REVEALED BY SCATTERED WAVE
G

almost all branches of physics. We are going to study waves, but
assume knowledge of physics and differential equations. We will
tions about the general principles of delay, continuity, and energy
he results will be directly applicable to sound waves, water waves,
s, normal incident elastic waves of both pressure and shear type,
waves, transmission lines, electrical ladder networks, and other
e methods can also be applied to diffusion problems. Our first
s to solve the problem of calculating wave fields given reflection
r second main objective is to gain the ability to calculate the
ients given the observed waves.

TION AND TRANSMISSION COEFFICIENTS

lfspaces (the sky above, the earth below). If a wave of unit ampli-
onto the boundary, there will be a transmitted wave of amplitude t
vave of amplitude c as depicted in Fig. 8-1.

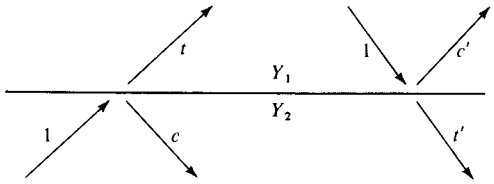


FIGURE 8-1
Waves incident, reflected c, and transmitted t at an interface.

A very simple relationship exists between t and c. The wave amplitudes have a physical meaning of something like pressure, material displacement, or tangential electric or magnetic fields; and these physical variables must be the same on either side of the boundary. Thus, we must have

$$t = 1 + c \qquad (8\text{-}1\text{-}1)$$

$$t' = 1 + c' \qquad (8\text{-}1\text{-}2)$$

It may be surprising that t may be greater than unity. However, this phenomenon may easily be seen at the ocean, where waves get larger as they approach the shore (until they break). Energy is not determined by wave height alone. Energy is equal to the squared wave amplitude multiplied by a proportionality factor Y depending upon the medium in which the wave is measured. If we denote the factor of the top medium by Y_1 and the bottom by Y_2, then the statement that the energy before incidence equals the energy after incidence is

$$Y_2 1^2 = Y_2 c^2 + Y_1 t^2 \qquad (8\text{-}1\text{-}3)$$

solving for c we get

$$0 = -Y_2 + Y_2 c^2 + Y_1(1 + c)^2$$

$$0 = (c - 1)Y_2 + (c + 1)Y_1$$

$$0 = c(Y_2 + Y_1) + (Y_1 - Y_2)$$

$$c = \frac{Y_2 - Y_1}{Y_2 + Y_1} \qquad (8\text{-}1\text{-}4)$$

In acoustics the up- and downgoing wave variables may be normalized to either pressure or velocity. When they measure velocity, the scale factor multiplying velocity squared is called the impedance I. When they measure pressure, the scale factor is called the admittance Y.

The wave c' which reflects when energy is incident from the other side is obtained from (8-1-4) if Y_1 and Y_2 are interchanged. Thus

$$c' = -c \qquad (8\text{-}1\text{-}5)$$

ly reflecting interface is one which does not allow energy through. out not only when $t = 0$ or $c = -1$, but also when $t = 2$ or $c = +1$. te that on the left in Fig. 8-1

$$\frac{\text{Energy transmitted}}{\text{Energy incident}} = \frac{Y_1 t^2}{Y_2 1^2} = \frac{Y_1}{Y_2}\left(1 + \frac{Y_2 - Y_1}{Y_1 + Y_2}\right)^2$$

$$= \frac{Y_1}{Y_2}\left(\frac{2Y_2}{Y_1 + Y_2}\right)^2 = \frac{4Y_1 Y_2}{(Y_1 + Y_2)^2} \qquad (8\text{-}1\text{-}6)$$

6) says that 100 percent of the incident energy is transmitted when he percentage of transmission is very small when Y_1 and Y_2 are

f caution: Occasionally special applications are described by authors ine reflection and transmission coefficients in terms of some variable nuous at a boundary. This is usually an oversight which unfortu- the relationship of the special application to wave theory in general r in particular. It is almost never an essential feature of the special t $t \neq 1 + c$ but just a result of an unwise choice of variables in the or example, material density is an unwise variable in acoustics ers a discontinuity at a material boundary. Pressure or normal ter descriptors of wave strength.

y there are two kinds of variables used to describe waves, and both continuous at a material discontinuity. One is a scalar like pressure, e, potential, stress, or temperature. The other is a vector of which cal component. Examples of the latter are velocity, stretch, electric ement, and heat flow. Occasionally a wave variable will be a tensor. ary condition is the vanishing of one of the motion components, lary is often said to be rigid. When it is the pressure or potential , then the boundary is often said to be free. Rigid and free bound- ves with unit magnitude reflection coefficients.

ose of this chapter is to establish fundamental mathematical prop- in layers and to avoid specialization to any particular physical type will be done in the next chapter. However, so as not to disguise pect of the mathematics, a precise definition of upgoing wave U and e D will now be given in terms of classical acoustics. In acoustics pressure P and vertical component of parcel velocity W (not to be wave velocity v). One possible definition for U and D (which will Chap. 9, Sec. 3) is

$$D = \frac{P + W/Y}{2} \qquad (8\text{-}1\text{-}7a)$$

$$U = \frac{P - W/Y}{2} \qquad (8\text{-}1\text{-}7b)$$

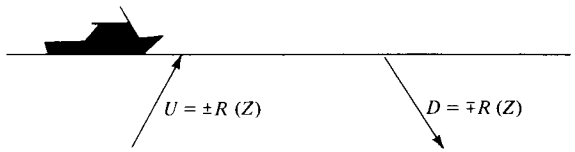


FIGURE 8-2
A waveform $R(Z)$ reflecting at the surface of the sea. Pressure equal to $U + D$ vanishes at the surface. The vertical velocity of the surface is proportional to $D - U$. Theoretically, waves are observed by measuring W at the surface; however, as a practical matter P is often observed a fraction of a wavelength below the surface.

with the inverse relations

$$P = D + U \qquad (8\text{-}1\text{-}8a)$$

$$W = (D - U)Y \qquad (8\text{-}1\text{-}8b)$$

Other definitions with different scale factors and signs are possible. With this definition, the relation $t = 1 + c$ is readily seen to be associated with (8-1-8a) and continuity of pressure at an interface. The minus signs in (8-1-7) and (8-1-8) are associated with the direction of the z axis. Reversal of the z axis changes W to $-W$ and switches the roles of U and D.

