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Short Note

Theoretical aspects of noise attenuation

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INTRODUCTION

In Guitton (2001) I presented an efficient algorithm that attenuates coherent noise based on the spatial predictability of noise and signal. I called this algorithm the subtraction method. In this paper I show that the subtraction approach is closely related to another described method, the filtering method (Brown and Clapp, 2000; Clapp and Brown, 2000; Spitz, 1999; Soubaras, 1994) if I use a preconditioning strategy (Claerbout and Fomel, 2001). In a second part I prove that the Spitz estimate for the signal PEF (Spitz, 1999) makes the inversion of the Hessian stable in the subtraction method.

FROM THE FILTERING TO THE SUBTRACTION OF NOISE

The filtering method is essentially based on the signal-preserving properties of the so-called "projection filters" (Soubaras, 1994). This idea has been widely used to attenuate a large variety of noise in seismic data. Abma (1995) developed a solid mathematical background that introduces these projection filters and showed that they are related to the classical Wiener estimator (Castleman, 1996). In this section I unravel the link between the filtering and subtraction method.

Definitions

First I introduce a set of important variables that will help us build the desired filters.

- **d**: the data vector; input to the problem.
- **n**: the noise vector; assumed to be known.
- **s**: the signal vector; output of the problem.
- **D**: annihilation filter for the data; a Prediction Error Filter (PEF).

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- **N**: annihilation filter for the noise; a PEF.
- **S**: annihilation filter for the signal; a PEF.

The leading assumption is that the data vector is the sum of the signal and noise vectors, i.e,

$$\mathbf{d} = \mathbf{s} + \mathbf{n}. \tag{1}$$

The filtering method

Abma (1995) solved a constrained least-squares problem to separate signal from spatially uncorrelated noise:

$$\begin{array}{rcl} \mathbf{Nn} & \approx & \mathbf{0} \\ \epsilon \mathbf{Ss} & \approx & \mathbf{0} \\ \text{subject to} & \leftrightarrow & \mathbf{d} = \mathbf{s} + \mathbf{n} \end{array} \tag{2}$$

The first equation defines mathematically the annihilation filter **N** whereas the second equation defines the annihilation filter **S**. Minimizing in a least-squares sense the fitting goals in equation (2) with respect to **s** leads to the following expression for the estimated signal:

$$\hat{\mathbf{s}} = \left(\mathbf{N^T N} + \epsilon^2 \mathbf{S^T S}\right)^{-\mathbf{1}} \mathbf{N^T N d}. \tag{3}$$

$\left(\mathbf{N}^T\mathbf{N} + \epsilon^2 \mathbf{S}^T\mathbf{S}\right)^{-1} \mathbf{N}^T\mathbf{N}$ is a projection filter. I call it the filtering method because the noise components are filtered out by the PEF **N** in equation (3).

Preconditioning the filtering method

There is a simple trick that modifies the fitting goals in equation (2). We can pose the following preconditioning transformations:

$$\begin{array}{rcl} \mathbf{n} & = & \mathbf{N^{-1} m_n}, \\ \mathbf{s} & = & \mathbf{S^{-1} m_s}, \end{array} \tag{4}$$

where $\mathbf{m_n}$ and $\mathbf{m_s}$ are new variables. Now we can derive a new system of fitting goals as follows:

$$\begin{array}{rcl} \mathbf{m_n} & \approx & \mathbf{0} \\ \epsilon \mathbf{m_s} & \approx & \mathbf{0} \\ \text{subject to} & \leftrightarrow & \mathbf{d} = \mathbf{S^{-1} m_s} + \mathbf{N^{-1} m_n}. \end{array} \tag{5}$$

This system is almost equivalent to what I introduced in Guitton (2001), except for the regularization that I omitted. With $\mathbf{L_n} = \mathbf{N^{-1}}$ the noise-modeling operator and $\mathbf{L_s} = \mathbf{S^{-1}}$ the signal-modeling operator, the least-squares inverse of equations (5) without the regularization terms is then given by

$$\begin{pmatrix} \hat{\mathbf{m}}_{\mathbf{n}} \\ \hat{\mathbf{m}}_{\mathbf{s}} \end{pmatrix} = \begin{pmatrix} (\mathbf{L}_{\mathbf{n}}'\overline{\mathbf{R}_{\mathbf{s}}}\mathbf{L}_{\mathbf{n}})^{-1}\mathbf{L}_{\mathbf{n}}'\overline{\mathbf{R}_{\mathbf{s}}} \\ (\mathbf{L}_{\mathbf{s}}'\overline{\mathbf{R}_{\mathbf{n}}}\mathbf{L}_{\mathbf{s}})^{-1}\mathbf{L}_{\mathbf{s}}'\overline{\mathbf{R}_{\mathbf{n}}} \end{pmatrix} \mathbf{d},$$

with

$$\begin{aligned} \overline{\mathbf{R}_{\mathbf{s}}} &= \mathbf{I} - \mathbf{L}_{\mathbf{s}}(\mathbf{L}_{\mathbf{s}}'\mathbf{L}_{\mathbf{s}})^{-1}\mathbf{L}_{\mathbf{s}}', \\ \overline{\mathbf{R}_{\mathbf{n}}} &= \mathbf{I} - \mathbf{L}_{\mathbf{n}}(\mathbf{L}_{\mathbf{n}}'\mathbf{L}_{\mathbf{n}})^{-1}\mathbf{L}_{\mathbf{n}}'. \end{aligned} \qquad (6)$$

I showed in Guitton et al. (2001) that $\overline{\mathbf{R}_{\mathbf{s}}}$ and $\overline{\mathbf{R}_{\mathbf{n}}}$ can also be interpreted in term of projection filters.

