Chapter 1

Fundamentals of data regularization

In this chapter, I develop a general theoretical framework for addressing the data regularization problem. The problem fundamentals are traced back to statistical estimation theory. Following Claerbout (1992, 1999), I formulate data regularization as a simple linear-estimation problem.

STATISTICAL ESTIMATION

Let $\mathbf{d}$ be the vector of observed data, and $\mathbf{m}$ be the ideal underlying model. The regularized data represent the model estimate $<\mathbf{m}>$. Taking into account the lack of information about $\mathbf{m}$, we can treat both $\mathbf{m}$ and $\mathbf{d}$ as random vectors and approach the problem of finding $<\mathbf{m}>$ statistically.

For any two random vectors $\mathbf{x}$ and $\mathbf{y}$, let us denote by $\mathbf{C}_{xy}$ the mathematical expectation of the random matrix $\mathbf{x}\,\mathbf{y}^T$, where $\mathbf{y}^T$ denotes the adjoint of y. Analogously, $\mathbf{C}_x$ will denote the mathematical expectation of $\mathbf{x}\,\mathbf{x}^T$. For zero-mean vectors, the matrices $\mathbf{C}_x$ and $\mathbf{C}_{xy}$ correspond to covariances. In a more general case, they are second-moment statistics of the corresponding random processes.

Applying the Gauss-Markoff theorem, one can obtain an explicit form of the estimate $<\mathbf{m}>$ under three very general assumptions (Liebelt, 1967):

1. The estimate has a linear relationship with the input data:

$$<\mathbf{m}> = \mathbf{A}\,\mathbf{d}\,, \tag{1.1}$$

   where $\mathbf{A}$ is a linear operator.

2. The estimate corresponds to the minimum of $\mathbf{C}_e = E\left(\mathbf{e}\,\mathbf{e}^T\right)$, where E is the mathematical expectation and $\mathbf{e}$ denotes the model error $\mathbf{e} = <\mathbf{m}> - \mathbf{m}$. For unbiased estimates (zero mathematical expectation of $\mathbf{e}$), the matrix $\mathbf{C}_e$ corresponds to the model error covariance. Although we do not make any explicit assumptions about the statistical distribution of the error, minimizing $\mathbf{C}_e$ is particularly meaningful in case of normal (Gaussian) distributions (Tarantola, 1987).

3. The square matrix $\mathbf{C}_d$ is invertible.

Doing a simple algebraic transformation, we find that

$$\begin{aligned}
\mathbf{C}_e &= E\left[(<\mathbf{m}> - \mathbf{m})(<\mathbf{m}> - \mathbf{m})^T\right] = E\left[(\mathbf{A}\,\mathbf{d} - \mathbf{m})\left(\mathbf{d}^T\,\mathbf{A}^T - \mathbf{m}^T\right)\right] = \\
& \mathbf{A}\,\mathbf{C}_d\,\mathbf{A}^T - \mathbf{C}_{md}\,\mathbf{A}^T - \mathbf{A}\,\mathbf{C}_{md}^T + \mathbf{C}_m = \\
& \left(\mathbf{A} - \mathbf{C}_{md}\,\mathbf{C}_d^{-1}\right)\mathbf{C}_d\left(\mathbf{A} - \mathbf{C}_{md}\,\mathbf{C}_d^{-1}\right)^T - \mathbf{C}_{md}\,\mathbf{C}_d^{-1}\,\mathbf{C}_{md} + \mathbf{C}_m\,.
\end{aligned} \tag{1.2}$$

It is evident from equation (1.2) that C_e will be minimized when $\mathbf{A} = \mathbf{C}_{md}\,\mathbf{C}_d^{-1}$. This leads immediately to the Gauss-Markoff result

$$<\mathbf{m}> = \mathbf{C}_{md}\,\mathbf{C}_d^{-1}\,\mathbf{d}\,. \tag{1.3}$$

Equation (1.3) has fundamental importance in different data regularization schemes. With some slight modifications, it appears as the basis for such methods as optimal interpolation in atmospheric data analysis (Gandin, 1965; Daley, 1991), least-squares collocation in geodesy (Moritz, 1980), and linear kriging in petroleum and mining engineering (Journel and Huijbregts, 1978; Hohn, 1999). In order to apply formula (1.3) in practice, one needs first to get an estimate of the matrices $\mathbf{C}_{md}$ and $\mathbf{C}_d$. In geostatistics, the covariance matrices are usually chosen from simple variogram models (Deutsch and Journel, 1997).

Unfortunately, a straightforward application of the Gauss-Markoff formula (1.3) is computationally unaffordable for typical seismic data applications. If the data vector contains N parameters, a straightforward application will lead to an N by N matrix inversion, which requires storage proportional to N^2 and a number of operations proportional to N^3. Although the data can be divided into local patches to reduce the computational requirements for an individual patch, the total computational complexity is still too high to be affordable for the values of N typical in 3-D seismic exploration (N as high as 10^{10}).

