IMPROVING THE ACCURACY OF DIFFERENCE APPROXIMATIONS TO THE MIGRATION EQUATION

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Part 1: Description of the Approximation Schemes

1.1 Introduction

When making finite difference approximations to the migration equation, we are always faced with the problem of *aliasing*, due to the fact that the initial data are sampled only at discrete points in x and t. If information in the initial data lies near the Nyquist frequency in x or t, we do not expect to be able to handle this information very accurately. When making new approximations to improve the accuracy, we hope that frequencies nearer the Nyquist will be approximated better than before. In Part 1 we discuss briefly some earlier work which concentrated on improvement of the x-derivative approximation, and then concentrate our discussion on the improvement of the approximation of the t-derivative in the migration equation. A family of approximations is suggested and discussed in which the t-derivative approximation is improved by including a few extra points when we average the $D_+^x D_-^x$-operator in t and z. In Part 2, we present more formally the numerical methods used to obtain the results given in Part 1. In particular, the stability and accuracy of the difference approximations are discussed in some detail.

1.2 Approximation of the x*-derivative*

We will be considering difference approximations to the migration equation,

$$P_{zt} + \frac{v}{2} P_{xx} = 0. \qquad (1.2\text{-}1)$$

The Crank-Nicolson scheme,*

$$\left(D_+^z D_+^t + \frac{v}{2} \begin{array}{c} \overset{\rightarrow t}{} \\ \overset{\downarrow z}{} \end{array} \boxed{\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}} D_+^x D_-^x\right) P_{j,k}^n = 0 , \tag{1.2-2}$$

is commonly used for solving this equation numerically and we will be using it for comparison with the methods suggested here.

The approximation to the x-derivative used is

$$D_+^x D_-^x P_k = \frac{1}{\Delta x^2}(P_{k-1} - 2P_k + P_{k+1}) = P_{xx} + O(\Delta x^2). \tag{1.2-3}$$

If a plane-wave solution to the differential equation (1.2-1) is written

$$P(x,z,t) = P'(x,t)\, e^{ik_z z}, \tag{1.2-4}$$

then neglecting errors due to discretization in z and t, the corresponding solution to the difference equation can be written

$$P(x_k,z,t) = P'(x_k,t)\, e^{i\hat{k}_z(k_x\Delta x)z}, \tag{1.2-5}$$

where $x_k = k\,\Delta x$ and by $\hat{k}_z(k_x\Delta x)$ we mean that some phase error is made in the evolution (z) direction which is a function of the sampling density in x. The Nyquist frequency is defined to be that frequency that is sampled at 2 points per wavelength and is given by $k_x^{(Ny)}\Delta x = \pi$. In Fig. (1.2.1), the curve marked "CN" is a plot of the phase error, $|k_z - \hat{k}_z|$, *vs.* the sampling frequency $k_x\Delta x$, where by "50% phase error" we would mean that after calculating for one wavelength in z we were 180° out of phase with the real solution. We can see that if we want to make calculations with 10% error or less, we must sample the initial data densely enough so that all the information of interest lies below $0.35\, k_x^{(Ny)}$. The other curves in Fig. (1.2.1) show the phase error that we expect if we replace $D_+^x D_-^x P_k$ by

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*This difference notation will be formally introduced in section 2.2. D_+^q and D_-^q are, respectively, the forward and backward single difference operators in the q-direction; $P_{j,k}^n$ is the approximation to $P(k\,\Delta x, n\,\Delta z, j\,\Delta t)$; the approximation is centered at $P_{j+1/2,\,k}^{n+1/2}$.

