Hz wavefield; b) 10 Hz wavefield; c) 15 Hz wavefield; d) 20 Hz wavefield; and e) smoothed salt body model velocity field. jeff1-G5paper [ER]

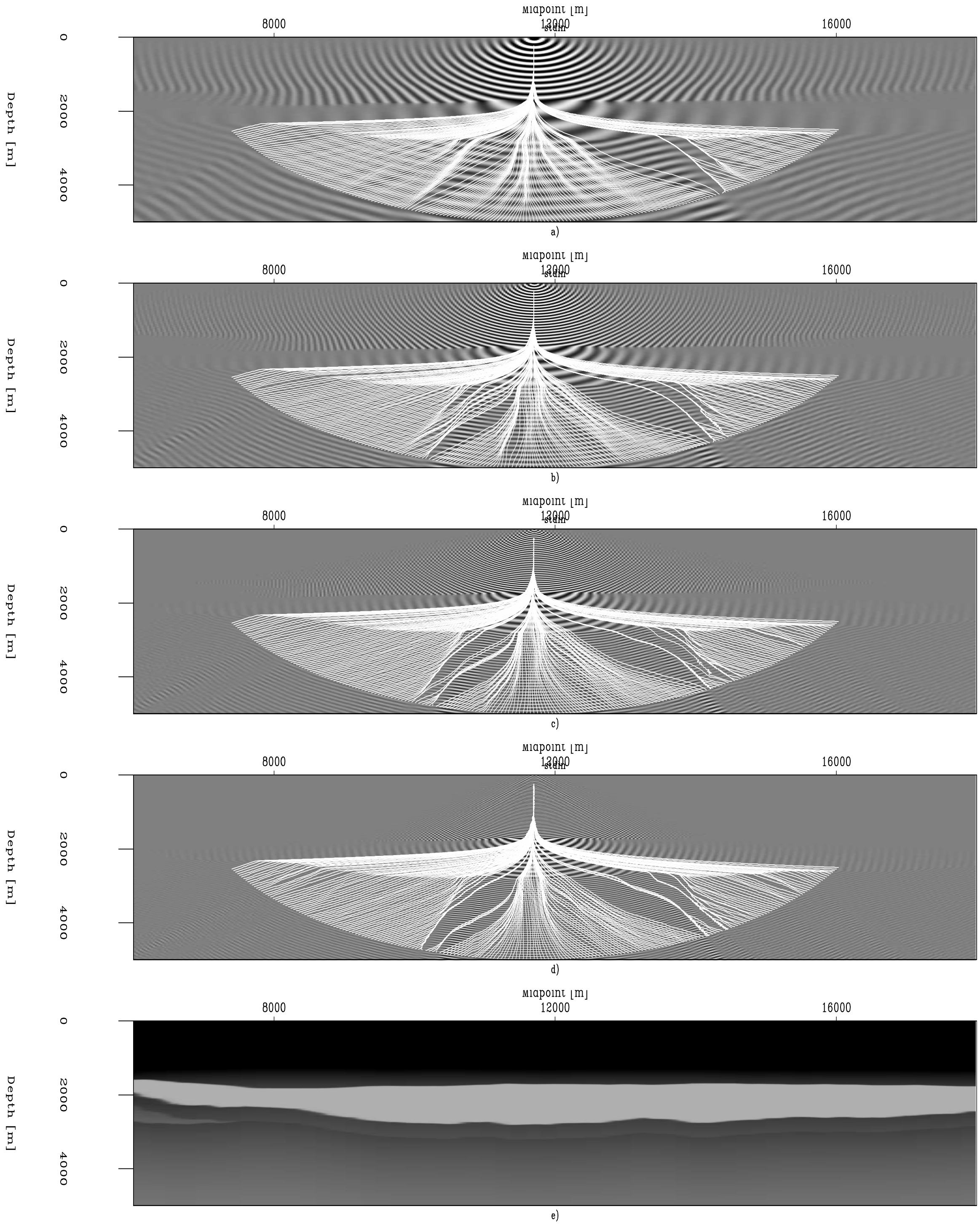

Figure 3: 200 phase-rays traced backwards through monochromatic wavefields for velocity model shown in e). Initial ray locations are points at regular intervals on a semicircular arc of radius 5000 m. a) 5 Hz wavefield; b) 10 Hz wavefield; c) 15 Hz wavefield; d) 20 Hz wavefield; and e) smoothed salt body model velocity field. jeff1-G5backpaper [ER]