We notice that a downgoing wave D all by itself with U vanishing provides a moving disturbance of both pressure P and velocity W, and the vanishing of U assures us that the ratio between the two $W/P = Y$ is the characteristic admittance Y of the material. The energy, we have said, is proportional to either YP^2 or IW^2 from which the ratio $W/P = Y$ allows us to deduce that the impedance of a material is the inverse of its admittance $I = 1/Y$.

For sound waves in the ocean the sea surface is a nearly perfect reflector because of the great contrast between air and water. If this interface is idealized to a perfect reflector, then it is a free surface. Since the pressure vanishes on a free surface, we have that $D = -U$ at the surface so the reflection coefficient is -1. If a wave is to be seen at the surface, it is necessary to measure not pressure but something proportional to velocity. In geophysical exploration practice, pressure-sensing hydrophones are used. They must be kept at a suitable distance below the sea surface. The situation can be depicted as in Fig. 8-2. The pressure normally

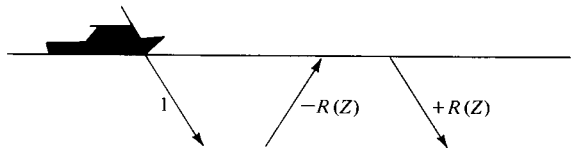


FIGURE 8-3
An initial downgoing disturbance 1 results in a later upgoing reflected wave $-R(Z)$ which reflects back down as $R(Z)$. The pressure at the surface is $D + U = 1 + R - R = 1$.

e sea surface, but if we wish to initiate an impulsive disturbance, nay momentarily take on some other value, say 1. This is depicted he total vertical component of velocity of the sea surface due to the the resulting acoustic wave is $D - U = 1 + 2R(Z)$.

n terms of Y_1 and Y_2.
application continuity is expressed by saying that $D - U$ is the same on of the interface. This implies that $t = 1 - c$. Derive an equation like he reflection coefficient in terms of the admittance Y.
flection and transmission coefficients in terms of the impedance I? (Clear m your result.)
rinciple of energy conservation we showed that $c' = -c$. It may also be m time reversal. To do this, copy Fig. 8-1 with arrows reversed. Scale superpose various figures in an attempt to create a situation where a he right-hand side of Fig. 8-1 has $-c'$ for the reflected wave. (HINT: Draw ormal incidence.)

Y FLUX IN LAYERED MEDIA

wave resonance in a layer. Let the travel time through the layer and given by the delay operator Z. The situation is shown in Fig. 8-4. above the layer has the form

$$t_2 t_1[1 + c_1 c_2' Z + (c_1 c_2')^2 Z^2 + \cdots] = \frac{t_2 t_1}{1 - c_1 c_2' Z}$$

nt that the infinite series may be summed. We will soon see that for vaves, which are of infinite duration, may be expressed as simple f degree n. We will consider many layers and the general problem

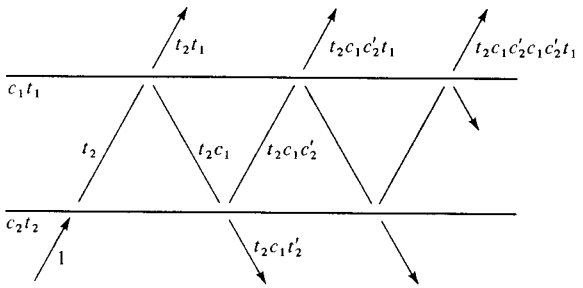


FIGURE 8-4
Some rays corresponding to resonance in a layer.

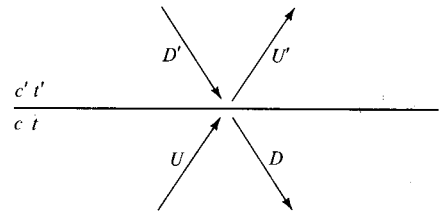


FIGURE 8-5
Waves incident and reflected from an interface.

of determining waves given reflection coefficients and determining reflection coefficients given waves.

The reflection and transmission coefficients show one how to calculate the waves resulting from a wave impinging on a layer. Equation (8-2-1) relates to Fig. 8-5 and shows how from the waves U and D' one extrapolates into the future to get U' and D.

$$\begin{aligned} U' &= tU + c'D' \\ D &= cU + t'D' \end{aligned} \tag{8-2-1}$$

Let us rearrange (8-2-1) to get U' and D' on the right and U and D on the left. Then we will have an equation which extrapolates from the primed medium to the unprimed medium. We get

$$\begin{aligned} -tU \quad &= -U' + c'D' \\ -cU + D = \quad &t'D' \end{aligned}$$

which may be arranged in the matrix form

$$\begin{bmatrix} -t & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'$$

Now premultiplying by the inverse of the left-hand matrix

$$\begin{bmatrix} U \\ D \end{bmatrix} = \frac{1}{-t} \begin{bmatrix} 1 & 0 \\ c & -t \end{bmatrix} \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'$$

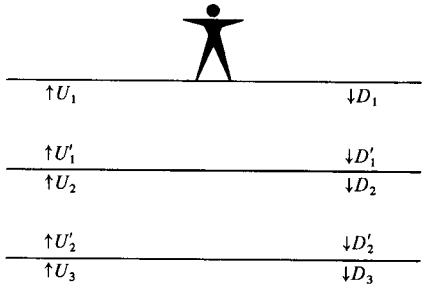
$$= \frac{1}{-t} \begin{bmatrix} -1 & c' \\ -c & cc' - tt' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'$$

finally getting the result, an equation to extrapolate from the primed medium to the unprimed medium.

$$\begin{bmatrix} U \\ D \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \tag{8-2-2}$$

Now let us consider the Goupillaud type [Ref. 30] layered medium shown in Fig. 8-6. For this arrangement of layers, (8-2-2) may be written

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k'$$



E 8-6
aud-type layered medium (layers
ual travel time).