We can take two major theoretical steps to reduce the computational complexity of the method. The first step is to approximate the covariance matrices with sparse operators so that the matrix multiplication is reduced from N^2 operations to something linear in N. The second step is to approach model estimation as an optimization problem and to use an iterative method for solving it. The goal is to obtain a reasonable model estimate after only a small number of iterations.

REPRESENTING COVARIANCE MATRICES BY SPARSE OPERATORS

In order to understand the structure of the matrices $\mathbf{C}_{md}$ and $\mathbf{C}_d$, we need to make some assumptions about the relationship between the true model $\mathbf{m}$ and the data $\mathbf{d}$. A natural assumption is that if the model were known exactly, the observed data would be related to it by a *forward interpolation operator* $\mathbf{L}$ as follows:

$$\mathbf{d} = \mathbf{L}\,\mathbf{m} + \mathbf{n}\,, \tag{1.4}$$

where $\mathbf{n}$ is an additive observational noise. For simplicity, we can assume that the noise is uncorrelated and normally distributed around zero:

$$\mathbf{C}_{mn} = 0\,; \quad \mathbf{C}_n = \sigma_n^2\,\mathbf{I}\,, \tag{1.5}$$

where $\mathbf{I}$ is an identity matrix of the data size, and σ_n is a scalar. Assuming that there is no linear correlation between the noise and the model, we arrive at the following expressions for the second moment matrices in formula (1.3):

$$\mathbf{C}_d = E\left[(\mathbf{L}\,\mathbf{m}+\mathbf{n})\left(\mathbf{m}^T\,\mathbf{L}^T+\mathbf{n}^T\right)\right] = \mathbf{L}\,\mathbf{C}_m\,\mathbf{L}^T+\sigma_n^2\,\mathbf{I}\,, \tag{1.6}$$

$$\mathbf{C}_{md} = E\left[\mathbf{m}\left(\mathbf{m}^T\,\mathbf{L}^T+\mathbf{n}^T\right)\right] = \mathbf{C}_m\,\mathbf{L}^T\;. \tag{1.7}$$

Substituting equations (1.6) and (1.7) into (1.3), we finally obtain the following specialized form of the Gauss-Markoff formula:

$$<\mathbf{m}> = \mathbf{C}_m\,\mathbf{L}^T\,\left(\mathbf{L}\,\mathbf{C}_m\,\mathbf{L}^T+\sigma_n^2\,\mathbf{I}\right)^{-1}\,\mathbf{d}\,. \tag{1.8}$$

Assuming that $\mathbf{C}_m$ is invertible, we can also rewrite equation (1.8) in a mathematically equivalent form

$$<\mathbf{m}> = \left(\mathbf{L}\,\mathbf{L}^T+\sigma_n^2\,\mathbf{C}_m^{-1}\right)^{-1}\,\mathbf{L}^T\,\mathbf{d}\,. \tag{1.9}$$

The equivalence of formulas (1.8) and (1.9) follows from the simple matrix equality

$$\mathbf{C}_m\mathbf{L}^T(\mathbf{L}\mathbf{C}_m\mathbf{L}^T+\sigma_n^2\mathbf{I})^{-1} \equiv (\mathbf{L}^T\mathbf{L}+\sigma_n^2\mathbf{C}_m^{-1})^{-1}\mathbf{L}^T\;. \tag{1.10}$$

It is important to note an important difference between equations (1.8) and (1.9): The inverted matrix has data dimensions in the first case, and model dimensions in the second case. I discuss the practical significance of this distinction in Chapter 4.

In order to simplify the model estimation problem further, we can introduce a local differential operator $\mathbf{D}$. A model $\mathbf{m}$ complies with the operator $\mathbf{D}$ if the residual after we apply this operator $\mathbf{r} = \mathbf{D}\,\mathbf{m}$ is uncorrelated and normally distributed. This means that

$$E\left[\mathbf{D}\,\mathbf{m}\,\mathbf{m}^T\,\mathbf{D}^T\right] = \mathbf{D}\,\mathbf{C}_m\,\mathbf{D}^T = \sigma_m^2\,\mathbf{I}\,, \tag{1.11}$$

where the identity matrix $\mathbf{I}$ has the model size. Furthermore, assuming that $\mathbf{D}$ is invertible, we can represent $\mathbf{C}_m$ as follows:

$$\mathbf{C}_m = \sigma_m^2 \left(\mathbf{D}^T \mathbf{D}\right)^{-1} . \tag{1.12}$$

Substituting formula (1.12) into (1.8) and (1.9), we can finally represent the model estimate in the following equivalent forms:

$$<\mathbf{m}> \;=\; \mathbf{P}\mathbf{P}^T \mathbf{L}^T \left(\mathbf{L}\mathbf{P}\mathbf{P}^T \mathbf{L}^T + \epsilon^2 \mathbf{I}\right)^{-1} \mathbf{d} \, ; \tag{1.13}$$

$$<\mathbf{m}> \;=\; \left(\mathbf{L}\mathbf{L}^T + \epsilon^2 \mathbf{D}^T \mathbf{D}\right)^{-1} \mathbf{L}^T \mathbf{d} \, , \tag{1.14}$$

where $\mathbf{P}\mathbf{P}^T = \left(\mathbf{D}^T \mathbf{D}\right)^{-1}$ and $\epsilon = \frac{\sigma_n}{\sigma_m}$.