WAVEFIELD EXTRAPOLATION IN PHASE-RAY COORDINATES

This section examines the use of phase-rays as a coordinate system for wavefield extrapolation. The velocity model examined here (shown in figure 4a) is characterized by a slow, Gaussian-shaped velocity anomaly (-600 m/s) in a medium of otherwise constant velocity (1200 m/s). This model was chosen to ensure that extrapolated wavefields triplicate, as illustrated by figure 4b. This wavefield was generated for a shot point located at 8000 m using a split-step Fourier operator in a Cartesian coordinate system. Figure 4c shows phase-rays traced through the wavefield of figure 4b. Phase-rays in the upper portions of the model have fairly smooth coverage. In areas of wavefield triplication, though, significant coverage gaps are noticeable.

The phase-rays shown in figure 4c were subsequently used as a coordinate system for the generalized coordinate wavefield extrapolation approach (Sava and Fomel, 2003). Figure 5a shows the velocity model of figure 4a in phase-ray coordinates. Figure 5b presents the results of wavefield extrapolation in phase-ray coordinates using a split-step Fourier operator and the velocity model shown in figure 5a. Figure 5c shows this result mapped to Cartesian coordinates. The wavefield presented in figure 5d was computed in Cartesian coordinates using a split-step Fourier operator. In areas where wavefields are present in significant amplitude, the phase-ray and Cartesian results are similar. Areas of low wavefield amplitude beneath the Gaussian velocity anomaly (e.g. [z,x]=[4200 m,6800 m]), though, are markedly different. This difference is related to the inability to map the results from ray coordinates to Cartesian in areas of minimal or non-existent ray coverage.

This experiment highlights a consequence of using phase-ray coordinates for wavefield extrapolation. Monochromatic wavefield triplication is generally identified by interference patterns created by converging wavefield components. (See, for example, the checkerboard pattern beneath the Gaussian velocity anomaly.) Because phase-ray direction is dependent on the total wavefield gradient, it is similarly dependent on the gradients of each converging wavefield component. The gradient vector, being unable to unwrap individual convergent phases, chooses a weighted average of individual gradients. Accordingly, phase-rays are usually steered in the direction of the convergent component with the largest individual gradient magnitude, but they will never triplicate since the weighted gradient is uniquely defined at each wavefield point. This fact suggests that phase-ray coordinates represent a trade off between introducing inaccuracy associated with triplicating coordinates and inaccuracy of wavefield extrapolation at greater angles to the phase-ray direction.

Figure 6 presents a comparison between wavefields extrapolated in phase-ray and conventional ray coordinates (Sava and Fomel, 2003). Figures 6a and 6b present wavefields extrapolated in phase-ray coordinates and after interpolation into Cartesian coordinates, respectively. Figures 6c and 6d present similar results, but with conventionally traced rays. Of the two ray coordinate systems tested, the phase-ray coordinate extrapolated wavefield (figure 6b) better resembles the wavefield calculated in Cartesian coordinates (figure 5c). However, the sampling of phase-ray extrapolated wavefields, and their Cartesian maps, must be greatly improved before a definitive comparison is possible.