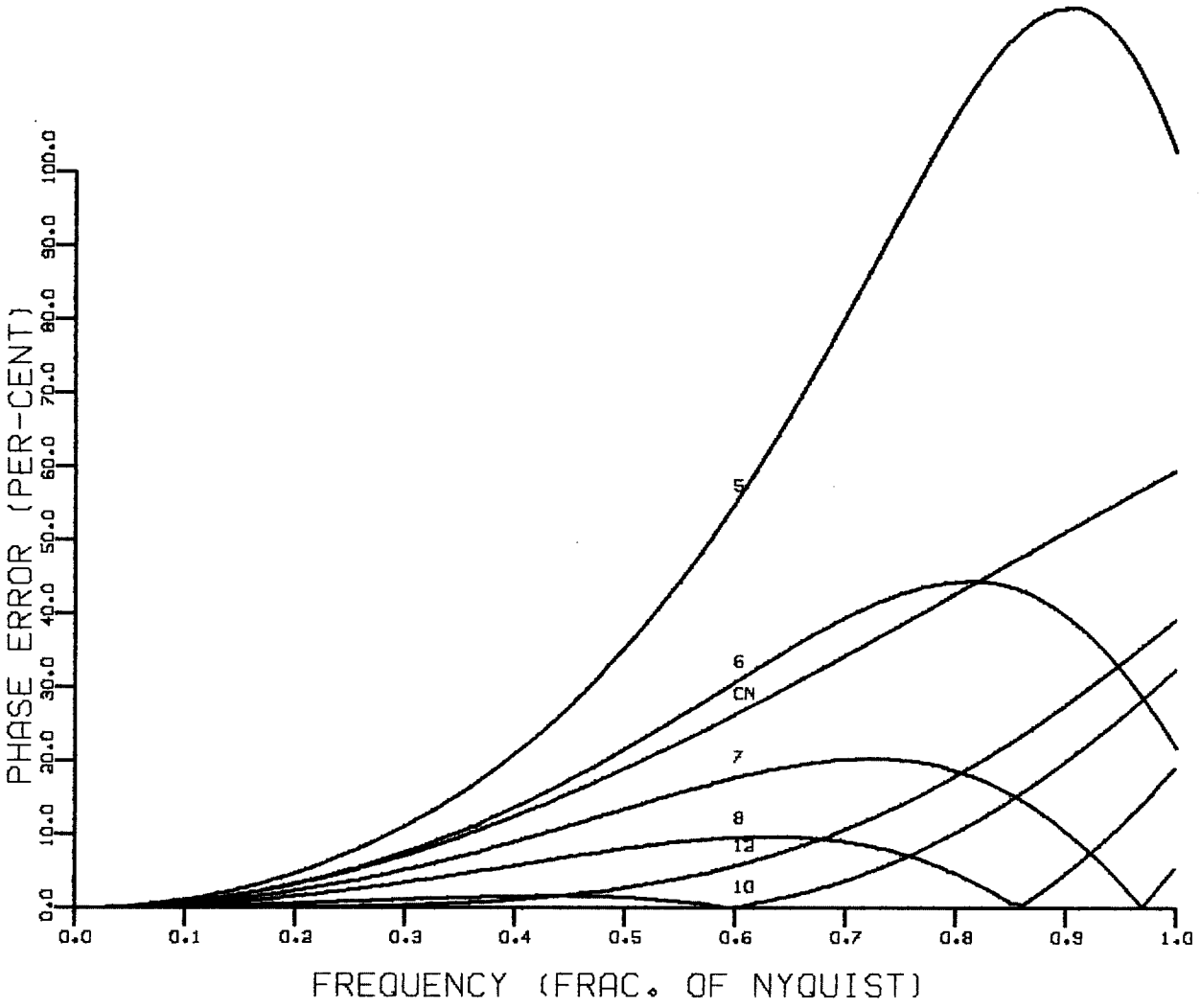

FIGURE 1.2.1.—Plot of *Phase Error* $\propto |k_z - \hat{k}_z(k_x \Delta x)|$ *vs.* sampling frequency in the x-direction, $k_x \Delta x$. This plot shows the theoretical error due to discretization in x when a plane wave solution to Eq. (1.2-1) is used, and the ∂_{xx}-derivative is approximated by

$$\partial_{xx} \approx \frac{D_+^x D_-^x}{1 + \frac{1}{n} \Delta x^2 D_+^x D_-^x} .$$

The curves are labeled with the value of n used. The curve labelled "CN" represents the phase error for $\partial_{xx} \approx D_+^x D_-^x$, the approximation used in the Crank-Nicolson scheme.

$$\left(\frac{D_+^x D_-^x}{1 + \frac{1}{n} \Delta x^2 D_+^x D_-^x} \right) P_k = \partial_{xx} P + O\begin{pmatrix} \Delta x^2 & \text{if } n \neq 12 \\ \Delta x^4 & \text{if } n = 12 \end{pmatrix}, \qquad (1.2\text{-}6)$$

for various values of n. The operator in which $n = 12$ is fourth-order in Δx, which means that as $\Delta x \to 0$, this difference operator will converge much faster to the derivative than the other members of this family. We can see that this is true; for frequencies less than $0.4\, k_x^{(Ny)}$, this method clearly looks the best, with a phase error of less than 1%. Often, however, we cannot choose how densely we sample in x, because the data have already been sampled. We may be faced with data in which information of interest lies much nearer to the Nyquist frequency. If this is the case, some of the second-order formulae shown here may be of more use to us. When $n = 7$, for example, we expect to make at most a 22% error for frequencies up to the Nyquist, while the fourth-order method, $n = 12$, makes a maximum of a 40% error over this range. Some workers like to choose $1/n = 1/4 - 1/\pi^2 \simeq 1/6.726$ in (1.2-6), since this value makes no phase error at all at the Nyquist frequency when treating plane waves, and is better than the standard Crank-Nicolson choice at all frequencies less than the Nyquist.

1.3 Approximation of the t-derivative

In this section we will discuss a family of approximations suggested by Muir [personal communication, February 1977] to (1.2-1) that are designed to improve the accuracy with which we represent the t-derivative. We do this by expanding the difference "star" to include a few more values of $P_{j,k}^n$. These approximations can be written as follows:

$$\left[D_+^z D_+^t + \frac{v}{4(N-1)} \begin{array}{|c|c|} \hline -1 & \\ \hline \frac{N-1}{2} & \frac{N+1}{2} \\ \hline \frac{N+1}{2} & \frac{N-1}{2} \\ \hline & -1 \\ \hline \end{array} D_+^x D_-^x \right] P_{j,k}^n = 0, \qquad (1.3\text{-}1)$$

where we get a different approximation for each N. These difference approximations are also centered at $P_{j+(1/2),k}^{n+(1/2)}$. As with the Crank-Nicolson scheme, they are unconditionally stable for all $N \geq 2$. The local truncation error [the difference between Eq. (1.2-1) and (1.3-1)] is in general