The first simplification step has now been accomplished. By introducing additional assumptions, we have approximated the covariance matrices $\mathbf{C}_d$ and $\mathbf{C}_{md}$ with the forward interpolation operator $\mathbf{L}$ and the differential operator $\mathbf{D}$. Both $\mathbf{L}$ and $\mathbf{D}$ act locally on the model. Therefore, they are sparse, efficiently computed operators. Different examples of operators $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{P}$ are discussed later in this dissertation. In the next section, I proceed to the second simplification step.

DATA REGULARIZATION AS AN OPTIMIZATION PROBLEM

The Gauss-Markoff equation (1.3) is derived as a solution of an optimization problem – minimizing the model error covariance matrix. After simplifying this equation to the forms (1.13) and (1.14), we can again recast it as a solution to an optimization problem of a different kind. In fact, equations (1.13) and (1.14) correspond to two fundamentally different optimization formulations.

Model-space regularization

Model-space regularization implies adding equations to system

$$\mathbf{L}\mathbf{m} \approx \mathbf{d} \tag{1.15}$$

to obtain a fully constrained (well-posed) inverse problem. The additional equations take the form

$$\epsilon \mathbf{D} \mathbf{m} \approx \mathbf{0} \,. \tag{1.16}$$

The full system of equations (1.15)-(1.16) can be written in a short notation as

$$\mathbf{G_m} \mathbf{m} = \left[\begin{array}{c} \mathbf{L} \\ \epsilon \mathbf{D} \end{array} \right] \mathbf{m} \approx \left[\begin{array}{c} \mathbf{d} \\ \mathbf{0} \end{array} \right] = \hat{\mathbf{d}} \,, \tag{1.17}$$

where $\hat{\mathbf{d}}$ is the effective data vector:

$$\hat{\mathbf{d}} = \left[\begin{array}{c} \mathbf{d} \\ \mathbf{0} \end{array} \right] \,, \tag{1.18}$$

and $\mathbf{G_m}$ is a *column* operator:

$$\mathbf{G_m} = \left[\begin{array}{c} \mathbf{L} \\ \epsilon \mathbf{D} \end{array} \right] \,. \tag{1.19}$$

The estimation problem (1.17) is fully constrained. We can solve it by means of unconstrained least-squares optimization, minimizing the squared power $\hat{\mathbf{r}}^T \hat{\mathbf{r}}$ of the compound residual vector

$$\hat{\mathbf{r}} = \hat{\mathbf{d}} - \mathbf{G_m} \mathbf{m} = \left[\begin{array}{c} \mathbf{d} - \mathbf{L} \mathbf{m} \\ -\epsilon \mathbf{D} \mathbf{m} \end{array} \right] \,. \tag{1.20}$$

The formal solution of the regularized optimization problem has a known form, which coincides with formula (1.14). One can carry out the optimization iteratively with the help of the conjugate-gradient method (Hestenes and Steifel, 1952) or its analogs (Paige and Saunders, 1982).

The next subsection introduces an alternative formulation of the optimization problem.

Data-space regularization (model preconditioning)

The data-space regularization approach is closely related to the concept of *model preconditioning* (Nichols, 1994). Regarding the operator $\mathbf{P}$ from equation (1.13) as a preconditioning operator, we can introduce a new model $\mathbf{p}$ with the equality

$$\mathbf{m} = \mathbf{P}\mathbf{p}\ . \tag{1.21}$$

The residual vector $\mathbf{r}$ for the data-fitting equation (1.4) can be defined by the relationship

$$\epsilon \mathbf{r} = \mathbf{d} - \mathbf{L}\mathbf{m} = \mathbf{d} - \mathbf{L}\mathbf{P}\mathbf{p}\ , \tag{1.22}$$

where ϵ is the scaling parameter from equation (1.13). Let us consider a compound model $\hat{\mathbf{p}}$, composed of the preconditioned model vector $\mathbf{p}$ and the residual $\mathbf{r}$. With respect to the compound model, we can rewrite equation (1.22) as

$$\begin{bmatrix} \mathbf{L}\mathbf{P} & \epsilon \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{r} \end{bmatrix} = \mathbf{G_d}\hat{\mathbf{p}} = \mathbf{d}\ , \tag{1.23}$$

where $\mathbf{G_d}$ is a *row* operator:

$$\mathbf{G_d} = \begin{bmatrix} \mathbf{L}\mathbf{P} & \epsilon \mathbf{I} \end{bmatrix}\ , \tag{1.24}$$

and $\mathbf{I}$ represents the data-space identity operator.