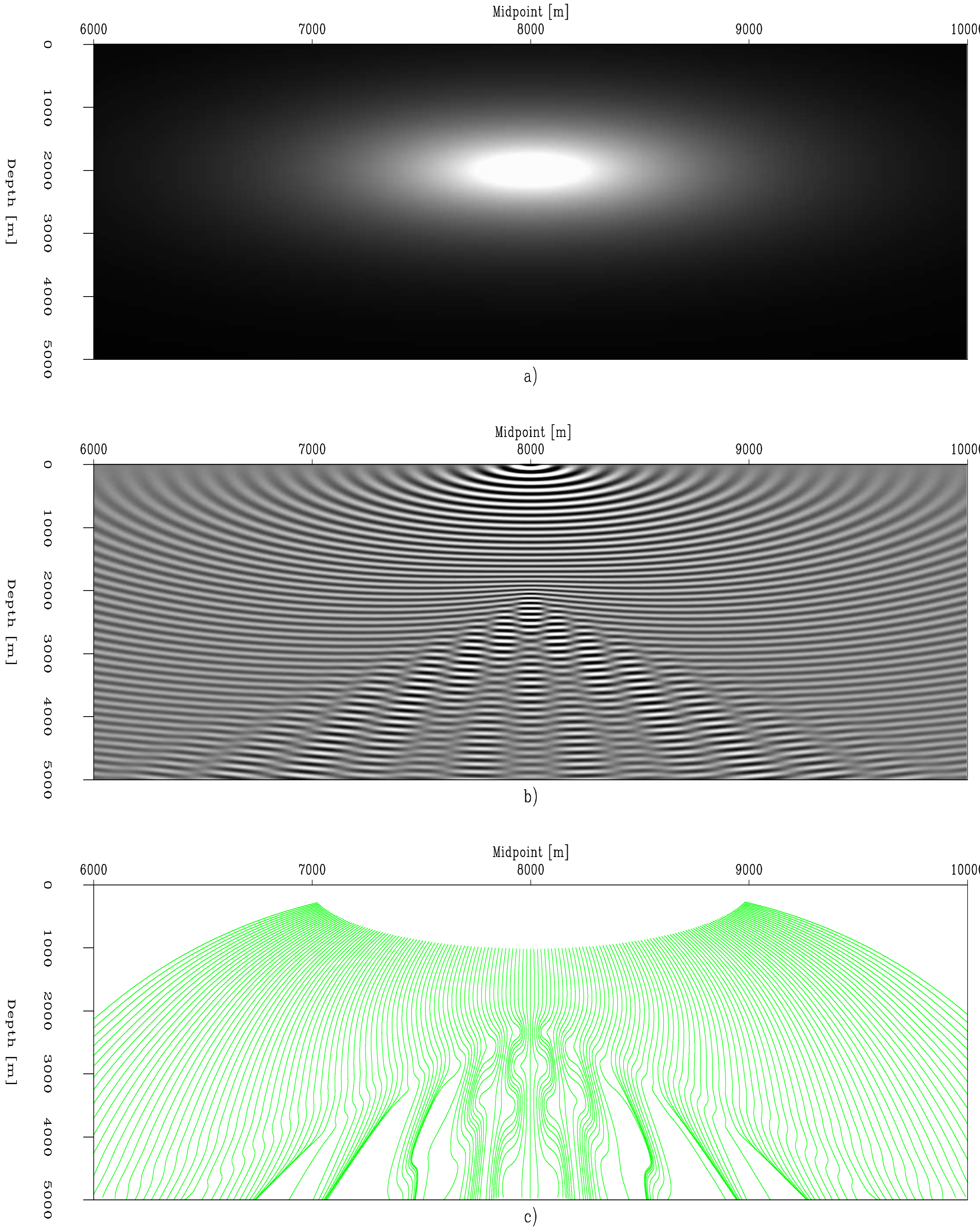

Figure 4: Gaussian-shaped velocity anomaly model. a) Gaussian-shaped anomaly of -600 m/s maximum velocity perturbation superposed on a constant 1200 m/s velocity field; b) 5 Hz wavefield computed for a shot located at 8000 m using a single-velocity split step Fourier operator in Cartesian coordinates; and c) phase-rays traced through the wavefield of b). jeff1-modelrays [ER]