$O(\Delta x^2 + \Delta z^2 + \Delta z\,\Delta t + \Delta t^2)$. When $N = 13$, however, the scheme becomes $O(\Delta x^2 + \Delta z^2 + \Delta z\,\Delta t + \Delta t^4)$, and is similar to a method suggested by Engquist [1]. As $N \to \infty$, the scheme converges to the Crank-Nicolson approximation with truncation error $O(\Delta x^2 + \Delta z^2 + \Delta t^2)$. As we mentioned earlier when discussing the ∂_{xx}-approximation, the local truncation error tells us only how well the approximation behaves as the step-sizes, Δx, Δz, and Δt, approach zero. We really wish to know how good the approximation is for sparsely-sampled initial data. Since this family of methods is designed to improve the accuracy of the ∂_t-approximation, we will look at the phase error in the direction of evolution which results from the discretization in t. We write this as $e = |k_z - \hat{k}_z(\omega\,\Delta t)|$, and plot it as a function of $\omega\,\Delta t$ in Fig. (1.3.1) for different values of N. The phase-error curve for the Crank-Nicolson scheme is also shown for comparison. We see that for frequencies less than about $0.25\,\omega^{(Ny)}$, we predict that the fourth-order ($N = 13$) scheme will perform the best. This is not the case for ω greater than $0.25\,\omega^{(Ny)}$, however. We see in fact that any of the methods with $N \geq 7$ should do as well as or better than the Crank-Nicolson scheme, and the methods with $7 \leq N \leq 12$ should be more accurate than the fourth-order scheme as we try to represent frequencies approaching the Nyquist. The actual choice of N would depend on the range of frequencies we want to represent well. We would probably try to choose a scheme which would give us the best overall accuracy for the frequency range we are interested in.

We ran two numerical experiments to test the behavior of these schemes. In the first test, the initial conditions and appropriate boundary conditions for the solution

$$P(x,z,t) = \sin\beta x \sin\beta(t+z) \tag{1.3-2}$$

were used in a calculation involving 13 sample points in x and t. We did the calculation for four steps in the z-direction with a step-length parameter $a = v\Delta z\Delta t/8\Delta x^2 = 0.025$. The relative L_2-error

$$e = \|P^n_{j,k} - P(x,z,t)\| \,/\, \|P(x,z,t)\|$$

was calculated and is plotted in Fig. 1.3.2 for several values of N and $\omega\,\Delta t$. The letters indicate calculated points, while the lines are plotted only for reference. We see that, as expected, the $N = 2$ scheme does worse than the

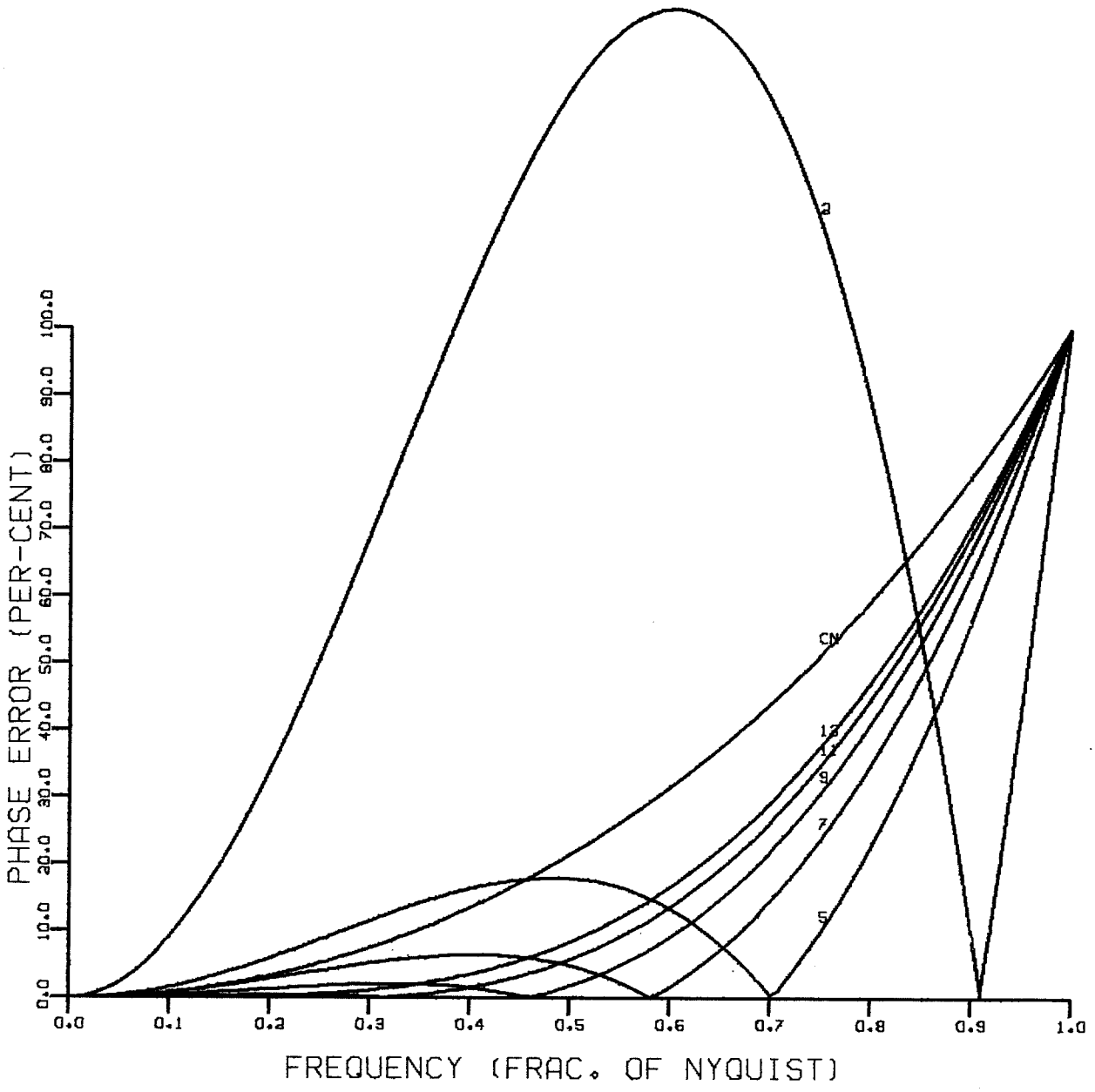

FIGURE 1.3.1.—Plot of phase error $\propto |k_z - \hat{k}_z(\omega\Delta t)|$ due to discretization in t vs. sampling frequency in the t-direction, $\omega\Delta t$, for the scheme given in Eq. (1.3-1). The curves are labelled with the value of N used in the scheme. The curve labelled "CN" corresponds to the Crank-Nicolson scheme.

