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One-way wave equation absorbing boundary condition

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ABSTRACT

In modeling and migration based on wave equation, the wavefield has to be extrapolated in a finite domain due to the limitations of our survey and computers. Absorbing boundary conditions must be introduced, otherwise some reflections will occur at the artificial grid boundary. In this paper, I will introduce an absorbing boundary condition based on the one-way wave equation, with some numerical examples.

INTRODUCTION

Several absorbing boundary conditions have been suggested to reduce the reflections at the artificial grid boundary (Engquist and Majda, 1977; Bayliss et al., 1982; Berenger, 1994). One kind of absorbing boundary condition is based on the one-way wave equation, and others are based on absorbing layers. In this paper, I introduce a high order one-way wave equation absorbing boundary condition, which can be solved using low order partial differential equations.

To simulate the wavefield in an open domain, absorbing boundary condition will be transparent to outgoing waves and be an obstacle to incoming waves. So, for a rectangular domain, the wavefield at the grid boundary satisfies the one-way wave equation. For example, the wavefield at the right boundary satisfies the leftgoing wave equation, and the wavefield at the left boundary satisfies the rightgoing wave equation. Solving the internal equation, which is a full wave equation in modeling and a one-way wave equation in migration, and using absorbing boundary conditions, we can simulate the wavefield in an open domain.

ONE-WAY WAVE EQUATION

For the one-way wave equation

$$\frac{\partial p}{\partial z} = \pm \frac{i\omega}{c}\sqrt{1+\frac{c^2}{\omega^2}\frac{\partial^2}{\partial x^2}}\,p,$$

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we can write its $(2n+1)$th order approximation (Zhang, 1985) in time domain

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c}\frac{\partial}{\partial t}\right) p = \pm \frac{1}{c}\frac{\partial}{\partial t}\sum_{k=1}^{n} a_k q(s_k;t,x,z), \tag{1}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - s_k^2 \frac{\partial^2}{\partial x^2}\right) q(s_k,t,x,z) = \frac{\partial^2}{\partial x^2} p(t,x,z), \tag{2}$$

where q is the auxiliary wavefield, c is the velocity, and

$$s_k = \cos(\frac{k\pi}{n+1}), \quad a_k = \frac{1}{n+1}\sin^2(\frac{k\pi}{n+1}), \quad k = 0,1,\cdots,n+1.$$

When $n = 0$, we obtain the 5^o one-way equation

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c}\frac{\partial}{\partial t}\right) p = 0. \tag{3}$$

When $n = 1$, we obtain the 15^o one-way equation in Claerbout (1999)

$$\left(\frac{\partial}{\partial z} \mp \frac{1}{c}\frac{\partial}{\partial t}\right) p = \pm \frac{1}{c}\frac{\partial q}{\partial t}, \tag{4}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} q = \frac{1}{2}\frac{\partial^2}{\partial x^2} p. \tag{5}$$

When $n = 2$, we obtain the 45^o one-way wave equation in Claerbout (1999)

$$\left(\frac{\partial}{\partial z} \mp \frac{1}{c}\frac{\partial}{\partial t}\right) p = \pm \frac{1}{c}\frac{\partial q}{\partial t}, \tag{6}$$

$$\left(\frac{1}{4}\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) q = \frac{1}{2}\frac{\partial^2}{\partial x^2} p. \tag{7}$$

ABSORBING BOUNDARY CONDITION FOR MODELING

Let us consider the explicit finite-difference scheme for the full wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p + f, \tag{8}$$

where p is the wavefield and f is the force. We can extrapolate the wavefield along t using the following explicit finite-difference scheme

$$\Delta_t^2 p_{x,z}^t = c^2 (\Delta t)^2 \left(\frac{\Delta_z^2}{(\Delta z)^2} + \frac{\Delta_x^2}{(\Delta x)^2}\right) p_{x,z}^t + (\Delta t)^2 f_{x,z}^t, \tag{9}$$

where Δ_t^2 and Δ_x^2 are the second order central finite-difference operators

$$\Delta_t^2 p_{x,z}^t = p_{x,z}^{t+1} + p_{x,z}^{t-1} - 2p_{x,z}^t,$$

$$\Delta_x^2 p_{x,z}^t = p_{x+1,z}^t + p_{x-1,z}^t - 2p_{x,z}^t.$$

Given the initial condition $p_{x,z}^{t-1}$ and $p_{x,z}^t$, we can solve equation (9) to get the wavefield at time $t+1$, $p_{x,z}^{t+1}$ from the wavefield at time $t-1$, $p_{x,z}^{t-1}$ and the wavefield at time $t, p_{x,z}^t$, except for the wavefield at the boundaries $p_{x=X_{min},z}^{t+1}$, $p_{x=X_{max},z}^{t+1}$, $p_{x,z=Z_{min}}^{t+1}$, $p_{x,z=Z_{max}}^{t+1}$.