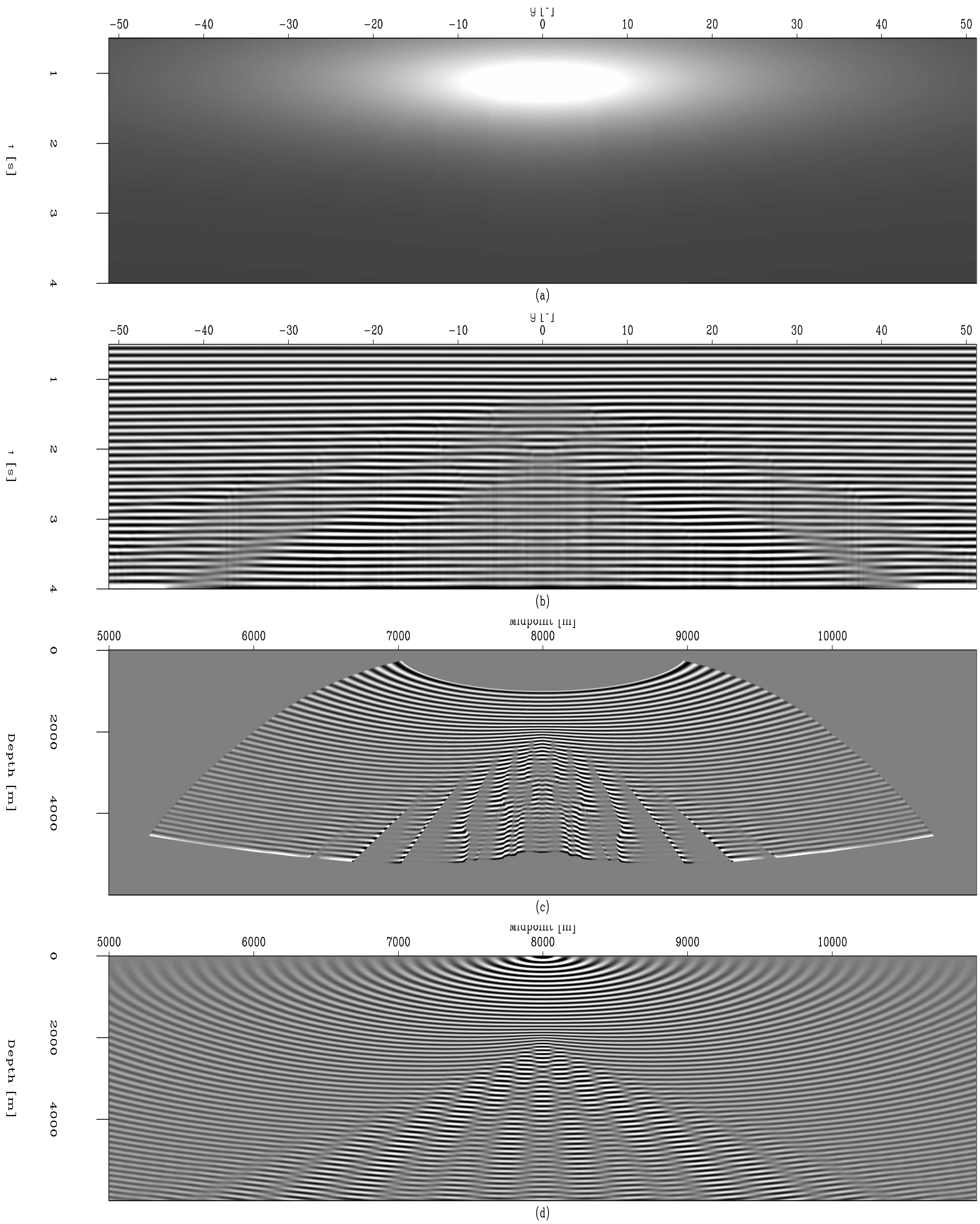

Figure 5: A comparison of wavefield extrapolation results computed in phase-ray and Cartesian coordinates. a) Velocity model of figure 4a mapped to phase-ray coordinates; b) wavefield extrapolated in phase-ray coordinates using a split-step Fourier method; c) the map of wavefield in b) to Cartesian coordinates; d) wavefield solved in Cartesian coordinates using a split-step Fourier operator. jeff1-Reg.ER1.ssf.ps [CR]