$e^{i\omega T}$ where T, the two-way travel time, equals the data sampling
rly, multiplication by $\sqrt{Z}$ is equivalent to delaying a function by
time across a layer. This gives in the kth layer a relation between
primed waves.

$$\begin{bmatrix} U \\ D \end{bmatrix}_k' = \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{8-2-3}$$

3) into (8-2-2) we get a *layer matrix*

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix}_{k+1} &= \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\ &= \frac{1}{t_k} \begin{bmatrix} 1/\sqrt{Z} & c_k\sqrt{Z} \\ c_k/\sqrt{Z} & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\ &= \frac{1}{\sqrt{Z}t_k} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \end{aligned} \tag{8-2-4}$$

s energy flowing through a stack of layers, there must be the same
ugh the kth layer as through the $(k+1)$st layer. Otherwise, there is
or source at the layer boundary. The net upward flow of energy
t any frequency ω in the kth layer is given by

$$\text{flux}(\omega) = Y_k\left(U(Z)\bar{U}\left(\frac{1}{Z}\right) - D(Z)\bar{D}\left(\frac{1}{Z}\right)\right)_k \tag{8-2-5}$$

at this is indeed independent of k, we take the Hermitian conjugate
conjugate with respect to real ω) of (8-2-4).

$$\left[U\left(\frac{1}{Z}\right)D\left(\frac{1}{Z}\right)\right]_{k+1} = \frac{1}{t_k}\left[U\left(\frac{1}{Z}\right)D\left(\frac{1}{Z}\right)\right]_k \begin{bmatrix} \sqrt{Z} & c_k\sqrt{Z} \\ c_k/\sqrt{Z} & 1/\sqrt{Z} \end{bmatrix} \tag{8-2-6}$$

Now combine (8-2-4) with (8-2-6) in the form

$$\begin{aligned} &\left[U\left(\frac{1}{Z}\right)D\left(\frac{1}{Z}\right)\right]_{k+1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} U(Z) \\ D(Z) \end{bmatrix}_{k+1} \\ &\quad = \frac{1}{t_k^2}\begin{bmatrix} \bar{U} & \bar{D} \end{bmatrix}_k \begin{bmatrix} \sqrt{Z} & \sqrt{Z}c_k \\ c_k/\sqrt{Z} & 1/\sqrt{Z} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{Z} & \sqrt{Z}c_k \\ c_k/\sqrt{Z} & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\ &\quad = \frac{1}{t_k^2}\begin{bmatrix} \bar{U} & \bar{D} \end{bmatrix}_k \begin{bmatrix} 1 - c_k^2 & 0 \\ 0 & c_k^2 - 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \end{aligned} \tag{8-2-7}$$

Since $(1 - c_k^2)/t_k^2 = t_k'/t_k = Y_k/Y_{k+1}$ this may be rewritten as the desired result, namely

$$Y_{k+1}\left[U\left(\frac{1}{Z}\right)U(Z) - D\left(\frac{1}{Z}\right)D(Z)\right]_{k+1} = Y_k\left[U\left(\frac{1}{Z}\right)U(Z) - D\left(\frac{1}{Z}\right)D(Z)\right]_k \tag{8-2-8}$$

Equation (8-2-8) says that at each frequency ω the energy flowing through the kth layer equals the energy flowing through the $(k+1)$st layer.

This energy flux theorem leads quickly to some sweeping statements about the waveforms scattered from layered structures. Figure 8-7 shows the basic geometry of reflection seismology. Applying the energy flux theorem to this geometry we may say that the energy flux in the top layer equals that in the lower halfspace so

$$Y_1\left\{R\left(\frac{1}{Z}\right)R(Z) - \left[1 + R\left(\frac{1}{Z}\right)\right][1 + R(Z)]\right\} = -Y_k E\left(\frac{1}{Z}\right)E(Z)$$

or

$$1 + R\left(\frac{1}{Z}\right) + R(Z) = \frac{Y_k}{Y_1}E\left(\frac{1}{Z}\right)E(Z) \tag{8-2-9}$$

This very remarkable result says that if we were to observe the escaping wave $E(Z)$, we could by autocorrelation construct the waveform seen at the surface. We will later see that $E(Z)$ is minimum-phase so that E could be constructed from R by spectral factorization.

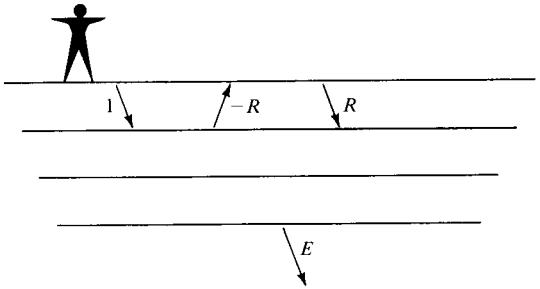


FIGURE 8-7
Basic reflection seismology geometry. The man initiates an impulse going downward. The earth sends back $-R(Z)$ to the surface. Since the surface is perfectly reflective, the surface sends $R(Z)$ back into the earth. Escaping from the bottom of the layers is a wave $E(Z)$ which is heading toward the other side of the earth.