System (1.23) is clearly underdetermined with respect to the compound model $\hat{\mathbf{p}}$. If from all possible solutions of this system we seek the one with the minimal power $\hat{\mathbf{p}}^T\hat{\mathbf{p}}$, the formal (ideal) result takes the well-known form

$$<\hat{\mathbf{p}}> = \begin{bmatrix} <\mathbf{p}> \\ <\mathbf{r}> \end{bmatrix} = \mathbf{G_d}^T \left(\mathbf{G_d}\,\mathbf{G_d}^T\right)^{-1} \mathbf{d} = \begin{bmatrix} \mathbf{P}^T\mathbf{L}^T\left(\mathbf{L}\mathbf{P}\mathbf{P}^T\mathbf{L}^T + \epsilon^2\mathbf{I}\right)^{-1}\mathbf{d} \\ \epsilon\left(\mathbf{L}\mathbf{P}\mathbf{P}^T\mathbf{L}^T + \epsilon^2\mathbf{I}\right)^{-1}\mathbf{d} \end{bmatrix}\ . \tag{1.25}$$

Applying equation (1.21), we obtain the corresponding estimate $<\mathbf{m}>$ for the initial model $\mathbf{m}$, which is precisely equivalent to equation (1.13). This proves the legitimacy of the alternative

Table 1.1: Comparison between model-space and data-space regularization

Regularization	*Model-space*	*Data-space*
effective model	$\mathbf{m}$	$\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p} \\ \mathbf{r} \end{bmatrix}$
effective data	$\hat{\mathbf{d}} = \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix}$	$\mathbf{d}$
effective operator	$\mathbf{G_m} = \begin{bmatrix} \mathbf{L} \\ \epsilon \mathbf{D} \end{bmatrix}$	$\mathbf{G_d} = \begin{bmatrix} \mathbf{LP} & \epsilon \mathbf{I} \end{bmatrix}$
optimization problem	minimize $\hat{\mathbf{r}}^T \hat{\mathbf{r}}$, where $\hat{\mathbf{r}} = \hat{\mathbf{d}} - \mathbf{G_m m}$	minimize $\hat{\mathbf{p}}^T \hat{\mathbf{p}}$ under the constraint $\mathbf{G_d} \hat{\mathbf{p}} = \mathbf{d}$
formal estimate for $\mathbf{m}$	$\left(\mathbf{L}^T \mathbf{L} + \epsilon^2 \mathbf{C}^{-1}\right) \mathbf{L}^T \mathbf{d}$, where $\mathbf{C}^{-1} = \mathbf{D}^T \mathbf{D}$	$\mathbf{CL}^T (\mathbf{LCL}^T + \epsilon^2 \mathbf{I})^{-1} \mathbf{d}$, where $\mathbf{C} = \mathbf{PP}^T$.

data-space approach to data regularization: the model estimation is reduced to least-square minimization of the specially constructed compound model $\hat{\mathbf{p}}$ under the constraint (1.22).

I summarize the differences between model-space and data-space regularization in Table 1.1.

Although the two approaches lead to similar theoretical results, they behave quite differently in the process of iterative optimization. In Chapter 4, I illustrate this fact with many examples and show that in the case of incomplete optimization, the second (preconditioning) approach is generally preferable.

The next chapter addresses the choice of the forward interpolation operator $\mathbf{L}$ – the necessary ingredient of the iterative data regularization algorithms.

ACKNOWLEDGMENTS

I am grateful to Jim Berryman, Bill Harlan, Dave Nichols, Gennady Ryzhikov, and Bill Symes for insightful discussions about preconditioning and optimization problems.

Bibliography

Babich, V. M., 1991, Short-wavelength diffraction theory: asymptotic methods: Springer-Verlag, Berlin; New York.

Bagaini, C., and Spagnolini, U., 1993, Common shot velocity analysis by shot continuation operator: 63rd Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 673–676.

Bagaini, C., and Spagnolini, U., 1996, 2-D continuation operators and their applications: Geophysics, **61**, no. 06, 1846–1858.

Bagaini, C., Spagnolini, U., and Pazienza, V. P., 1994, Velocity analysis and missing offset restoration by prestack continuation operators: 64th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1549–1552.

Bale, R., and Jakubowicz, H., 1987, Post-stack prestack migration: 57th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, Session:S14.1.

Ben-Avraham, Z., Amit, G., Golan, A., and Begin, Z. B., 1990, The bathymetry of Lake Kinneret and its structural significance: Israel Journal of Earth Sciences, **39**, 77–84.

Beylkin, G., 1985, Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform: Journal of Mathematical Physics, **26**, 99–108.

Biondi, B., and Chemingui, N., 1994a, Transformation of 3-D prestack data by Azimuth Moveout: SEP–**80**, 125–143.

Biondi, B., and Chemingui, N., 1994b, Transformation of 3-D prestack data by azimuth moveout (AMO): 64th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1541–1544.

Biondi, B., and Palacharla, G., 1996, 3-D prestack migration of common-azimuth data: Geophysics, **61**, no. 6, 1822–1832.

Biondi, B., Fomel, S., and Chemingui, N., 1996, Azimuth moveout for 3-D prestack imaging: SEP–**93**, 15–44.

Biondi, B., Fomel, S., and Chemingui, N., 1998, Azimuth moveout for 3-D prestack imaging: Geophysics, **63**, no. 02, 574–588.

