Stanford Exploration Project, Report 114, October 14, 2003, pages 67–71

Short Note

WKBJ and amplitude preserving one-way wave equation

Guojian Shan and Biondo Biondi[1]

INTRODUCTION

The standard one-way wave equation (Claerbout, 1971) produces the correct phase of the wavefield, but it is not equivalent to the acoustic wave equation in terms of amplitude. Zhang (1993) suggests that to improve the dynamics information of one-way wave equation, an amplitude correction term should be included into the standard one-way wave equation. Zhang et al. (2001) apply the one-way wave equation with the amplitude correction to shot-profile migration and shows that it can provide the same amplitude as an image produced by the true amplitude Kirchoff migration (Hanitzsch, 1997) by asymptotic analysis. Vlad et al. (2003) compare the amplitude of one-way wave equation with amplitude correction with acoustic wave equation and WKBJ amplitude correction, and concludes that it has the same effect as the WKBJ correction (Clayton and Stolt, 1981). In this short note, we will demonstrate in theory that the one-way wave equation with the amplitude correction term is equivalent to the first order approximation of WKBJ amplitude correction.

AMPLITUDE PRESERVING ONE-WAY WAVE EQUATION

The standard one-way wave equation is

$$\frac{\partial P}{\partial z} = i\frac{\omega}{v}\sqrt{1-\frac{v^2k_x^2}{\omega^2}}P, \tag{1}$$

where k_x is the wave number in the x direction, and v is the velocity. From the dispersion relation, we have

$$k_z = \frac{\omega}{v}\sqrt{1-\frac{v^2k_x^2}{\omega^2}}, \tag{2}$$

where k_z is the wavenumber in the z direction. Equation (1) mimics the phase behavior of the acoustic wave equation:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}P - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)P = 0, \tag{3}$$

[1]**email:** shan@sep.stanford.edu, biondo@sep.stanford.edu

although the amplitude is not correct. Zhang (1993) suggests that to maintain the dynamics information of equation (3), an extra amplitude correction term needs to be included into the standard one-way wave equation. The new one-way wave equation is

$$\frac{\partial P}{\partial z} = \left[ik_z - \frac{1}{2v}\frac{\partial v}{\partial z}\left(1+\frac{(vk_x)^2}{\omega^2-(vk_x)^2}\right)\right] P. \tag{4}$$

We call it the amplitude preserving one-way wave equation. By equation (4), the wavefield can be extrapolated as the following:

$$P(z+\Delta z) = e^{ik_z\Delta z - \frac{1}{2v}\frac{\partial v}{\partial z}\left(1+\frac{(vk_x)^2}{\omega^2-(vk_x)^2}\right)\Delta z} P(z) \tag{5}$$

$$= e^{-\frac{1}{2v}\frac{\partial v}{\partial z}\left(1+\frac{(vk_x)^2}{\omega^2-(vk_x)^2}\right)\Delta z} \cdot e^{ik_z\Delta z} P(z). \tag{6}$$

In equation (6), first term defines the amplitude information of the wavefield, and second term $e^{ik_z\Delta z}$ defines the phase information of the wavefield.

WKBJ AND ITS FIRST ORDER APPROXIMATION

In this section, we demonstrate that the first order approximation of WKBJ is the same as the amplitude preserving one-way wave equation for $v = v(z)$.

The one-way wave equation with the WKBJ amplitude correction is

$$P(z+\Delta z) = \sqrt{\frac{k_z(z+\Delta z)}{k_z(z)}} e^{ik_z\Delta z} P(z). \tag{7}$$

The WKBJ amplitude correction term $\sqrt{\frac{k_z(z+\Delta z)}{k_z(z)}}$ can be rewritten as

$$\sqrt{\frac{k_z(z+\Delta z)}{k_z(z)}} = e^{\frac{1}{2}\ln\frac{k_z(z+\Delta z)}{k_z(z)}}. \tag{8}$$

Then $k_z(z+\Delta z)$ in equation (8) can be linearized to $k_z(z) + \frac{\partial k_z}{\partial z}\Delta z$, so we have

$$\frac{1}{2}\ln\frac{k_z(z+\Delta z)}{k_z(z)} \approx \frac{1}{2}\ln(1+\frac{1}{k_z}\frac{\partial k_z}{\partial z}\Delta z). \tag{9}$$