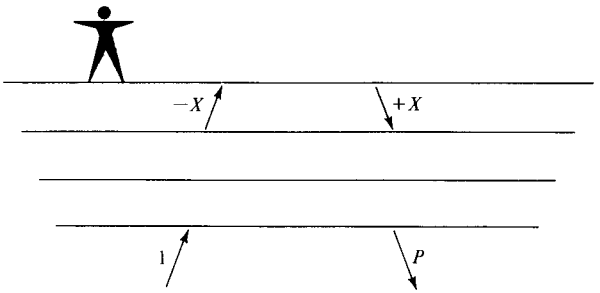


FIGURE 8-8
Earthquake seismology geometry. An impluse 1 is incident from below. The waveform $X(Z)$ is incident upon the free surface and is reflected back down. The waveform $P(Z)$ scatters back into the earth.

us turn our attention to the earthquake seismology geometry
g. 8-8. Applying the energy flux theorem to this geometry we obtain

$$Y_1\left[X\left(\frac{1}{Z}\right)X(Z) - X\left(\frac{1}{Z}\right)X(Z)\right] = Y_k\left[1 - P(Z)P\left(\frac{1}{Z}\right)\right]$$

$$1 = P(Z)P\left(\frac{1}{Z}\right) \qquad (8\text{-}2\text{-}10)$$

tion of the result is that the backscattered waveform $P(Z)$ has the pass filter. This result may have been anticipated on physical grounds nergy which is incident is ultimately reflected without attenuation; thing which can happen is that there will be frequency-dependent

we will derive a theorem which relates energy flux to impedance and ctions (these functions have Fourier transforms with a positive real e that a downgoing wave $D(Z)$ is stronger than an upgoing wave quencies, i.e.

$$D\left(\frac{1}{Z}\right)D(Z) - U\left(\frac{1}{Z}\right)U(Z) > 0 \qquad \text{on } |Z| = 1 \qquad (8\text{-}2\text{-}11)$$

does not imply $|d_t| > |u_t|$.) We will abbreviate (8-2-11) by

$$\bar{D}D - \bar{U}U > 0 \qquad (8\text{-}2\text{-}12)$$

or (8-2-12) we will deduce that $(D - U)/(D + U)$ has a Fourier a positive real part. We have

$$\frac{D-U}{D+U} + \frac{\bar{D}-\bar{U}}{\bar{D}+\bar{U}} = \frac{(D-U)(\bar{D}+\bar{U}) + (\bar{D}-\bar{U})(D+U)}{(D+U)(\bar{D}+\bar{U})}$$

$$= \frac{2(\bar{D}D - \bar{U}U)}{(D+U)(\bar{D}+\bar{U})} \qquad (8\text{-}2\text{-}13)$$

The numerator of (8-2-13) is positive by hypothesis (8-2-12) and the denominator of (8-2-13) is positive, since it is the spectrum of the time function $d_t + u_t$ and any spectrum is always positive. Thus $(D - U)/(D + U)$ is called "positive real." The acoustical interpretation of $(D - U)/(D + U)$ is that $(D - U)$ represents the vertical component of material velocity and $(D + U)$ represents the material pressure.

8-3 GETTING THE WAVES FROM THE REFLECTION COEFFICIENTS

A layered material may be specified by giving the reflection coefficient at each interface. Alternate descriptions are to give any one of the scattered waves $R(Z)$, $E(Z)$, $X(Z)$, or $P(Z)$. Our ultimate objective is to get such a good grip on the algebra of this kind of problem that we will be able to start with any descriptor of the layers and from it deduce all the other descriptors.

An important result of the last section was the development of a "layer matrix" (8-2-4) that is, a matrix which can be used to extrapolate waves observed in one layer to the waves observed in the next layer. This process may be continued indefinitely. To see how to extrapolate from layer 1 to layer 3 substitute (8-2-4) with $k = 1$ into (8-2-4) with $k = 2$, obtaining

$$\begin{bmatrix} U \\ D \end{bmatrix}_3 = \frac{1}{\sqrt{\bar{Z}t_2}}\begin{bmatrix} 1 & Zc_2 \\ c_2 & Z \end{bmatrix}\frac{1}{\sqrt{\bar{Z}t_1}}\begin{bmatrix} 1 & Zc_1 \\ c_1 & Z \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1$$

$$= \frac{1}{\sqrt{\bar{Z}^2 t_1 t_2}}\begin{bmatrix} 1 + Zc_1c_2 & Zc_1 + Z^2c_2 \\ c_2 + Zc_1 & Zc_1c_2 + Z^2 \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \qquad (8\text{-}3\text{-}1)$$

Inspection of this example suggests the general form for a product of k layer matrices

$$\frac{1}{\sqrt{\bar{Z}^k \prod_{j=1}^{k} t_j}}\begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \qquad (8\text{-}3\text{-}2)$$

Now let us verify that (8-3-2) is indeed the general form. We assume (8-3-2) is correct for $k - 1$; then we multiply (8-3-2) by another layer matrix and see if the product retains the same form with $k - 1$ increased to k. The product is

$$\frac{1}{\sqrt{\bar{Z}t_k}}\begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix}\frac{1}{\sqrt{\bar{Z}^{k-1}\prod_{j=1}^{k-1} t_j}}\begin{bmatrix} F(Z) & Z^{k-1}G\left(\frac{1}{Z}\right) \\ G(Z) & Z^{k-1}F\left(\frac{1}{Z}\right) \end{bmatrix}$$

$$= \frac{1}{\sqrt{\bar{Z}^k \prod_1^k t_j}}\left[\begin{array}{c|c} F(Z) + c_k Z G(Z) & Z^{k-1}G\left(\frac{1}{Z}\right) + c_k Z^k F\left(\frac{1}{Z}\right) \\ \hline c_k F(Z) + Z G(Z) & c_k Z^{k-1} G\left(\frac{1}{Z}\right) + Z^k F\left(\frac{1}{Z}\right) \end{array}\right] \qquad (8\text{-}3\text{-}3)$$