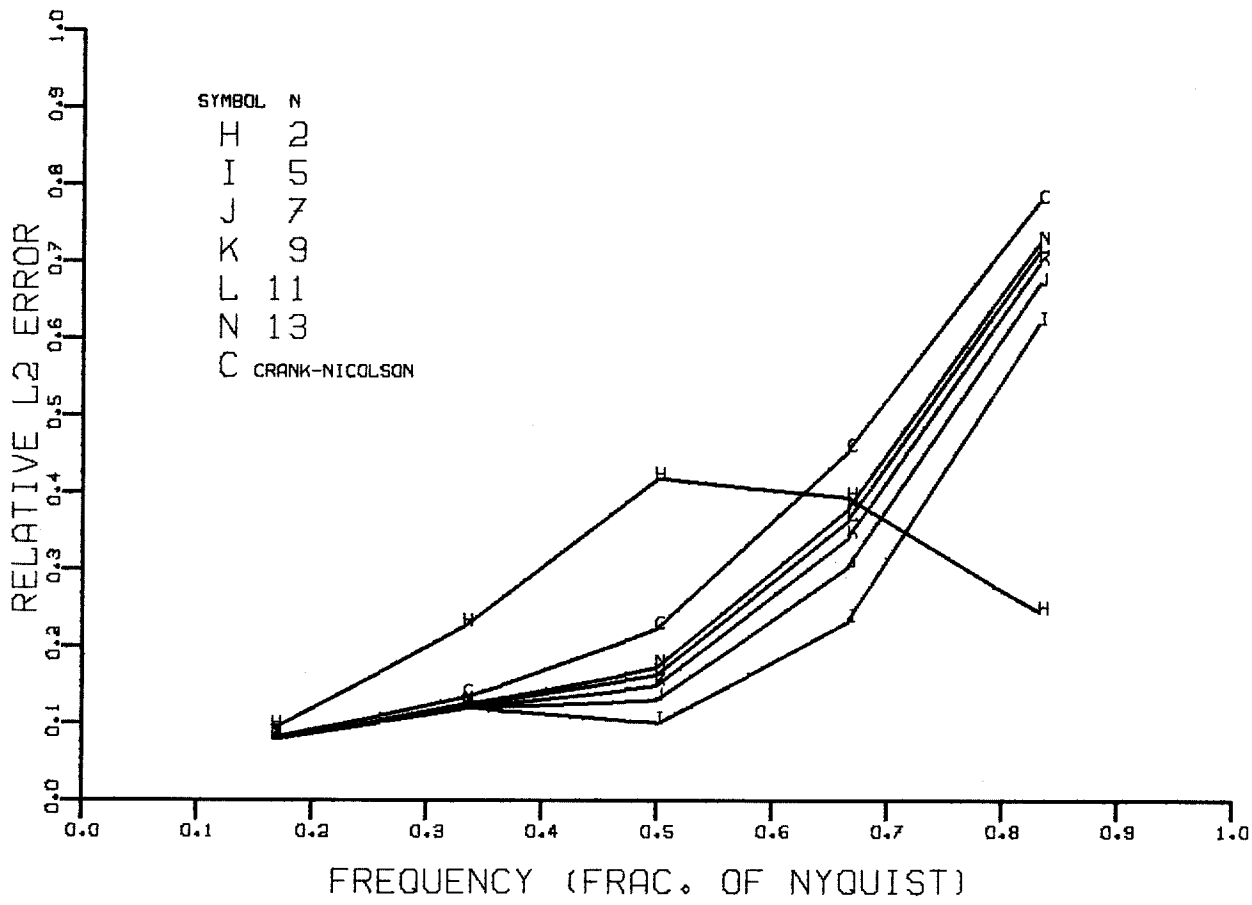

FIGURE 1.3.2.—Plot of calculated relative L_2-error using the scheme given in Eq. (1.3-1). Initial and boundary conditions were given for the solution $P(x,z,t) = \sin\beta(x)\ \sin\beta(z+t)$. Thirteen sample points were used in x and t, 4 steps were taken in the z-direction. The calculated points are indicated by the letters.

others at all frequencies except near the Nyquist. The curves for $N = 5$, $N = 7$, $N = 9$, $N = 11$ and $N = 13$ all did better than the Crank-Nicolson method for the frequencies calculated.

In the second test, shown in Fig. 1.3.3, point-source initial conditions were used and the solution was calculated for 15 steps in z for the *downward* wave equation (forward problem). There were 50 sample points in t and 40 sample points in x. Only every other trace was plotted, and the plots show only the first 25 sample points in t. The step-length parameter $a = 1.0$ was chosen which corresponds to step sizes $\Delta x = \Delta z = \frac{1}{8} v \Delta t$. The initial data were thus sampled much more densely in x than in t, so that we expect the results to emphasize the effect of the t-discretization. We see that the $N = 2$ scheme shows very poor performance compared with Crank-Nicolson, as we would expect, while the other schemes all show sharper arrivals far away from the source than does the Crank-Nicolson method. This can be attributed to the decreased dispersion which we predicted earlier in the phase error calculations.