Biondi, B., 1996, Common-azimuth prestack depth migration of a North Sea data set: SEP–**93**, 1–14.

Black, J. L., Schleicher, K. L., and Zhang, L., 1993, True-amplitude imaging and dip moveout: Geophysics, **58**, no. 1, 47–66.

Bleistein, N., and Cohen, J. K., 1995, The effect of curvature on true amplitude DMO: Proof of concept: ACTI, 4731U0015-2F; CWP-193, Colorado School of Mines.

Bleistein, N., 1984, Mathematical methods for wave phenomena: Academic Press Inc. (Harcourt Brace Jovanovich Publishers), New York.

Bleistein, N., 1990, Born DMO revisited: 60th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1366–1369.

Blu, T., Thévenaz, P., and Unser, M., 1998, Minimum support interpolators with optimum approximation properties: Proc. IEEE Int. Conf. Image Processing, Chicago, IL, USA, October 4-7, 242–245.

Bolondi, G., Loinger, E., and Rocca, F., 1982, Offset continuation of seismic sections: Geophys. Prosp., **30**, no. 6, 813–828.

Briggs, I. C., 1974, Machine contouring using minimum curvature: Geophysics, **39**, no. 1, 39–48.

Brown, M., and Claerbout, J., 2000a, Ground roll and the Radial Trace Transform - revisited: SEP–**103**, 219–237.

Brown, M., and Claerbout, J., 2000b, A pseudo-unitary implementation of the radial trace transform: 70th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 2115–2118.

Brown, M., Clapp, R. G., and Marfurt, K., 1999, Predictive signal/noise separation of groundroll-contaminated data: SEP–**102**, 111–128.

Burg, J. P., 1972, The relationship between maximum entropy spectra and maximum likelihood spectra (short note): Geophysics, **37**, no. 2, 375–376.

Burg, J. P., 1975, Maximum entropy spectral analysis: Ph.D. thesis, Stanford University.

Canales, L. L., 1984, Random noise reduction: 54th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, Session:S10.1.

Canning, A., and Gardner, G. H. F., 1996, Regularizing 3-D data sets with DMO: Geophysics, **61**, no. 04, 1103–1114.

Chemingui, N., and Biondi, B., 1994, Coherent partial stacking by offset continuation of 2-D prestack data: SEP–**82**, 117–126.

Chemingui, N., and Biondi, B., 1996, Handling the irregular geometry in wide azimuth surveys: 66th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 32–35.

Chemingui, N., 1999, Imaging irregularly sampled 3D prestacked data: Ph.D. thesis, Stanford University.

Claerbout, J. F., 1976, Fundamentals of geophysical data processing: Blackwell.

Claerbout, J. F., 1983, Ground roll and radial traces: SEP–**35**, 43–54.

Claerbout, J. F., 1985, Imaging the Earth's Interior: Blackwell Scientific Publications.

Claerbout, J. F., 1992, Earth Soundings Analysis: Processing Versus Inversion: Blackwell Scientific Publications.

Claerbout, J. F., 1993, 3-D local monoplane annihilator: SEP–**77**, 19–25.

Claerbout, J., 1997, Multidimensional recursive filters via a helix: SEP–**95**, 1–13.

Claerbout, J., 1998a, Multidimensional recursive filters via a helix: Geophysics, **63**, no. 5, 1532–1541.

Claerbout, J., 1998b, Multidimensional recursive filters via a helix: SEP–**97**, 319–335.

Claerbout, J. F., 1998c, Multidimensional recursive filters via a helix with application to velocity estimation and 3-D migration: 68th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1995–1998.

Claerbout, J., 1999, Geophysical estimation by example: Environmental soundings image enhancement: Stanford Exploration Project, http://sepwww.stanford.edu/sep/prof/.

Clapp, R. G., and Biondi, B. L., 1998, Regularizing time tomography with steering filters: SEP–**97**, 137–146.

Clapp, R. G., and Biondi, B. L., 2000, Tau tomography with steering filters: 2-D field data example: SEP–**103**, 1–19.

Clapp, R. G., and Brown, M., 2000, $(t-x)$ domain, pattern-based multiple separation: SEP–**103**, 201–210.

Clapp, R. G., Fomel, S., and Claerbout, J., 1997, Solution steering with space-variant filters: SEP–**95**, 27–42.

Clapp, R., Biondi, B., Fomel, S., and Claerbout, J., 1998, Regularizing velocity estimation using geologic dip information: 68th Ann. Internat. Meeting, Soc. Expl. Geophys., Expanded Abstracts, 1851–1854.

Clapp, R. G., Fomel, S., Crawley, S., and Claerbout, J. F., 1999, Directional smoothing of non-stationary filters: SEP–**100**, 197–209.

Clapp, R., 2000a, 3-D steering filters: SEP–**105**, 109–116.

Clapp, R., 2000b, Multiple realizations using standard inversion techniques: SEP–**105**, 67–78.

Cole, S. P., 1995, Passive seismic and drill-bit experiments using 2-D arrays: Ph.D. thesis, Stanford University.