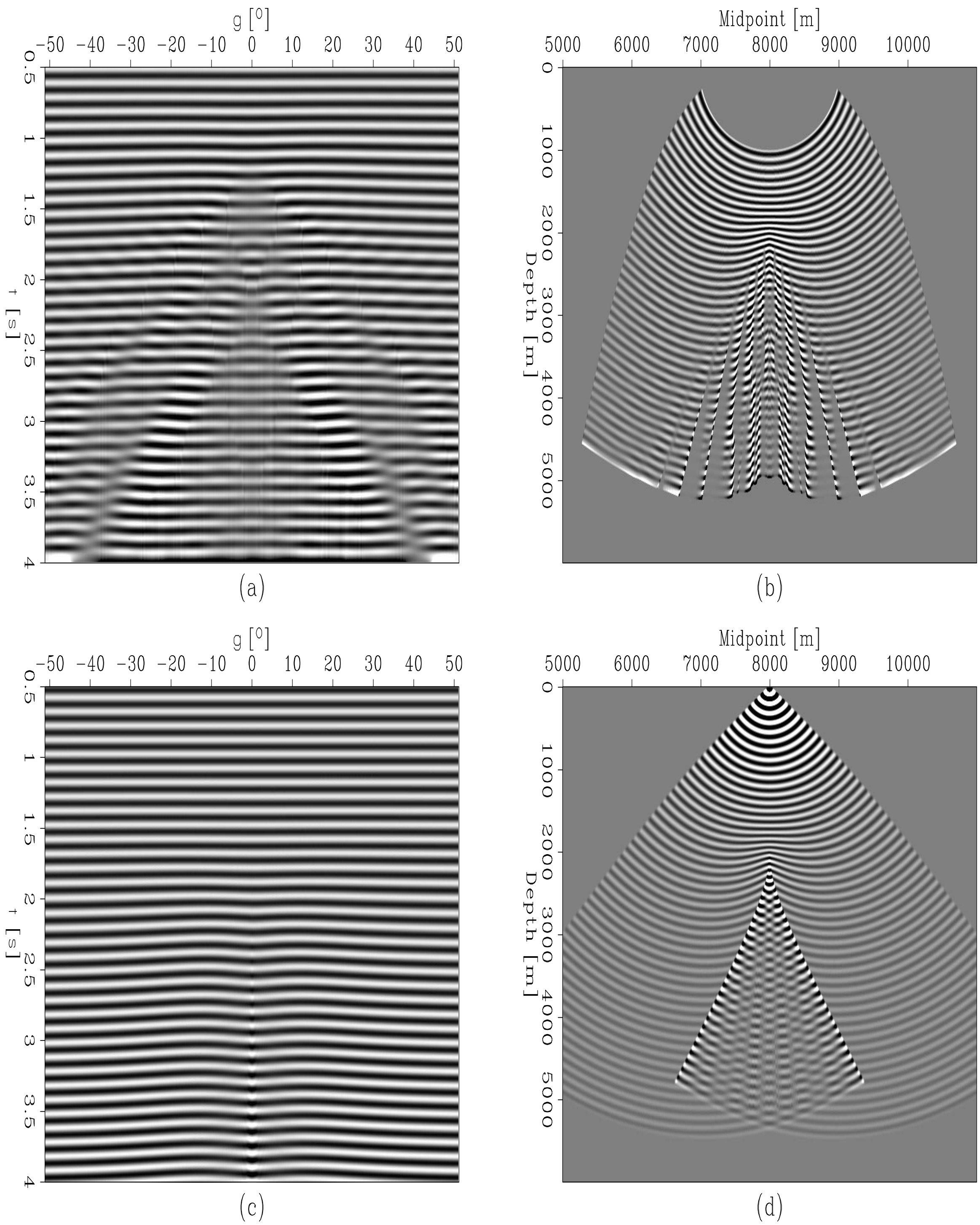

Figure 6: A comparison of wavefield extrapolation results computed in phase-ray and conventional ray coordinates. a) wavefield extrapolated in phase-ray coordinates; b) wavefield of a) interpolated into Cartesian coordinates; c) wavefield extrapolated in conventional ray coordinates (Sava and Fomel, 2003); and d) wavefield of c) interpolated in Cartesian coordinates. jeff1-Jeff.ER1.ssf.ps [CR]

Toward broadband propagation

The goal of using phase-rays as a coordinate system for wavefield extrapolation is to enable a more accurate propagation of broadband wavefields through the subsurface. However, this requires extrapolating wavefields at many frequencies. One interesting observation is that phase-rays exhibit frequency-dependent behavior. Thus, one strategy for wavefield extrapolation is to use phase-ray coordinates that adapt to frequency-dependent illumination.