e product we see that the scaling factor is of the same form with
k. Also the 22 matrix element can be obtained from the 11 element
ith $1/Z$ and multiplying by Z^k. Likewise, the 21 element is obtained
nent; thus (8-3-2) does indeed represent a general form. The poly-
d $G(Z)$ of order k are built up in the following way [from the first
ght-hand side of (8-3-3)]:

$$F_k(Z) = F_{k-1}(Z) + c_k Z G_{k-1}(Z) \qquad (8\text{-}3\text{-}4a)$$

$$G_k(Z) = c_k F_{k-1}(Z) + Z G_{k-1}(Z) \qquad (8\text{-}3\text{-}4b)$$

ing (8-3-4) we can see some of the details of F and G. From (8-3-4a)
lead coefficient f_0 of $F(Z)$ does not change with k. It is always
ing this from (8-3-4b) we see that $(g_0)_k = c_k$. Also with knowledge
(Z) are of the same degree in Z, we see that (8-3-4b) implies that
icient of $G(Z)$, say $(g_k)_k$ does not change with k and therefore it
g value of c_1. Finally, with this knowledge and (8-3-4a) we deduce
coefficient in $F(Z)$ will always be $c_1 c_k$. Thus, in summary

$$F(Z) = 1 + f_1 Z + f_2 Z^2 + \cdots + c_1 c_k Z^{k-1} \qquad (8\text{-}3\text{-}5a)$$

$$G(Z) = c_k + g_1 Z + g_2 Z^2 + \cdots + c_1 Z^{k-1} \qquad (8\text{-}3\text{-}5b)$$

in (8-3-5) and proved from the recurrence relations (8-3-4) that
f F contain even powers of c and that G contains odd powers of c.
if all c change sign, G will change sign but F is unchanged.
mials $F(Z)$ and $G(Z)$ are not independent and a surprising energy-
ship exists between them. By substitution from (8-3-4) one may
at

$$\left[F(Z)G\left(\frac{1}{Z}\right)\right]_k = (1 - c_{k-1}^2)\left[F(Z)F\left(\frac{1}{Z}\right) - G(Z)G\left(\frac{1}{Z}\right)\right]_{k-1} \qquad (8\text{-}3\text{-}6)$$

and $G_1(Z) = c_1$ we have by iterative application of (8-3-6) that

$$\left[F(Z)F\left(\frac{1}{Z}\right) - G(Z)G\left(\frac{1}{Z}\right)\right]_k = \prod_1^k (1 - c_k^2) = \prod_1^k t't \qquad (8\text{-}3\text{-}7)$$

is a surprising equation because on the left-hand side we have two
trum of f_t and the spectrum of g_t, but the right-hand side is a
cy-independent constant. Since the spectrum of f_t is thus greater
m of g_t, we may apply the theorem of adding garbage to a mini-
let to deduce from (8-3-4a) and from knowledge that $|c_k| < 1$ that
-phase if $F_{k-1}(Z)$ is minimum-phase. Since $F_1(Z) = 1$ is minimum-
at all $F_k(Z)$ are minimum-phase. Since $F(Z)$ is minimum-phase,
e calculated from its spectrum $F(Z)F(1/Z)$ or the spectrum of g_t
ingle number $\Pi t't$). However, we cannot get G from F. Before
lgebraic discussion we take up an example.

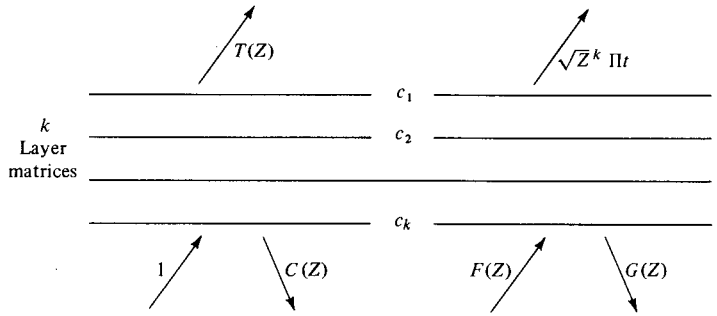


FIGURE 8-9
Waves incident, reflected, and transmitted from a stack of layers between two half-spaces.

Let a stack of layers be sandwiched in between two halfspaces (Fig. 8-9). An impulse is incident from below. The backscattered wave is called $C(Z)$ and the transmitted wave is called $T(Z)$.

Mathematically, we describe the situation with the equations

$$\begin{bmatrix} 1 \\ C(Z) \end{bmatrix} = \frac{1}{\sqrt{Z^k \Pi t}} \begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \begin{bmatrix} T(Z) \\ 0 \end{bmatrix} \qquad (8\text{-}3\text{-}8)$$

We may solve the first of (8-3-8) for the transmitted wave $T(Z)$

$$T(Z) = \frac{\sqrt{Z^k \Pi t}}{F(Z)} \qquad (8\text{-}3\text{-}9)$$

and introduce the result back into the second of (8-3-8) to obtain the backscattered wave

$$C(Z) = \frac{G(Z)T(Z)}{\sqrt{Z^k \Pi t}} = \frac{G(Z)}{F(Z)} \qquad (8\text{-}3\text{-}10)$$