Courant, R., 1962, Methods of mathematical physics: Interscience Publishers, New York.

Crawley, S., Clapp, R., and Claerbout, J., 1998, Decon and interpolation with nonstationary filters: SEP–**97**, 183–192.

Crawley, S., Clapp, R., and Claerbout, J., 1999, Interpolation with smoothly nonstationary prediction-error filters: 69th Ann. Internat. Meeting, Soc. Expl. Geophys., Expanded Abstracts, 1154–1157.

Crawley, S., 1995a, An example of inverse interpolation accelerated by preconditioning: SEP–**84**, 289–294.

Crawley, S., 1995b, Multigrid nonlinear SeaBeam interpolation: SEP–**84**, 279–288.

Crawley, S., 1999, Interpolation with smoothly nonstationary prediction-error filters: SEP–**100**, 181–196.

Crawley, S., 2000, Seismic trace interpolation with nonstationary prediction-error filters: Ph.D. thesis, Stanford University.

Daley, R., 1991, Atmospheric data analysis: Cambridge University Press.

Daubechies, I., 1992, Ten lectures on wavelets: SIAM, Philadelphia, Pennsylvania.

de Boor, C., 1978, A practical guide to splines: Springer-Verlag.

Deregowski, S. M., and Rocca, F., 1981, Geometrical optics and wave theory of constant offset sections in layered media: Geophys. Prosp., **29**, no. 3, 374–406.

Deregowski, S. M., 1986, What is DMO: First Break, **4**, no. 7, 7–24.

Deutsch, C. V., and Journel, A. G., 1997, Gslib: geostatistical software library and user's guide: Oxford University Press.

Dubois, G., 1999, Spatial interpolation comparison 97: Foreword and introduction: Journal of Geographic Information and Decision Analysis, **2**, 1–10.

Fomel, S., and Biondi, B., 1995a, The time and space formulation of azimuth moveout: SEP–**84**, 25–38.

Fomel, S., and Biondi, B., 1995b, The time and space formulation of azimuth moveout: 65th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1449–1452.

Fomel, S., and Bleistein, N., 1996, Amplitude preservation for offset continuation: Confirmation for Kirchhoff data: SEP–**92**, 219–227.

Fomel, S., and Claerbout, J., 1995, Searching the Sea of Galilee: The splendors and miseries of iteratively reweighted least squares: SEP–**84**, 259–270.

Fomel, S., Bleistein, N., Jaramillo, H., and Cohen, J. K., 1996, True amplitude DMO, offset continuation and AVA/AVO for curved reflectors: 66th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1731–1734.

Fomel, S., Clapp, R., and Claerbout, J., 1997, Missing data interpolation by recursive filter preconditioning: SEP–**95**, 15–25.

Fomel, S. B., 1994, Kinematically equivalent differential operator for offset continuation of seismic sections: Russian Geology and Geophysics, **35**, no. 9, 122–134.

Fomel, S., 1995a, Amplitude preserving offset continuation in theory Part 1: The offset continuation equation: SEP–**84**, 179–198.

Fomel, S., 1995b, Amplitude preserving offset continuation in theory Part 2: Solving the equation: SEP–**89**, 109–132.

Fomel, S., 1996a, Least-square inversion with inexact adjoints. Method of conjugate directions: A tutorial: SEP–**92**, 253–265.

Fomel, S., 1996b, Stacking operators: Adjoint versus asymptotic inverse: SEP–**92**, 267–292.

Fomel, S., 2000a, Applications of plane-wave destructor filters: SEP–**105**, 1–26.

Fomel, S., 2000b, Seismic data interpolation with the offset continuation equation: SEP–**103**, 237–254.

Fung, Y. C., 1965, Foundations of solid mechanics: Prentice-Hall.

Gandin, L. S., 1965, Objective analysis of meteorological fields: Jerusalem, Israel Program for Scientific Translations.

Garcia, A. G., 2000, Orthogonal sampling formulas: A unified approach: SIAM Review, **42**, no. 3, 499–512.

Gazdag, J., 1978, Wave equation migration with the phase shift method: Geophysics, **43**, 1342–1351.

Goldin, S. V., and Fomel, S. B., 1995, Estimation of reflection coefficient in DMO: Russian Geology and Geophysics, **36**, no. 4, 103–115.

Goldin, S. V., 1988, Transformation and recovery of discontinuities in problems of tomographic type: Institute of Geology and Geophysics, Novosibirsk (in Russian).

Goldin, S., 1990, A geometric approach to seismic processing: the method of discontinuities: SEP–**67**, 171–210.

Goldin, S. V., 1994, Superposition and continuation of tranformations used in seismic migration: Russian Geology and Geophysics, **35**, no. 9, 131–145.

Golub, G. H., and Van Loan, C. F., 1996, Matrix computations: The John Hopkins University Press.

Gradshtein, I. S., and Ryzhik, I. M., 1994, Table of integrals, series, and products: Boston: Academic Press.