Part 2: Numerical Methods

In this section we will present methods which may be used for investigating the stability and accuracy of difference approximations to the "upward" and "downward" wave equations using the difference scheme introduced in Part 1 as an example.

2.1 The Differential Equations

We will consider the differential equations

$$P_{zt}(x,z,t) = -\frac{v}{2} P_{xx}(x,z,t), \qquad (2.1\text{-}1a)$$

and

$$P_{zt}(x,z,t) = \frac{v}{2} P_{xx}(x,z,t), \qquad (2.1\text{-}1b)$$

in the region $0 \leq x \leq 1$, $0 \leq t \leq t_{max}$, $z \geq 0$; with the initial conditions

$$P(x,z=0,t) = P_0(x,t).$$

We will refer to Eq. (2.1-1a) as the "upward" wave equation, or as the "migration"

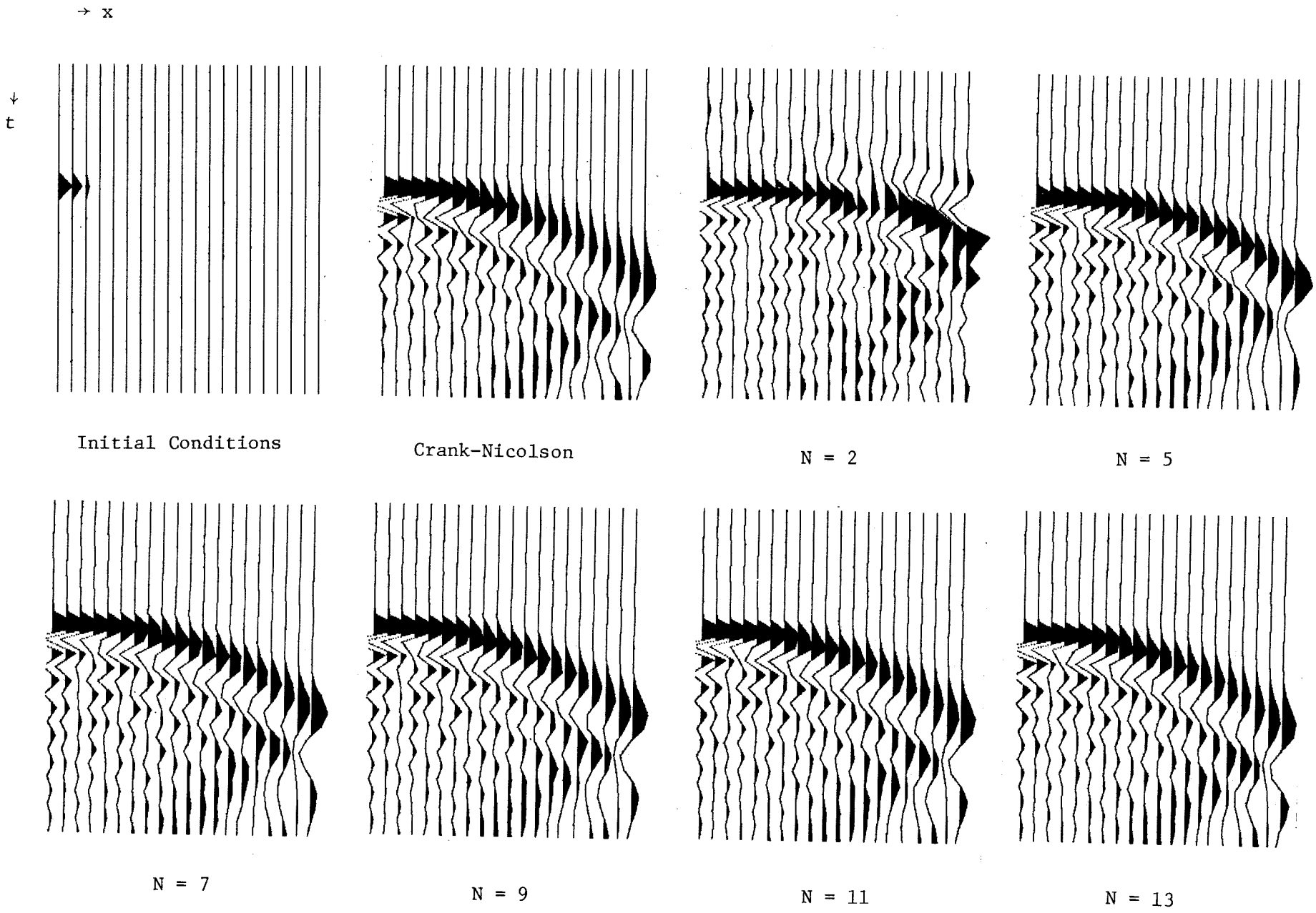

FIGURE 1.3.3.—The scheme of Eq. (1.3-1) was used on the initial conditions shown. As expected (see Fig. 1.3.1), all methods but N=2 show sharper main arrivals for the far traces than does the Crank-Nicolson scheme.

equation, and (2.1-1b) as the "downward" wave equation. These equations are "paraxial" approximations to the scalar wave equation that describe waves travelling only in one direction (see Claerbout [2]). The boundary conditions are usually chosen to be as simple as possible, for example,

$$P_x(x=0,z,t) = P_x(x=1,z,t) = 0,$$

and

$$P(x,z,t=0) = P(x,z,t=t_{max}) = 0.$$

The x-boundary conditions given here correspond physically to reflective boundaries. The choice of t-boundary conditions is not important in practice, and so zero-value conditions are often specified. More complicated x-boundary conditions such as the "absorbing" boundary conditions of Clayton and Engquist [3] may also be posed.