Let us consider the wavefield on the boundary $z = Z_{max}$. There are only outgoing waves at $z = Z_{max}$, so the wavefield satisfies the downgoing wave equation, for which we can write its approximate equations:

$$\frac{\partial p}{\partial z} = -\frac{1}{c}\frac{\partial p}{\partial t} - \frac{1}{c}\frac{\partial q}{\partial t}, \tag{10}$$

$$\left(\beta\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) q = \alpha\frac{\partial^2}{\partial x^2} p. \tag{11}$$

For compatibility with the explicit finite-difference scheme at internal points, we apply the explicit finite-difference scheme for the boundaries using equation (10) and (11) and get

$$\frac{1}{2}\Delta_z^-\left(p_{x,Z_{max}}^{t+1} + p_{x,Z_{max}}^t\right) = -\frac{\Delta z}{2c\Delta t}\Delta_t^+\left(p_{x,Z_{max}}^t + p_{x,Z_{max}-1}^t\right) - \frac{\Delta z}{c\Delta t}\Delta_t^+ q_{x,Z_{max}-\frac{1}{2}}^t, \tag{12}$$

$$\Delta_t^2 q_{x,Z_{max}-\frac{1}{2}}^t = \beta\frac{c^2(\Delta t)^2}{(\Delta x)^2}\Delta_x^2 q_{x,Z_{max}-\frac{1}{2}}^t - \frac{1}{2}\alpha\frac{c^2(\Delta t)^2}{(\Delta x)^2}\Delta_x^2(p_{x,Z_{max}}^t + p_{x,Z_{max}-1}^t). \tag{13}$$

where Δ^- is the first order backward finite-difference operator, Δ^+ is the first order forward finite-difference opertor:

$$\Delta_z^- p_{x,z}^t = p_{x,z}^t - p_{x,z-1}^t, \quad \Delta_t^+ p_{x,z}^t = p_{x,z}^{t+1} - p_{x,z}^t,$$

and Δ^2 is the second order central finite-difference operator:

$$\Delta_t^2 q_{x,z}^t = q_{x,z}^{t+1} - 2q_{x,z}^t + q_{x,z}^{t-1}, \quad \Delta_x^2 q_{x,z}^t = q_{x+1,z}^t - 2q_{x,z}^t + q_{x-1,z}^t.$$

Assuming that the wavefield $p_{x,z}^k$ for $k \le t$ is known, then we solve the internal equation (9) to get the wavefield for the internal points $X_{min} < x < X_{max}$, $Z_{min} < z < Z_{max}$ at time $t+1$, $p_{x,z}^{t+1}$ first. Then, the auxiliary wavefield $q_{x,Z_{max}-\frac{1}{2}}^{t+1}$ can be solved by equation (13) since the wavefield of the boundary at time t, $p_{x,Z_{max}}^t$ and $p_{x,Z_{max}-1}^t$ are known. Finally, we solve equation (12) to get the wavefield at the boundary $p_{x,z=Z_{max}}^t$. Figure 1 illustrates how the boundary conditions are solved.

The method of solving the wavefield at the other three boundaries $z = Z_{min}$, $x = X_{min}$, and $x = X_{max}$, is similar to that of boundary $z = Z_{max}$. The only difference is that the boundary condition equation is an upgoing wave equation at $z = Z_{min}$, leftgoing wave equation at $x = X_{min}$, and right-going wave equation at $x = X_{max}$.

According to Zhang and Wei (1998), this absorbing boundary condition is stable.

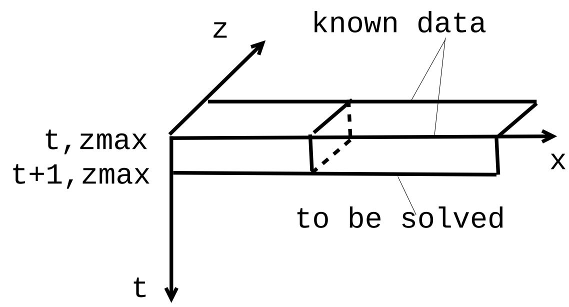

Figure 1: solution at the boundary $z = Z_{max}$ shan-boundary [NR]