The mathematical fact that $F(Z)$ is minimum-phase corresponds to the physical fact that the $C(Z)$ and $T(Z)$ have finite energy; therefore the denominators of (8-3-9) and (8-3-10) cannot have zeros inside the unit circle. Since we know that the backscattered wave $C(Z)$ contains less energy than the incident wave by reference to (8-2-13) we know that a positive real function is given by

$$\frac{1 - C(Z)}{1 + C(Z)} = \frac{1 - G(Z)/F(Z)}{1 + G(Z)/F(Z)} \qquad (8\text{-}3\text{-}11)$$

e how to reconstruct the reflection coefficients c_j from the observed
$C(Z)$. Referring to Fig. 8-9 we have

$$\begin{bmatrix} 1 \\ C(Z) \end{bmatrix} = \frac{1}{t_k\sqrt{Z}} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_{k-1} \tag{8-3-12}$$

cient of $C(Z)$ is c_k [this is physically obvious but may also be seen
Thus the layer matrix in (8-3-12) is known. Multiplying (8-3-12)
e inverse of the layer matrix we will have obtained $U_{k-1}(Z)$ and
next reflection coefficient c_{k-1} is obviously d_0/u_0. Thus we may
ll the c_k are determined.
us reconsider the reflection seismology geometry. We have

$$\begin{bmatrix} 0 \\ E(Z) \end{bmatrix} = \frac{1}{\sqrt{Z^k}\Pi t} \begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \begin{bmatrix} -R \\ 1+R \end{bmatrix} \tag{8-3-13}$$

equation we may solve for $R(Z)$

$$R(Z) = +\frac{Z^k G(1/Z)}{F(Z) - Z^k G(1/Z)} \tag{8-3-14}$$

tor occurs so often that we give it the name $A(Z)$

$$A(Z) = F(Z) - Z^k G\left(\frac{1}{Z}\right) \tag{8-3-15}$$

), is minimum-phase. The second of (8-3-13) gives the escaping

$$\frac{F(1/Z) + [-G(Z) + Z^k F(1/Z)]R}{\sqrt{Z^k}\Pi t}$$

$$\frac{F(1/Z)[F(Z) - Z^k G(1/Z)] + [-G(Z) + Z^k F(1/Z)]Z^k G(1/Z)}{A(Z)\sqrt{Z^k}\Pi t}$$

h (8-3-7) we get

$$E(Z) = \frac{\sqrt{Z^k}\Pi t'}{A(Z)} \tag{8-3-16}$$

eal function is

$$\frac{D-U}{D+U} = \frac{1+R-(-R)}{1+R-R} = 1 + 2R(Z) \tag{8-3-17}$$

$$= \frac{\text{Vertical velocity} = 1 + 2R.}{\text{Pressure} = 1}$$

As mentioned earlier, if the equations are interpreted in terms of acoustics, then $Y(D-U)/(D+U)$ is interpreted as vertical velocity divided by pressure. It is called the *admittance* which is the inverse of the impedance.

We have now completed the task of solving for the waves given the reflection coefficients. In the subsequent section we attack the inverse problems of getting the reflection coefficients from knowledge of various waves.

EXERCISES

1 In Fig. 8-9 let $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, and $c_3 = \frac{1}{3}$. What are the polynomial ratios $T(Z)$ and $C(Z)$?

2 For a simple interface, we had the simple relations $t = 1 + c$, $t' = 1 + c'$, and $c = -c'$. What sort of analogous relations can you find for the generalized interface of Fig. 8-9? [For example, show $1 - T(Z)T'(1/Z) = C(Z)C(1/Z)$ which is analogous to $1 - tt' = c^2$.]

3 Show that $T(Z)$ and $T'(Z)$ are the same waveforms within a scale factor. Deduce that many different stacks of layers may have the same $T(Z)$.

4 Let an impulse be incident on a stack of layers and let a wave $C(Z)$ be reflected. What is the reflection coefficient at the first layer encountered? What would be the reflected wave as a function of C for a situation which differs from the above by the removal of the first reflector?

5 Consider the earth to be modeled by layers over a halfspace. Let an impulse be incident from below (Fig. E8-3-5). Given $F(Z)$ and $G(Z)$, elements of the product of the layer

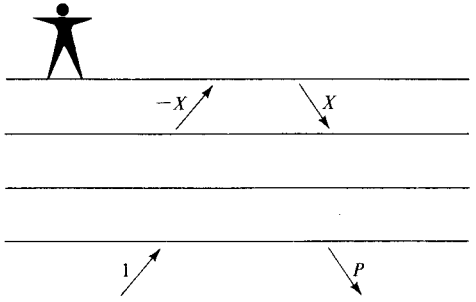


FIGURE E8-3-5

matrices, solve for X and for P. Check your answer by showing that $P(Z)\bar{P}(1/Z) = 1$. How is X related to E? This relation illustrates the principle of reciprocity which says source and receiver may be interchanged.

6 Show that $1 + R(1/Z) + R(Z) = (\text{scale factor})\ X(Z)\ X(1/Z)$, which shows that one may autocorrelate the transmission seismogram to get the reflection seismogram.

7 Refer to Fig. E8-3-7. Calculate R' from R.

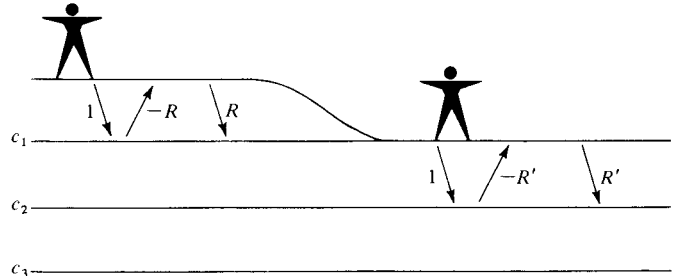


FIGURE E8-3-7

G THE REFLECTION COEFFICIENTS FROM THE

ng point for inverse problems is the Kunetz equation [Ref. 31]

$$1 + R\left(\frac{1}{Z}\right) + R(Z) = \frac{Y_k}{Y_1} E\left(\frac{1}{Z}\right) E(Z) \qquad (8\text{-}4\text{-}1)$$

ne expression for the escaping wave (8-3-16)

$$E(Z) = \frac{\sqrt{Z}^k \Pi t'}{A(Z)} \qquad (8\text{-}4\text{-}2)$$

o recall that $Y_k/Y_1 = \Pi t/t'$. With this (8-4-1) becomes

$$1 + R(Z) + R\left(\frac{1}{Z}\right) = \frac{\Pi t' t}{A(Z)A(1/Z)} \qquad (8\text{-}4\text{-}3)$$

ough by $A(Z)$ we get

$$\left[1 + R(Z) + R\left(\frac{1}{Z}\right)\right] A(Z) = \frac{\Pi t' t}{A(1/Z)} \qquad (8\text{-}4\text{-}4)$$

