2

ONE-SIDED FUNCTIONS

systems share the property that they do not respond before they are
us the impulse response of any physical system is a one-sided time
vanishes before $t = 0$). In system theory such a filter function is called
In wave propagation this property is associated with causality in that
ıy begin to arrive before it is transmitted. The lag-time point $t = 0$
ıliar and an important role. For this reason, many subtle matters will
ore clearly understood with sampled time than with continuous time.
r responds at and after lag time $t = 0$, we will say the filter is realizable
The word *causal* is appropriate in physics where stress may cause (prac-
antaneous strain and vice versa, but one should revert to the more
ds *realizable* or *one-sided* when using filter theory to describe economic
stems where simultaneity is quite different from cause and effect.

RSE FILTERS

and causal filters better, we now take up the task of undoing what a
has done. Consider that the output y_t of a filter b_t is known but the
unknown. See Fig. 2-1.

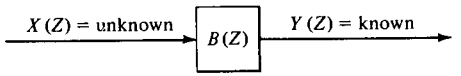


FIGURE 2-1
Sometimes the input to a filter is unknown.

This is the problem that one always has with a transducer/recorder system. For example, the output of a seismometer is a wiggly line on a piece of paper from which the seismologist may wish to determine the displacement, velocity, or acceleration of the ground. To undo the filtering operation of the filter $B(Z)$, we will try to find another filter $A(Z)$ as indicated in Fig. 2-2.

To solve for the coefficients of the filter $A(Z)$, we merely identify coefficients of powers of Z in $B(Z)A(Z) = 1$. For $B(Z)$, a three-term filter, this is

$$(a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3 + \cdots)(b_0 + b_1 Z + b_2 Z^2) = 1 \qquad (2\text{-}1\text{-}1)$$

The coefficients of $Z^0, Z^1, Z^2, \ldots$ in (2-1-1) are

$$a_0 b_0 = 1 \qquad (2\text{-}1\text{-}2)$$

$$a_1 b_0 + a_0 b_1 = 0 \qquad (2\text{-}1\text{-}3)$$

$$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0 \qquad (2\text{-}1\text{-}4)$$

$$a_3 b_0 + a_2 b_1 + a_1 b_2 = 0 \qquad (2\text{-}1\text{-}5)$$

$$a_4 b_0 + a_3 b_1 + a_2 b_2 = 0 \qquad (2\text{-}1\text{-}6)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$a_k b_0 + a_{k-1} b_1 + a_{k-2} b_2 = 0 \qquad (2\text{-}1\text{-}7)$$

From (2-1-2) one may get a_0 from b_0. From (2-1-3) one may get a_1 from a_0 and the b_k. From (2-1-4) one may get a_2 from a_1, a_0, and the b_k. Likewise, in the general case a_k may be found from a_{k-1}, a_{k-2}, and the b_k. Specifically, from (2-1-7) the a_k may be determined recursively by

$$a_k = \frac{-\sum_{i=1}^{2} a_{k-i} b_i}{b_0} \qquad (2\text{-}1\text{-}8)$$

Consider the example where $B(Z) = 1 - Z/2$; then, by equations like (2-1-2) to (2-1-7), by the binomial theorem, by polynomial division, or by Taylor's power series formula we obtain

$$A(Z) = \frac{1}{1 - Z/2} = 1 + \frac{Z}{2} + \frac{Z^2}{4} + \frac{Z^3}{8} + \cdots \qquad (2\text{-}1\text{-}9)$$

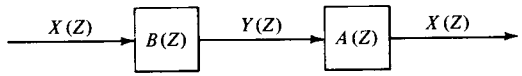


FIGURE 2-2
The filter $A(Z)$ is inverse to the filter $B(Z)$.

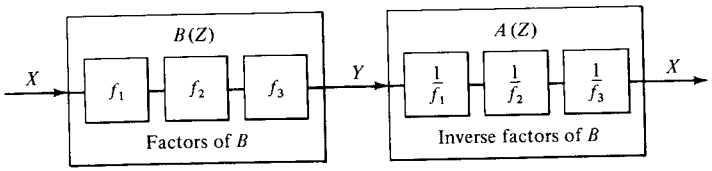


FIGURE 2-3
Factoring the polynomial $B(Z)$ breaks the filter into many two-term filters. Each one should have a bounded inverse.

there are an infinite number of filter coefficients but that they drop off ze so that approximation in a computer presents no problem. The ıot so rosy with the filter $B(Z) = 1 - 2Z$. Here we obtain

$$A(Z) = \frac{1}{1-2Z} = 1 + 2Z + 4Z^2 + 8Z^3 + 16Z^4 + 32Z^5 + \cdots \qquad (2\text{-}1\text{-}10)$$

ents of the series increase without bound. The outputs of the filter l infinitely strongly on inputs of the infinitely distant past. [Recall that ıutput of $A(Z)$ is a_0 times the present input x_t plus a_1 times the previous etc., so a_n represents memory of n time units earlier.] The implication at some filters $B(Z)$ will not have useful finite approximate inverses ıined from (2-1-2) to (2-1-8). We now seek ways to identify the good he bad ones. With a two-pulse filter, the criterion is merely that the first Z) be larger than the second. A more mathematical description of the irs results from solving for the roots of $B(Z)$, that is, find values of Z_0 $(Z_0) = 0$. For the example $1 - Z/2$ we find $Z_0 = 2$. For the example find $Z_0 = \frac{1}{2}$. The general case for wavelets with complex coefficients e solution value Z_0 of $B(Z_0) = 0$ lies inside the unit circle in the complex $1/B(Z)$ will have coefficients which blow up; and if the root lies outside cle, then the inverse $1/B(Z)$ will be bounded.