2.2 The Difference Approximations

We introduce a *mesh function* $P^n_{j,k}$ to represent the solution of the difference equation at discrete points on a uniform rectangular mesh in (x,z,t)-space. $P^n_{j,k}$ is an approximation to the wave function $P(x_k,z_n,t_j)$, where $x_k = k\Delta x$, $z_n = n\Delta z$, and $t_j = j\Delta t$; $k = 0,1,\ldots,K$; $n = 0,1,\ldots$; $j = 0,1,\ldots,J$. We will use the following difference operators:

$$D^x_+ P_k = \frac{1}{\Delta x}(P_{k+1} - P_k),$$

$$D^x_- P_k = \frac{1}{\Delta x}(P_k - P_{k-1}),$$

both of which are approximations to $\partial P/\partial x$. It may be quickly verified that

$$D^x_+ D^x_- P_k = \frac{1}{\Delta x^2}(P_{k+1} - 2P_k + P_{k-1}),$$

which is an approximation to $\partial^2 P/\partial x^2$. The operators D^z_+, D^z_-, D^t_+, and D^t_- are defined analogously. We will also use the averaging operators

$$A_N^+ P_j^n = \frac{1}{2(N-1)} \left(- P_{j+2}^n + \frac{N-1}{2} P_{j+1}^n + \frac{N+1}{2} P_j^n + \frac{N+1}{2} P_{j+1}^{n+1} \right.$$

$$\left. + \frac{N-1}{2} P_j^{n+1} - P_{j-1}^{n+1} \right),$$

$$= \frac{1}{2(N-1)}$$

t \ z	(n)	(n+1)	
		-1	(j - 1)
	$\frac{N+1}{2}$	$\frac{N-1}{2}$	(j)
	$\frac{N-1}{2}$	$\frac{N+1}{2}$	(j + 1)
	-1		(j + 2)

P_j^n ,

and

$$A_N^- P_j^n = \frac{1}{2(N-1)}$$

-1		
$\frac{N-1}{2}$	$\frac{N+1}{2}$	
$\frac{N+1}{2}$	$\frac{N-1}{2}$	
	-1	

P_j^n .

We note here also that

$$\lim_{N\to\infty} A_N^- = \lim_{N\to\infty} A_N^+ = \frac{1}{4}$$

(n)	(n+1)	
1	1	(j)
1	1	(j+1)

$\equiv A.$

The Crank-Nicolson approximation to (2.1-1) may then be written as

$$D_+^z D_+^t P_{j,k}^n = \pm \frac{v}{2} A D_+^x D_-^x P_{j,k}^n. \qquad (2.2\text{-}1)$$

This is an implicit scheme with local truncation error which is $O(\Delta x^2 + \Delta z^2 + \Delta t^2)$ and is unconditionally stable for evolution in the t or z direction. It is a standard method used for integration of the wave equation and is described in some detail by Claerbout [2].

In Part 1 we approximated Eq. (2.1-1b) with a family of schemes that we rewrite here as

$$D_+^z\, D_+^t\, P_{j,k}^n = -\frac{v}{2} A_N^- D_+^x D_-^x P_{j,k}^n. \tag{2.2-2}$$

These were suggested as methods for improving the accuracy of the approximation in the t-direction. The schemes

$$D_+^z\, D_+^t\, P_{j,k}^n = \frac{v}{2} A_N^+ D_+^x D_-^x P_{j,k}^n \tag{2.2-3}$$

are the corresponding approximations to Eq. (2.1-1a). Schemes (2.2-2) and (2.2-3) are also implicit in x with local truncation error which is $O(\Delta x^2 + \Delta z^2 + \Delta z\, \Delta t + \Delta t^2)$ in general, although the schemes are $O(\Delta x^2 + \Delta z^2 + \Delta z\, \Delta t + \Delta t^4)$ if $N = 13$. The schemes are unconditionally stable for all $N \geq 2$.

The initial conditions we use are

$$P_{j,k}^0 = P_0(t_j, x_k), \tag{2.2-4}$$

and the boundary conditions are

$$P_{j,0}^n = P_{j,1}^n, \quad P_{j,K-1}^n = P_{j,K}^n$$

and

$$P_{0,k}^{n+1} = P_{1,k}^{n+1} = P_{J,k}^n = 0.$$

(We need three conditions at the t-boundaries because of the averaging operator $A_N^{\pm}$.)

2.3 *Stability of the Difference Approximations*

We investigate the stability of the difference approximation (2.2-2) by first Fourier-transforming the difference equation with respect to x and t and considering the stability of the resulting initial-value problem in z. By *stability* we mean that the L_2-norm of the solution does not exhibit unbounded growth. We may express this mathematically by the following inequality:

$$\|P^{n+1}\| \leq K_0 \|P^0\|, \qquad \text{for } z_n \leq z_{max}, \tag{2.3-1}$$

where K_0 is some finite constant and the L_2-norm is defined by

$$\|P^n\|^2 = \frac{\Delta x\, \Delta t}{t_{max}} \sum_j \sum_k (P^n_{j,k})^2.$$

Once z-stability has been shown, we Fourier-transform the original equation (2.2-2) with respect to x only and look at the last z-level as an initial-value problem in t. If this is also stable, then we are assured that the difference approximation (2.2-2) is also stable.*