NUMERICAL EXAMPLE

I test the absorbing boundary condition on plane waves with different incident angles. I compare the results of a low order absorbing boundary condition (5^o one-way wave equation, equivalent to the method in Engquist and Majda (1977)) and a high order absorbing boundary condition (45^o one-way wave equation). The results show little benefit in using a high order absorbing boundary condition for small incident angle plane waves. However, we get a much better absorbing result of the high order absorbing boundary condition than that of the low order absorbing boundary condition for large incident angle plane waves. For small incident angle plane waves (Figure 2), both low order and high order absorbing boundary condition equations do very well with the reflection. Theoretically, the low order absorbing boundary condition equation can only handle the reflection with angles less than 5^o. So for reflection with a large angle (middle panels in Figure 3 and Figure 4), there is still a lot of reflected energy left after absorbing. High order absorbing boundary conditions (bottom panels in Figure 3 and Figure 4), still do well with the large angle reflection, and the results show that most reflection energy vanishes for both big and small incident angle plane waves.

CONCLUSION

High order one-way wave equation absorbing boundary conditions handle well the reflections from the boundaries. The method described in this paper is usable for explicit finite-difference schemes. However, it can be used for both explicit and implicit finite-difference schemes, both for modeling and migration.

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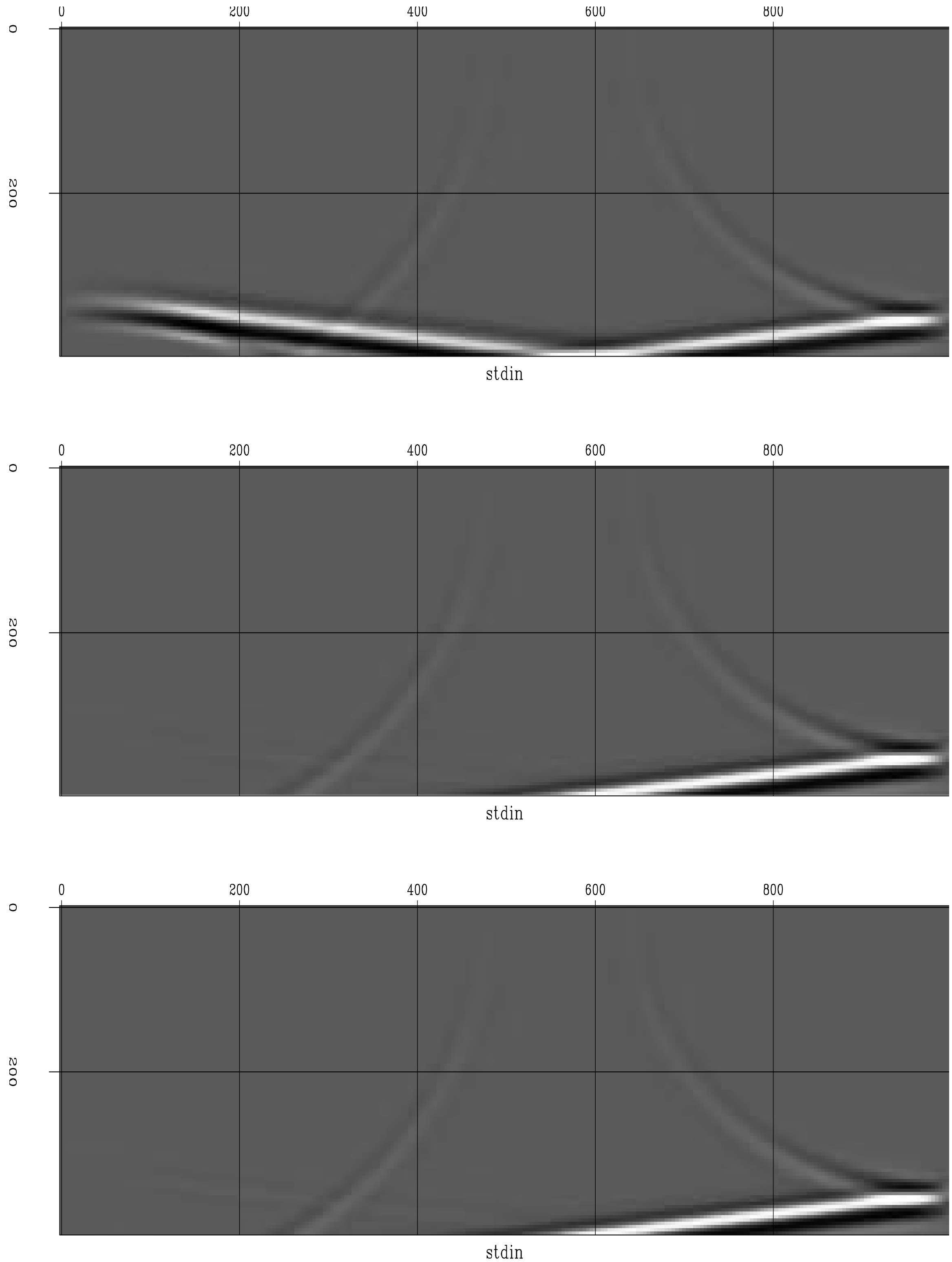