Haddon, R. A. W., and Buchen, P. W., 1981, Use of Kirchhoff's formula for body wave calculations in the earth: Geophys. J. Roy. Astr. Soc., **67**, 587–598.

Hale, I. D., 1980, Resampling irregularly sampled data: SEP–**25**, 39–58.

Hale, I. D., 1983, Dip moveout by Fourier transform: Ph.D. thesis, Stanford University.

Hale, D., 1984, Dip-moveout by Fourier transform: Geophysics, **49**, no. 6, 741–757.

Hale, D., 1991, Course notes: Dip moveout processing: Soc. Expl. Geophys.

Hale, D., Ed. **DMO processing**. Society Of Exploration Geophysicists, 1995.

Harlan, W. S., 1982, Avoiding interpolation artifacts in Stolt migration: SEP–**30**, 103–110.

Harlan, W. S., 1995, Regularization by model redefinition: http://sepwww.stanford.edu/oldsep/harlan/papers/regularization.ps.gz.

Henley, D. C., 1999, The radial trace transform: an effective domain for coherent noise attenuation and wavefield separation: 69th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1204–1207.

Henley, D., 2000, Wavefield separation and other useful applications in the radial trace domain: 70th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 2111–2114.

Hestenes, M. R., and Steifel, E., 1952, Methods of conjugate gradients for solving linear systems: J. Res. Nat. Bur. Stand., **49**, 409–436.

Hohn, M. E., 1999, Geostatistics and petroleum geology: Kluwer Academic Publishers.

Journel, A. G., and Huijbregts, C. J., 1978, Mining geostatistics: Academic Press.

Kaiser, J. F., and Shafer, R. W., 1980, On the use of the I0-Sinh window for spectrum analysis: IEEE Trans. Acoustics, Speech and Signal Processing, **ASSP-28(1)**, 105.

Karrenbach, M., 1995, Elastic tensor wave fields: Ph.D. thesis, Stanford University.

Keys, R. G., 1981, Cubic convolution interpolation for digital image processing: IEEE Trans. Acoust., Speech, Signal Process., **ASSP-29**, 1153–1160.

Kolmogoroff, A. N., 1939, Sur l'interpolation et extrapolation des suites stationnaires: C.R. Acad.Sci., **208**, 2043–2045.

Kotel'nikov, V. A., 1933, On the transmission capacity of "ether" and wire in electrocommunications: Izd. Red. Upr. Svyazi RKKA.

Liebelt, P. B., 1967, An introduction to optimal estimation: Addison-Wesley, Reading, Mass.

Lin, J., Teng, L., and Muir, F., 1993, Comparison of different interpolation methods for Stolt migration: SEP–**79**, 255–260.

Liner, C. L., and Cohen, J. K., 1988, An amplitude-preserving inverse of Hale's DMO: 58th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1117–1120.

Liner, C., 1990, General theory and comparative anatomy of dip moveout: Geophysics, **55**, no. 5, 595–607.

Liner, C. L., 1991, Born theory of wave-equation dip moveout: Geophysics, **56**, no. 2, 182–189.

Mazzucchelli, P., and Rocca, F., 1999, Regularizing land acquisitions using shot continuation operators: effects on amplitudes: 69th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1995–1998.

Moritz, H., 1980, Advanced physical geodesy: Abacus Press.

Nichols, D., 1994, A simple example of a null space and how to modify it: SEP–**82**, 177–182.

Nolet, G., Ed., 1987, Seismic tomography: with applications in global seismology and exploration geophysics D. Reidel.

Notfors, C. D., and Godfrey, R. J., 1987, Dip moveout in the frequency-wavenumber domain (short note): Geophysics, **52**, no. 12, 1718–1721.

Ottolini, R., 1982, Migration of reflection seismic data in angle-midpoint coordinates: Ph.D. thesis, Stanford University.

Paige, C. C., and Saunders, M. A., 1982, Algorithm 583, LSQR: Sparse linear equations and least squares problems: Assn. Comp. Mach. Trans. Mathematical Software, **8**, 195–209.

Petkovsek, M., Wilf, H. S., and Zeilberger, D., 1996, $A = B$: A K Peters Ltd., Wellesley, MA.

Popovici, A. M., Blondel, P., and Muir, F., 1993, Interpolation in Stolt migration: SEP–**79**, 261–264.

Popovici, A. M., Muir, F., and Blondel, P., 1996, Stolt redux: A new interpolation method: Journal of Seismic Exploration, **5**, no. 4, 341–347.

Rickett, J., and Claerbout, J., 1998, Helical factorization of the Helmholtz equation: SEP–**97**, 353–362.

Rickett, J., and Claerbout, J., 1999a, Acoustic daylight imaging via spectral factorization: Helioseismology and reservoir monitoring: SEP–**100**, 171–180.

Rickett, J., and Claerbout, J., 1999b, Acoustic daylight imaging via spectral factorization: Helioseismology and reservoir monitoring: 69th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1675–1678.

Ronen, S., Sorin, V., and Bale, R., 1991, Spatial dealiasing of 3-D seismic reflection data: Geophysical Journal International, pages 503–511.