We first transform the individual operators:

$$(\hat{D^t_+ P^n_j}) = \frac{1}{\Delta t}(e^{i\omega\Delta t} - 1)\hat{P}^n,$$

$$(\hat{A^-_N P^n_j}) = \frac{1}{2(N-1)}\left[\left(e^{-i\omega\Delta t} + \frac{N-1}{2} + \frac{N+1}{2}e^{i\omega\Delta t}\right)\hat{P}^n + \left(\frac{N+1}{2} + \frac{N-1}{2}e^{i\omega\Delta t} - e^{2i\omega\Delta t}\right)\hat{P}^{n+1}\right],$$

and

$$(\hat{D^x_+ D^x_- P^n_k}) = \frac{-4}{\Delta x^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)\hat{P}^n. \qquad (2.3\text{-}2)$$

In order to simplify the notation we will let $Z = e^{i\omega\Delta t}$ and

$$b = \frac{v\Delta t\Delta z}{8\Delta x^2}\sin^2(k_x\Delta x/2).$$

We will also refer to the step-length parameter, $a = v\Delta t\Delta z/8\Delta x^2$. Making these substitutions, the Fourier-transformed difference equation becomes

$$\left\{-\left[1 + 4\left(\frac{N+1}{N-1}\right)b\right] + (1-4b)Z + \frac{8b}{(N-1)}Z^2\right\}\hat{P}^{n+1} = -Z^{-1}\left\{\frac{8b}{(N-1)} + (1-4b)Z - \left[1+4\left(\frac{N+1}{N-1}\right)b\right]Z^2\right\}\hat{P}^n. \qquad (2.3\text{-}3)$$

By letting

$$A(Z) = \frac{8b}{(N-1)}Z^2 + (1-4b)Z - \left[1+4\left(\frac{N+1}{N-1}\right)b\right],$$

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*We do not consider the stability of the boundary conditions in x since this is usually not a problem. The reader is referred to reference [5] for a discussion of boundary condition stability.

we can rewrite (2.3-3) as

$$P^{n+1} = -\frac{Z\,A(1/Z)}{A(Z)}\,\hat{P}^{n}. \tag{2.3-4}$$

We recognize that $Z\,A(1/Z)/A(Z)$ is an all-pass filter (see [2], p. 40), which means that its magnitude is unity for all $\omega\,\Delta t$. Hence we have

$$|\hat{P}^{n+1}| = \left|\frac{-Z\,A(1/Z)}{A(Z)}\right| |\hat{P}^{n}| = |\hat{P}^{n}|,$$

and using Parseval's relation,

$$\|P^{n+1}\| = \|P^{n}\|. \tag{2.3-5}$$

An estimate of the form (2.3-1) follows immediately with $K_0 = 1$.

For stability in t, we transform (2.2-2) with respect to x, which gives us at the (n+1)-level,

$$\hat{P}_{j+1} - \hat{P}_{j} = \frac{8b}{(N-1)}\left[\frac{(N+1)}{2}\,P_{j} + \frac{(N-1)}{2}\,\hat{P}_{j+1} - \hat{P}_{j+2}\right]. \tag{2.3-6}$$

This ordinary difference equation is solved by assuming a solution of the form $P_j = Z^{J-j}$, whereby the characteristic polynomial equation

$$\frac{8b}{N-1} + (1-b)Z - \left[1+\left(\frac{N+1}{N-1}\right)b\right]Z^{2} = 0 \tag{2.3-7}$$

is obtained. For the solution of this equation to be bounded as we evolve in the -t-direction, we must have an estimate of the form

$$|\hat{P}_{j-1}| \leq |\hat{P}_{j}|, \tag{2.3-8}$$

which implies that $|Z| \leq 1$ is required for stability, i.e., the roots of the characteristic polynomial must lie within the unit circle. If we recognize that the characteristic polynomial is identical to $Z^{-2}A(1/Z)$, using the notation from above, we see that the requirement of roots within the unit circle is the same as requiring that $A(Z)$ be *minimum phase* (see Claerbout [1], ch. 2).

To check that the roots of (2.3-7) lie within the unit circle, we can solve the quadratic equation directly for its two roots. We find that both roots of (2.3-7) lie within the unit circle for all $N \geq 2$ and all values of $b \geq 0$. Hence the difference approximation is *unconditionally stable* for evolution in the $-t$ direction. Clearly we may not evolve the difference equation in the $+t$ direction at the n+1 level, since (2.3-8) implies that the solution could increase without bound.

We can also show why the same averaging operator cannot be used for both the upward and downward equations. Suppose that instead of using (2.2-2) to approximate the migration equation, we tried the approximation

$$D_+^z D_+^t P_{j,k}^n = -\frac{v}{2} \frac{1}{2(N-1)} \begin{array}{|c|c|} \hline & -1 \\ \hline \frac{N+1}{2} & \frac{N-1}{2} \\ \hline \frac{N-1}{2} & \frac{N+1}{2} \\ \hline -1 & \\ \hline \end{array} D_+^x D_-^x P_{j,k}^n \qquad (2.3\text{-}9)$$

instead. We find that the z-stability would still hold, but that the resulting characteristic polynomial for the t-stability would have one root inside and one root outside for all values of N and b. This implies that the approximation is unconditionally unstable.

We can briefly summarize what we have said so far to give sufficient conditions for stability of an approximation to Eq. (2.1-1):

THEOREM (2.3-1): If the difference approximation to (2.1-1) is Fourier-transformed with respect to x and t, and the resulting one-step ordinary equation is written in the form

$$\hat{P}^{n+1} = \frac{B(Z)}{C(Z)} \hat{P}^n ,$$

where $Z = e^{i\omega\Delta t}$, then a sufficient condition for stability of the difference approximation is that $B(Z)/C(Z)$ be *all-pass* and $C(Z)$ be *minimum phase*.