Figure 2: From top to bottom: modeling with no boundary condition; modeling with low order absorbing boundary condition; modeling with high order absorbing boundary condition shan-absorb4 [ER]

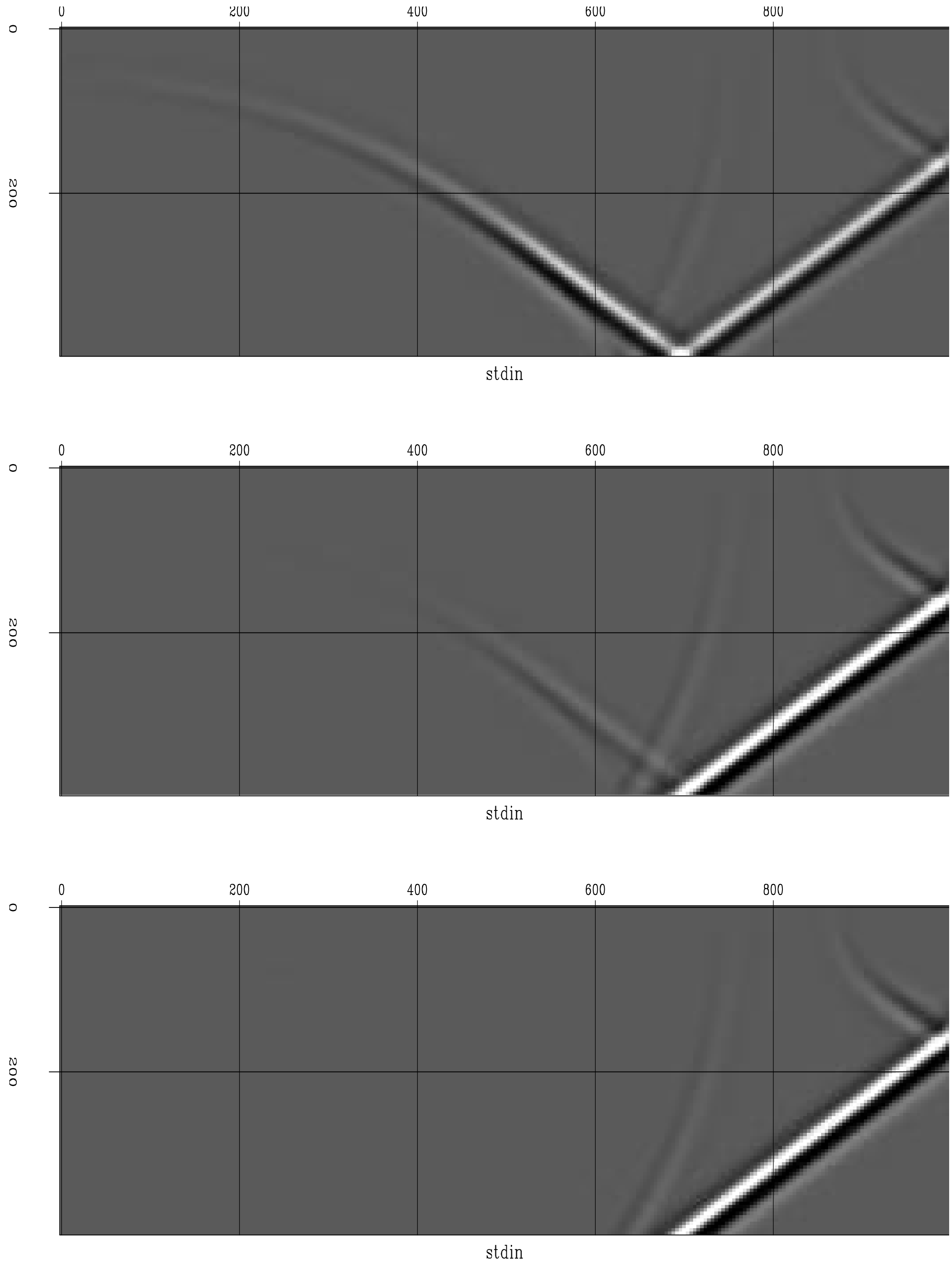

Figure 3: From top to bottom: modeling with no boundary condition; modeling with low order absorbing boundary condition; modeling with high order absorbing boundary condition shan-absorb2 [ER]