The proposed approach for frequency-dependent wavefield extrapolation is as follows: i) an initial wavefield extrapolation in Cartesian or polar coordinate at the lowest frequency; ii) phase-ray tracing using the current wavefield solution; iii) extrapolating the next wavefield using the traced phase-rays as the new coordinate system; iv) a mapping of the latest wavefield result from phase-ray to Cartesian coordinates; and v) repeating steps ii), iii) and iv) until wavefields at all frequencies are calculated. The broadband wavefield is then obtained by a summation of wavefields over all extrapolated frequencies. Alternatively, because frequency-dependent wavefield illumination usually varies slowly over individual frequency steps, phase-rays could be traced only periodically to save computational cost.

CONCLUSIONS

This paper has introduced a method for tracing phase-rays through monochromatic wavefield solutions of the Helmholtz equation. The resulting phase-rays exhibit a number of attractive characteristics, including: i) a triplication-free ray-field; ii) an ability to shoot rays forward or backward from areas of strong or weak wavefield amplitude alike; and iii) an ability to easily infill rays to ensure adequate phase-ray density. Phase-rays may then be successfully used as a coordinate system on which to extrapolate wavefields. These coordinates avoid coordinate system triplication that can debilitate wavefields extrapolated using conventional ray-field coordinates.

The phase-ray formulation, though, cannot unwrap individual triplicating phases, and chooses a weighted average between interfering phases. Because of this fact phase-ray coordinates represent a trade off between introducing inaccuracy associated with triplicating coordinates and inaccuracy of wavefield extrapolation at greater angles to the ray direction. However, before a critical comparison of the relative merits and drawbacks of phase-ray and conventional ray coordinate extrapolated wavefields is attempted, phase-ray extrapolated wavefields need to be better sampled so that their Cartesian maps are more comparable.

ACKNOWLEDGMENTS

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APPENDIX A

FREQUENCY DEPENDENCE OF THE RAY PATH EQUATION

The frequency-dependent eikonal equation is,

$$K^2 = k^2 + \frac{\nabla^2 A}{A} = \nabla\phi \cdot \nabla\phi, \tag{A-1}$$

where A and ϕ are the amplitude and phase functions, respectively. Defining parameter $\gamma = \frac{\nabla^2 A}{A}$ and using $k = \omega s$, where ω is angular frequency and s is slowness, one may use the definition of K in equation (A-1) to write the following as the ray path equation (see equation (4))

$$\nabla K = \frac{\omega s \nabla s + \nabla \frac{\gamma}{2}}{\sqrt{\omega^2 s^2 + \gamma}} = \frac{\mathrm{d}}{\mathrm{d}s}\left(K \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{r}\right). \tag{A-2}$$

To examine how equation (A-2) varies as a function of frequency, a frequency derivative is applied to yield,

$$\frac{\partial}{\partial\omega}(\nabla K) = \frac{\gamma\omega s \nabla s - \nabla\frac{\gamma}{2}\left(\omega s^2 + \frac{\partial}{\partial\omega}\frac{\gamma}{2}\right) + \frac{\partial}{\partial\omega}\left(\nabla\frac{\gamma}{2}\right)\left(\omega^2 s^2 + \gamma - \omega^2 s \nabla s\right)^{\frac{3}{2}}}{\left(\omega^2 s^2 + \gamma\right)}. \tag{A-3}$$

High frequency approximation

The ray theoretic ansatz assumes that $K \approx k = \omega s$, which involves setting the contribution of γ and its derivates to zero. Thus, the frequency dependence of the ray path equation is,

$$\frac{\partial}{\partial\omega}(\nabla K) = 0, \tag{A-4}$$

which requires that the ray paths are stationary with respect to frequency.