From

$$\ln(1+x) = x - x^2 + \cdots,$$

and

$$\ln(1+x) \approx x,$$

we have

$$\frac{1}{2}\ln(1+\frac{1}{k_z}\frac{\partial k_z}{\partial z}\Delta z) \approx \frac{1}{2}\frac{1}{k_z}\frac{\partial k_z}{\partial z}\Delta z. \tag{10}$$

Because

$$\frac{1}{k_z}\frac{\partial k_z}{\partial z} = \frac{\partial}{\partial z}\ln k_z, \tag{11}$$

and from the dispersion relation $k_z = \sqrt{\frac{\omega^2}{v^2} - k_x^2}$, we have

$$\frac{1}{2}\frac{1}{k_z}\frac{\partial k_z}{\partial z}\Delta z = \frac{1}{2}\frac{\partial}{\partial z}\ln\sqrt{\frac{\omega^2}{v^2} - k_x^2}\Delta z = \frac{1}{4}\frac{\partial}{\partial z}\ln\left(\frac{\omega^2}{v^2} - k_x^2\right)\Delta z, \tag{12}$$

and

$$\frac{1}{4}\frac{\partial}{\partial z}\ln\left(\frac{\omega^2}{v^2} - k_x^2\right) = \frac{1}{4}\frac{\frac{\partial}{\partial z}\left(\frac{\omega^2}{v^2}\right)}{\frac{\omega^2}{v^2} - k_x^2} = -\frac{1}{2v}\frac{\partial v}{\partial z}\frac{\omega^2}{\omega^2 - (k_x v)^2}. \tag{13}$$

From equation (8) to equation (13), we have

$$\sqrt{\frac{k_z(z+\Delta z)}{k_z(z)}} \approx e^{-\frac{1}{2v}\frac{\partial v}{\partial z}\frac{\omega^2}{\omega^2-(k_x v)^2}}\Delta z. \tag{14}$$

So equation (7) can be rewritten as

$$P(z+\Delta z) = \sqrt{\frac{k_z(z+\Delta z)}{k_z(z)}}e^{ik_z\Delta z}P(z) \tag{15}$$

$$\approx e^{ik_z\Delta z - \frac{1}{2v}\frac{\partial v}{\partial z}\frac{\omega^2}{\omega^2-(k_x v)^2}\Delta z}P(z). \tag{16}$$

Comparing the amplitude preserving one-way wave equation (equation (4)) with first order approximation of WKBJ (equation (16)), we find they are same. So we demonstrate theoretically that the amplitude preserving one-way wave equation is equivalent to the first order approximation of WKBJ.

CONCLUSION

With the extra amplitude correction term, the amplitude preserving one-way wave equation improves the amplitude of the wavefield. The extra amplitude correction in the equation is mathematically equivalent to the first order approximation of WKBJ amplitude correction. But unlike the WKBJ correction, the amplitude preserving one-way wave equation can be used for a laterally varying velocity. The accuracy of this correction in the presence of lateral velocity variation has not been analyze yet.

REFERENCES

Claerbout, J. F., 1971, Toward a unified theory of reflector mapping: Geophysics, **36**, no. 03, 467–481.

Clayton, R. W., and Stolt, R. H., 1981, A Born-WKBJ inversion method for acoustic reflection data: Geophysics, **46**, no. 11, 1559–1567.

Hanitzsch, C., 1997, Comparison of weights in prestack amplitude-preserving Kirchhoff depth migration: Geophysics, **62**, no. 06, 1812–1816.

Vlad, I., Tisserant, T., and Biondi, B., 2003, Improving the amplitude accuracy of downward continuation operators: SEP–**113**, 163–176.

Zhang, Y., Sun, J., Notfors, C., Bleistein, N., and Zhang, G., 2001, Towards accurate amplitudes for one-way wavefield extrapolation of 3-D common shot records: 71st Ann. Internat. Mtg., Soc. of Expl. Geophys., Workshop presentation.

Zhang, G., 1993, System of coupled equations for upgoing and downgoing waves: Acta Math. Appl. Sinica, **16**, 251–263.