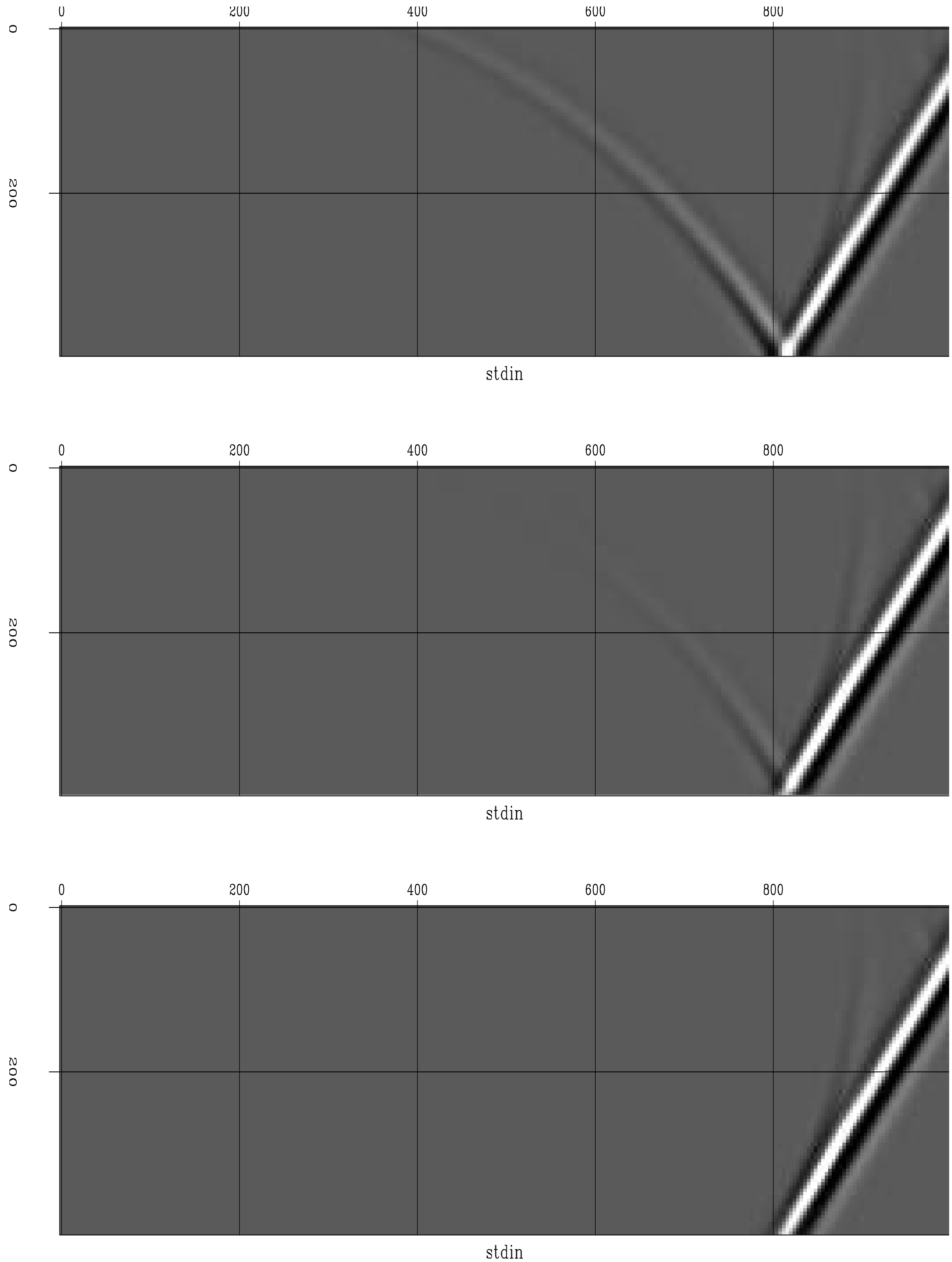

Figure 4: From top to bottom: modeling with no boundary condition; modeling with low order absorbing boundary condition; modeling with high order absorbing boundary condition shan-absorb1 [ER]

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