2.4 *Accuracy of the Difference Approximations*

In this section we will discuss the accuracy with which the difference approximation treats information at different frequencies in the initial data. For simple initial data, we might ask how many points per wavelength we need to sample to obtain a prescribed accuracy, but when the data contain information over a large bandwidth of frequencies, we wish to look for a method that will give the best overall accuracy over the bandwidth of interest.

It has been a common mistake in the past to confuse the *order* or *local truncation error* of the difference approximation with the *accuracy* of the difference approximation. We must emphasize that the local truncation error tells only how well the approximation does in the limit of infinite sampling density. We can calculate the local truncation error T of (2.2-3) by writing it as a Taylor series expansion about the midpoint of the approximation $(x_k, z_{n+1/2}, t_{j+1/2})$. This can be written as

$$P_{zt} - \frac{v}{2}P_{xx} = T$$

$$= -\frac{\Delta t^2}{12}P_{zzzt} - \frac{\Delta x^2}{12}P_{xxtz} - \frac{4\Delta z\Delta t}{N-1}P_{xxxx} + \frac{(13-N)}{12(N-1)}\Delta t^2 P_{tttz}$$

$$+ O(\Delta z^4 + \Delta z^3\Delta t + \Delta z^2\Delta t^2 + \Delta z\Delta t^3 + \Delta t^4). \qquad (2.4\text{-}1)$$

It is common to simply say that the approximation is $O(\Delta x^2 + \Delta z^2 + \Delta z\Delta t + \Delta t^2)$ without specifying the actual constants. We remark here that if $N = 13$, the Δt^2 term vanishes and the approximation becomes $O(\Delta x^2 + \Delta z^2 + \Delta z\Delta t + \Delta t^4)$. When $N \to \infty$, the scheme converges to the Crank-Nicolson scheme and is then $O(\Delta x^2 + \Delta z^2 + \Delta t^2)$. Since the local truncation error is calculated by means of a Taylor series expansion, it clearly holds only in the limit of zero step size. We saw in Part 1 that, in fact, the accuracy of the $O(\Delta t^4)$ scheme is actually worse than some of the $O(\Delta t^2)$ schemes when the step size becomes large (see Fig. 1.3.1).

To consider the accuracy as a function of sampling density we follow the method used by Kreiss and Oliger [4] in the discussion of hyperbolic equations. We will concentrate here on the error that results from the discretization in t when we propagate a plane wave in the z-direction. To do this we write our difference equation using difference operators in the t-direction,

but keeping the differential operators in the x- and z-directions. We consider the differential equation

$$P_{zt} = P_{xx}. \tag{2.4-2}$$

The difference approximation corresponding to Eq. (2.2-3) is then

$$D_+^t P_z^j = \frac{1}{2(N-1)} \left(- P_{xx}^{j-1} + N P_{xx}^{j} + N P_{xx}^{j+1} - P_{xx}^{j+2}\right) . \tag{2.4-3}$$

If we use plane-wave initial conditions

$$P(x,z=0,t) = e^{i\omega(t+\alpha x)}, \tag{2.4-4}$$

then the solution to (2.4-2) becomes

$$P(x,z,t) = e^{i\omega(t+\alpha x+\alpha^2 z)}. \tag{2.4-5}$$

As we evolve in the z-direction using the difference approximation (2.4-3), we would then expect to make some *phase error* in z due to the discretization in t. This manifests itself in the form of dispersion. We write the solution to (2.4-3) as

$$P^j(x,z) = e^{i\omega[j\Delta t+\alpha x+\alpha_N^2(\omega)z]}. \tag{2.4-6}$$

The phase error is then given by

$$e_N(\omega) = \omega\left|\alpha^2 - \alpha_N^2(\omega)\right| , \tag{2.4-7}$$

which we might rewrite as

$$e_N(\omega) = \left|k_z - \hat{k}_z(\omega)\right| \tag{2.4-7a}$$

using the notation of Part 1.

We can calculate $\alpha_N^2(\omega)$ by substituting (2.4-6) into the difference approximation (2.4-3). We find that

$$D_+^t = \frac{1}{\Delta t}(e^{i\omega\Delta t} - 1) ,$$

$$\partial_z = i\omega\alpha_N^2(\omega),$$

$$\partial_{xx} = -\omega^2\alpha^2 ,$$

and the averaging operator becomes

$$A_N^+ = \frac{1}{2(N-1)}(-e^{-i\omega\Delta t} + N + N e^{i\omega\Delta t} - e^{2i\omega\Delta t}).$$

Substituting and cancelling P^j, we get

$$\alpha_N^2(\omega) = \frac{\omega\Delta t\,\alpha^2}{2(N-1)} \frac{N\cos(\omega\Delta t/2) - \cos(3\omega\Delta t/2)}{\sin(\omega\Delta t/2)} . \qquad (2.4\text{-}8)$$

The phase error per wavelength in the z-direction is then given by

$$e_N(\omega\Delta t) = \left| 1 - \frac{(\omega\Delta t/2)}{(N-1)} \frac{N\cos(\omega\Delta t/2) - \cos(3\omega\Delta t/2)}{\sin(\omega\Delta t/2)} \right| . \qquad (2.4\text{-}9)$$

We saw this function plotted for several values of N in Fig. 1.3.1. Note in particular that at $\omega\Delta t = 0.5\pi$ which corresponds to sampling in the t-direction at four points per wavelength, we predict that the methods with $N = 7$, $N = 9$, and $N = 11$, which are $O(\Delta t^2)$ make *less* error than the $O(\Delta t^4)$ method ($N = 13$).

References

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