The estimated noise and signal are then computed as follows

$$\begin{aligned} \hat{\mathbf{n}} &= \mathbf{L}_{\mathbf{n}}\hat{\mathbf{m}}_{\mathbf{n}}, \\ \hat{\mathbf{s}} &= \mathbf{L}_{\mathbf{s}}\hat{\mathbf{m}}_{\mathbf{s}}. \end{aligned} \qquad (7)$$

Because of the relationship that exists between the filtering and subtraction methods, the estimated noise or signal should be equivalent for both. This has been observed in a multiple attenuation problem by Guitton et al. (2001).

Discussion

The preceding section derives the relationship between the filtering and subtraction methods. The preconditioning changes the nature of the problem quite deeply: from a filtering algorithm we end-up with a prediction/subtraction method. The least-squares inverses in equations (3) and () are also very different: from a problem with one unknown, **s**, we end-up with a problem with two unknowns, $\mathbf{m}_{\mathbf{n}}$ and $\mathbf{m}_{\mathbf{s}}$. Fortunately, this preconditioning should speed-up the convergence toward the signal vector **s**. In addition, this transformation separates the data space in its natural components, e.g, the signal and noise vectors more explicitely than with the filtering method.

STABILITY OF THE HESSIAN WITH THE SPITZ ESTIMATE

Nemeth (1996) shows that the Hessians in equations () can be unstable if the signal and noise operators $\mathbf{L}_{\mathbf{s}}$ and $\mathbf{L}_{\mathbf{n}}$ predict similar components of the data space. In this section I show that the Spitz estimate of the signal PEF makes this overlap impossible.

The Spitz estimate

Equation (5) assumes that the signal PEF is known in advance. This argument is circular since we are estimating the signal needed to compute **S**! Nonetheless, Spitz (1999) shows that **S** can be estimated directly from the data PEF **D** and the noise PEF **N**. Again, we assume that we are able to compute a model for the coherent noise we wish to attenuate. In equation, the signal PEF can be estimated as follows:

$$\mathbf{S} \approx \mathbf{D}/\mathbf{N}. \tag{8}$$

In the next section, I show on a simple 1D example that this estimate makes the overlap of the signal and noise operators $\mathbf{L_s}$ and $\mathbf{L_n}$ impossible

A 1D example

Now I consider the Z-transforms of the data, signal and noise PEFs for a 1D case. For the data PEF **D**, I assume that the filter has the form

$$D(Z) = \alpha(Z - Z_1)(Z - Z_2)(Z - Z_3)(Z - Z_4), \tag{9}$$

with $\alpha = 1/(Z_1 Z_2 Z_3 Z_4)$. The Z_i correspond to the roots of the filter. In this example I consider that Z_1 and Z_2 are the roots for the noise and Z_3 and Z_4 the roots for the signal. Now I assume that we have for the noise PEF **N**

$$N(Z) = \beta(Z - Z_1)(Z - Z_2), \tag{10}$$

with $\beta = 1/(Z_1 Z_2)$. The Spitz estimate yields for the signal PEF **S**

$$S(Z) = \gamma(Z - Z_3)(Z - Z_4), \tag{11}$$

with $\beta = 1/(Z_3 Z_4)$. We see that by construction, the signal and noise PEF annihilate different parts of the data space and can't overlap.

Now, If we assume that the noise PEF is a "bad" estimate of the noise with one erroneous root, i.e,

$$N(Z) = \beta(Z - Z_1)(Z - Z_5), \tag{12}$$

with $\beta = 1/(Z_1 Z_5)$, we find for the signal PEF **S**

$$S(Z) = \frac{Z_5}{Z_2 Z_3 Z_4} \frac{(Z - Z_2)(Z - Z_3)(Z - Z_4)}{Z - Z_5}. \tag{13}$$

Because the PEFs are minimum phase, we can write

$$\begin{aligned} \frac{1}{Z - Z_5} &= \frac{-1}{Z_5} \frac{1}{1 - Z/Z_5}, \\ &= \frac{-1}{Z_5} \left(1 + \frac{Z}{Z_5} + \frac{Z^2}{Z_5^2} + ... \right), \\ &\approx \frac{-1}{Z_5} \left(1 + \frac{Z}{Z_5} \right), \\ &\approx \frac{-1}{Z_5^2} (Z + Z_5). \end{aligned} \tag{14}$$

Then we obtain for the signal PEF

$$S(Z) \approx \frac{-1}{Z_2 Z_3 Z_4 Z_5}(Z - Z_2)(Z - Z_3)(Z - Z_4)(Z + Z_5). \tag{15}$$

The wrong root in the noise PEF leaks in the signal PEF but with an opposite sign. Again, the Spitz estimate makes it impossible for the signal and noise operators to overlap in the data space. This simple example in 1D can be easily expendable in 2D via the helical coordinates (Claerbout, 1998).

CONCLUSION

I have shown that the filtering and subtraction method are linked by a simple preconditioning transformation. I have also demonstrated that the Spitz estimate for the signal PEF estimation prevent stability issues when the Hessians in equation () are computed.

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