Ronen, J., 1982, Stolt migration; interpolation artifacts: SEP–**30**, 95–102.

Ronen, J., 1987, Wave equation trace interpolation: Geophysics, **52**, no. 7, 973–984.

Ronen, S., 1994, Handling irregular geometry: Equalized DMO and beyond: 64th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1545–1548.

Salvador, L., and Savelli, S., 1982, Offset continuation for seismic stacking: Geophys. Prosp., **30**, no. 6, 829–849.

Samko, S. G., Kilbas, A. A., and Marichev, O. I., 1993, Fractional integrals and derivatives: theory and applications: Gordon and Breach Science Publishers.

Sandwell, D. T., 1987, Biharmonic spline interpolation of GEOS-3 and SEASAT altimeter data: Geophys. Res. Letters, **14**, no. 2, 139–142.

Santos, L. T., Schleicher, J., and Tygel, M., 1997, 2.5-D true-amplitude offset continuation: Journal of Seismic Exploration, **6**, no. 2-3, 103–116.

Sava, P., and Fomel, S., 1999, Spectral factorization revisited: SEP–**100**, 227–234.

Sava, P., Rickett, J., Fomel, S., and Claerbout, J., 1998, Wilson-Burg spectral factorization with application to helix filtering: SEP–**97**, 343–351.

Schwab, M., and Claerbout, J., 1995, The interpolation of a 3-D data set by a pair of 2-D filters: SEP–**84**, 271–278.

Schwab, M., 1993, Shot gather continuation: SEP–**77**, 117–130.

Schwab, M., 1998, Enhancement of discontinuities in seismic 3-D images using a Java estimation library: Ph.D. thesis, Stanford University.

Schweikert, D. G., 1966, An interpolation curve using a spline in tension: Journal of Mathematics and Physics, **45**, 312–313.

Shannon, C. E., 1949, Communication in the presense of noise: Proc. I.R.E., **37**, 10–21.

Smith, W. H. F., and Wessel, P., 1990, Gridding with continuous curvature splines in tension: Geophysics, **55**, no. 3, 293–305.

Spagnolini, U., and Opreni, S., 1996, 3-D shot continuation operator: 66th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 439–442.

Spitz, S., 1991, Seismic trace interpolation in the f-x domain: Geophysics, **56**, no. 6, 785–794.

Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, **43**, no. 1, 23–48.

Stovas, A. M., and Fomel, S. B., 1993, Kinematically equivalent DMO operators: Presented at the SEG-Moscow, SEG-Moscow.

Stovas, A. M., and Fomel, S. B., 1996, Kinematically equivalent integral DMO operators: Russian Geology and Geophysics, **37**, no. 2, 102–113.

Swain, C. J., 1976, A FORTRAN IV program for interpolating irregularly spaced data using the difference equations for minimum curvature: Computers and Geosciences, **1**, 231–240.

Symes, W. W., and Carazzone, J. J., 1991, Velocity inversion by differential semblance optimization: Geophysics, **56**, no. 5, 654–663.

Symes, W. W., 1999, All stationary points of differential semblance are asymptotic global minimizers: Layered acoustics: SEP–**100**, 71–92.

Tarantola, A., 1987, Inverse problem theory: Elsevier.

Tenenbaum, M., and Pollard, H., 1985, Ordinary differential equations : an elementary textbook for students of mathematics, engineering, and the sciences: Dover Publications.

Thévenaz, P., Blu, T., and Unser, M., 2000, *in* Handbook of Medical Image Processing , in press.

Tikhonov, A. N., and Arsenin, V. Y., 1977, Solution of ill-posed problems: John Wiley and Sons.

Timoshenko, S., and Woinowsky-Krieger, S., 1968, Theory of plates and shells: McGraw-Hill.

Unser, M., Aldroubi, A., and Eden, M., 1993, B-spline signal processing: Part I – Theory: IEEE Transactions on Signal Processing, **41**, 821–832.

Unser, M., 1999, Splines: a perfect fit for signal and image processing: IEEE Signal Processing Magazine, **16**, no. 6, 22–38.

Červený, V., Molotkov, I. A., and Pšenčik, I., 1977, Ray method in seismology: Univerzita Karlova, Praha.

Watson, G. N., 1952, A treatise on the theory of Bessel functions: Cambridge University Press, 2nd edition.

Wilson, G., 1969, Factorization of the covariance generating function of a pure moving average process: SIAM J. Numer. Anal., **6**, no. 1, 1–7.

Wolberg, G., 1990, Digital image warping: IEEE Computer Society Press.

Woodward, M. J., Farmer, P., Nichols, D., and Charles, S., 1998, Automated 3-D tomographic velocity analysis of residual moveout in prestack depth migrated common image point gathers: 68th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1218–1221.

Zhao, Y., 1999, Helix derivative and low-cut filters' spectral feature and application: SEP–**100**, 235–250.

Zhou, B., Mason, I. M., and Greenhalgh, S. A., 1996, An accurate formulation of log-stretch dip moveout in the frequency-wavenumber domain: Geophysics, **61**, no. 03, 815–820.