) is minimum-phase, $A(Z)$ may be written as $1/B(Z)$ or $A(1/Z) =$ (8-4-4) becomes

$$[1 + R(Z) + R(1/Z)]A(Z) = (\Pi t' t)\left(1 + \frac{b_1}{Z} + \frac{b_2}{Z^2} + \cdots\right) \qquad (8\text{-}4\text{-}5)$$

ficients of zero and positive powers of Z as simultaneous equations, equations which for a three-layer model looks like ($r_0 = 1$).

$$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_1 & r_2 \\ r_2 & r_1 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \\ r_4 & r_3 & r_2 & r_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ -c_3 \end{bmatrix} = \begin{bmatrix} \Pi t' t \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{bmatrix} \qquad (8\text{-}4\text{-}6)$$

we see our old friend the Toeplitz matrix. It used to work for a and predicting time series. Notice that $-c_3$ has been inserted in ghest coefficient of $A(Z)$. This is justified by reference back to the Z) in terms of $F(Z)$ and $G(Z)$ which were in turn defined from the c_k. ning the Toeplitz simultaneous equations (8-4-6) and the Levinson ion (3-3-10) that we will learn how to compute the reflection co- he waves.

our equations in (8-4-6) would normally be thought of as follows: ree reflected pulses r_1, r_2, and r_3 we may solve the equations for A, ing the reflection coefficient c_3. Knowing A, the 5th equation in used to compute r_4. If the model were truly a three-layer model,

it would come out right; if not, the discrepancy would be indicative of another reflector c_4 which could be found by expanding equation (8-4-6) from 4th order to 5th order. In summary, given the reflected pulses r_k, the Levinson recursion successively turns out the reflection coefficients c_k.

Now suppose we begin by observation of the escaping wave $E(Z)$. One way to determine the reflection coefficients would be to form $1 + R(Z) + R(1/Z)$ by the autocorrelation of $E(Z)$; then, the Levinson recursion could be used to solve for the reflection coefficients. The only disadvantage of this method is that $E(Z)$ contains an infinite number of coefficients so that in practice some truncation must be done. The truncation is avoided by an alternative method. Given $E(Z)$ polynomial division will find $A(Z)$. The heart of the Levinson recursion is the building up of $A(Z)$ by $A_k(Z) = A_{k-1}(Z) - c_k Z^k A_{k-1}(1/Z)$. In particular, from (3-3-12) we have

$$\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_3 = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 - c_3 \begin{bmatrix} 0 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_2 \qquad (8\text{-}4\text{-}7)$$

which shows how to get $A_3(Z)$ from $A_2(Z)$ and c_3. To do it backwards, we see first that c_3 is $-a_3$. Then write (8-4-7) upside-down

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 0 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_2 - c_3 \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 \qquad (8\text{-}4\text{-}8)$$

Next multiply (8-4-7) by $1/(1 - c_3^2)$ and add the product to (8-4-8) multiplied by $c_3/(1 - c_3^2)$. Notice that the upside-down vectors on the right-hand side cancel, leaving

$$\frac{1}{1 - c_3^2} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_3 + \frac{c_3}{1 - c_3^2} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 \qquad (8\text{-}4\text{-}9)$$

Equation (8-4-9) is the desired result which shows how to reduce $A_{k+1}(Z)$ to $A_k(Z)$ while learning c_{k+1}. A program to continue this process is given in Fig. 8-10. An inverse program to get R and A from c is in Fig. 8-11.

```
      COMPLEX A,C,AL,BE,TOP,CONJG
      C(1)=-1.; R(1)=1.; A(1)=1.; V(1)=1.
  300 DO 310 I=1,N
  310 C(I)=A(I)
      DO 330 K=1,N
      J=N-K+2
      AL=1./(1.-C(J)*CONJG(C(J)))
      BE=C(J)*AL
      JH=(J+1)/2
      DO 320 I=1,JH
      TOP=AL*C(I)-BE*CONJG(C(J-I+1))
      C(J-I+1)=AL*C(J-I+1)-BE*CONJG(C(I))
  320 C(I)=TOP
  330 C(J)=-BE/AL
```

FIGURE 8-10
A program to compute reflection coefficients c_k from the prediction-error filter $A(Z)$. The complex arithmetic is optional.

```
      COMPLEX C,R,A,BOT,CONJG
      C(1)=-1.; R(1)=1.; A(1)=1.; V(1)=1.
100   DO 120 J=2,N
      A(J)=0.
      R(J)=C(J)*V(J-1)
      V(J)=V(J-1)*(1.-C(J)*CONJG(C(J)))
      DO 110 I=2,J
110   R(J)=R(J)-A(I)*R(J-I+1)
      JH=(J+1)/2
      DO 120 I=1,JH
      BOT=A(J-I+1)-C(J)*CONJG(A(I))
      A(I)=A(I)-C(J)*CONJG(A(J-I+1))
120   A(J-I+1)=BOT
```

E 8-11
am inverse to the program of
). It computes both R and A

t us see how to do a problem where there are random sources.
ws the "earthquake geometry." However, in order to introduce a
nt, the pulse incident from below has been convolved with a white-
f random numbers. Consequently, all the waves internal to Fig.
y the convolution of w_t with the corresponding wave in the impulse-
Now suppose we are given the top-layer waves $D = -U = XW$
sider downward continuation. We have the layer matrix

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{\sqrt{Z t_k}} \begin{bmatrix} 1 & cZ \\ c & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{8-4-10}$$

-written as

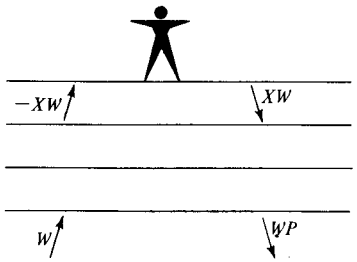
$$\begin{bmatrix} -U \\ D \end{bmatrix}_{k+1} = \frac{1}{\sqrt{Z t_k}} \begin{bmatrix} 1 & -cZ \\ -c & Z \end{bmatrix} \begin{bmatrix} -U \\ D \end{bmatrix}_k \tag{8-4-11}$$

tion-error scheme can be written in the form

$$\begin{bmatrix} e^+ \\ e^- \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & -c \\ -c & 1 \end{bmatrix} \begin{bmatrix} e^+ \\ e^- \end{bmatrix}_k \tag{8-4-12}$$

equivalent within a scale factor to downward continuing surface
e remaining question is whether Burg's estimate of the reflection
ly,

$$\hat{c}_k = \frac{2 \sum_t e_k^+ e_k^-}{\sum_t e_k^+ e_k^+ + e_k^- e_k^-} \tag{8-4-13}$$



URE 8-12
quake seismogram geometry with
light incident from below. In the
yer, the sum of the waves vanishes
senting zero pressure at the free
e. The difference of up- and down-
waves is the observed vertical
onent of velocity.

turns out to estimate the reflection coefficient c_k in the physical model. To see how Burg's $\hat{c}_k$ is related to the c_k arising in the Levinson recursion, we define $\mathbf{f}^+$ and $\mathbf{f}^-$ for $k = 2$ as

$$\begin{bmatrix} \\ \mathbf{f}^+ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \\ \mathbf{f}^- \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ 1 \end{bmatrix} \tag{8-4-14}$$

Next form the dot product

$$(\mathbf{f}^-)^T \mathbf{f}^+ = [0 \quad a_1 \quad 1] \begin{bmatrix} x_0 & x_1 & x_2 & & \\ & x_0 & x_1 & x_2 & \\ & & x_0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix}$$

$$= [0 \quad a_1 \quad 1] \begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix} \tag{8-4-15}$$

Now utilize the fact that $(1, a_1)$ satisfies the 2×2 system. Following the Levinson recursion (8-4-15) can be written as

$$(\mathbf{f}^-)^T \mathbf{f}^+ = [0 \quad a_1 \quad 1] \begin{bmatrix} v \\ 0 \\ e \end{bmatrix} = e \tag{8-4-16}$$

Likewise we can deduce that $\mathbf{f}^+ \cdot \mathbf{f}^+ = \mathbf{f}^- \cdot \mathbf{f}^- = v$. Thus, the Levinson calculation of the reflection coefficient can be written as

$$c = \frac{2(\mathbf{f}^+ \cdot \mathbf{f}^-)}{(\mathbf{f}^+ \cdot \mathbf{f}^+) + (\mathbf{f}^- \cdot \mathbf{f}^-)} = \frac{2e}{2v} \tag{8-4-17}$$

The Burg treatment differs from the Levinson treatment in that Burg omits end-effect terms on (8-4-14). Instead of (8-4-14) he has

$$\begin{bmatrix} \\ \mathbf{e}^+ \\ \\ \end{bmatrix} = \begin{bmatrix} x_1 & x_0 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \\ \mathbf{e}^- \\ \\ \end{bmatrix} = \begin{bmatrix} x_1 & x_0 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} a_1 \\ 1 \end{bmatrix} \tag{8-4-18}$$

For a sufficiently long data sequence the Burg method and the Levinson technique thus become indistinguishable. For a data sample of finite duration we must make a choice. The Levinson technique with (8-4-14) is equivalent to assuming the data sample vanishes off the ends of the interval in which it is observed. In most applications this is untrue, and so the Burg technique is usually preferable.

and the first part of a reflection seismogram, that is, $1 + 2R(Z)$ is $Z^2/16 + Z^3/4 + \cdots$). What are the first three reflection coefficients? ere are no more reflectors what is the next point in the reflection seis-

n $X(Z) = 1/(1 - .1Z + .9Z^2)$ is observed at the surface of some layers ace. Sketch the time function and indicate its resonance frequency and Find the reflection coefficients if $X(Z)$ is due to an impulsive source of gnitude in the halfspace below the layers.

$- b_1Z$ deep in the halfspace produces a seismogram $B(Z)X(Z) = 1 - Z +$ $+ Z^4/4 - Z^5/4 + Z^6/8 - Z^7/8 + \cdots$. What are the layered structure and ne function?

9

MATHEMATICAL PHYSICS IN STRATIFIED MEDIA

In stratified media there are many common mathematical aspects in phenomena so physically diverse as acoustics, electromagnetic waves, magnetostatics, gravitational-elastic spherical resonance, heat flow, gas diffusion, electric current in a resistive material, seismic waves, water waves, and atmospheric gravity waves, among many others. We will present the general theory and work out some of the details for the case of simple acoustics.

By a stratified medium we mean one in which material properties, compressibility, conductivity, density, etc., are functions of one spatial coordinate only. The usual situation is cartesian coordinates, but when geophysics is done on a global scale spherical coordinates may be used.

9-1 FROM PHYSICS TO MATHEMATICS

First step:
The first step is to write down all the basic partial differential equations of classical physics which relate to the problem of interest. Do not write down equations containing second space derivatives which are derived from first-derivative equations.