ling earlier discussion that a polynomial $B(Z)$ of degree N may be o N subsystems and that the ordering of subsystems is unimportant 3), we suspect that if any of the N roots of $B(Z)$ lies inside the unit ay have difficulty with $A(Z)$. Actual proof of this suspicion relies on a ɔm complex-variable theory about absolutely convergent series. The that the product of absolutely convergent series is convergent, and he product of any convergent series with a divergent series is divergent. oof may be based upon the fact that a power series for $1/B(Z)$ converges ıbout the origin with a radius from the origin out to the first pole [the) of smallest magnitude]. Convergence of $A(Z)$ on the unit circle means, filters, that the coefficients of $A(Z)$ are decreasing. Thus, if all the Z) are outside the unit circle, we will get a convergent filter from (2-1-8). anything at all be done if there is one root or more inside the circle? is suggested by the example

$$\frac{1}{1-2Z} = -\frac{1}{2Z}\frac{1}{1-1/2Z} = -\frac{1}{2Z}\left[1 + \frac{1}{2Z} + \frac{1}{(2Z)^2} + \cdots\right] \qquad (2\text{-}1\text{-}11)$$

Equation (2-1-11) is a series expansion in $1/Z$, that is, a Taylor series about infinity. It converges from $Z = \infty$ all the way in to a circle of radius 1/2. This means that the inverse converges on the unit circle where it must, if the coefficients are to be bounded. In terms of filters it means that the inverse filter must be one of those filters which responds to future inputs and hence is not physically realizable but may be used in computer simulation.

In the general case, then, one must factor $B(Z)$ into two parts: $B(Z) = B_{out}(Z)B_{in}(Z)$ where B_{out} contains roots outside the unit circle and B_{in} contains the roots inside. Then the inverse of B_{out} is expressed as a Taylor series about the origin and the inverse of B_{in} is expressed as a Taylor series about infinity. The final expression for $1/B(Z)$ is called a Laurent expansion for $1/B(Z)$, and it converges on a ring surrounding the unit circle. Cases with zeros exactly on the unit circle present special problems. Sometimes you can argue yourself out of the difficulty but at other times roots on or even near the circle may mean that a certain computing scheme won't work out well in practice.

Finally, let us consider a mechanical interpretation. The stress (pressure) in a material may be represented by x_t, and the strain (volume change) may be represented by y_t. The following two statements are equivalent; that is, in some situations they are both true, and in other situations they are both false:

STATEMENT A The stress in a material may be expressed as a linear combination of present and past strains. Likewise, the strain may be deduced from present and past stresses.

STATEMENT B The filter which relates stress to strain and vice versa has all poles and zeros outside the unit circle.

EXERCISES

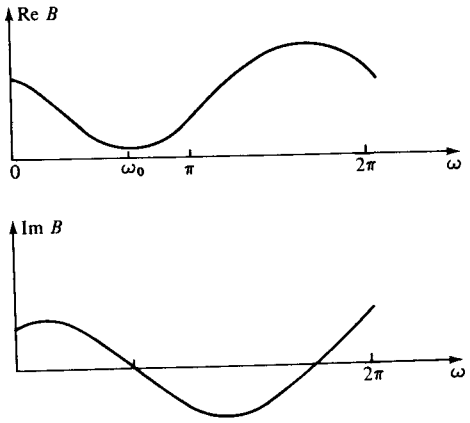
1 Find the filter which is inverse to $(2 - 5Z + 2Z^2)$. You may just drop higher-order powers of Z, but an exact expression for the coefficients of any power of Z is preferred. (Partial fractions is a useful, though not necessary, technique.) Sketch the impulse response.

2 Show that multiplication by $(1 - Z)$ in discretized time is analogous to time differentiation in continuous time. Show that dividing by $(1 - Z)$ is analogous to integration. What are the limits on the integral?

3 Describe a general method for determining $A(Z)$ and $B(Z)$ from a Taylor series of $B(Z)/A(Z) = C_0 + C_1Z + C_2Z^2 + \cdots + C_\infty Z^\infty$ where $B(Z)$ and $A(Z)$ are polynomials of unknown degree n and m, respectively. Work out the case $C(Z) = \frac{1}{2} - \frac{3}{4}Z - \frac{3}{8}Z^2 - \frac{3}{16}Z^3 - \frac{3}{32}Z^4 - \cdots$. Don't try this problem unless you are quite familiar with determinants. [HINT: Identify coefficients of $B(Z) = A(Z)C(Z)$.]

2-2 MINIMUM PHASE

In Sec. 2-1 we learned that knowledge of convergence of the Taylor series of $1/B(Z)$ on $|Z| = 1$ is equivalent to knowledge that $B(Z)$ has no roots inside the



2-4
maginary parts of the Z trans-
$Z/$ $(1.25\, e^{i2\pi/3})$.

ow we will see that these conditions are also equivalent to a certain
e phase of $B(Z)$ on the unit circle.
onsider the phase shift of the two-term filter

$$1 - \frac{Z}{Z_0} \qquad (Z_0 = \rho e^{i\omega_0})$$

$$1 - \rho^{-1} e^{i(\omega - \omega_0)}$$

$$1 - \rho^{-1}\cos(\omega - \omega_0) - i\rho^{-1}\sin(\omega - \omega_0)$$

phase is the arctangent of the ratio of the imaginary part to the real

of phase as a function of frequency looks radically different for $\rho < 1$
1. See Fig. 2-4 for the case $\rho > 1$.

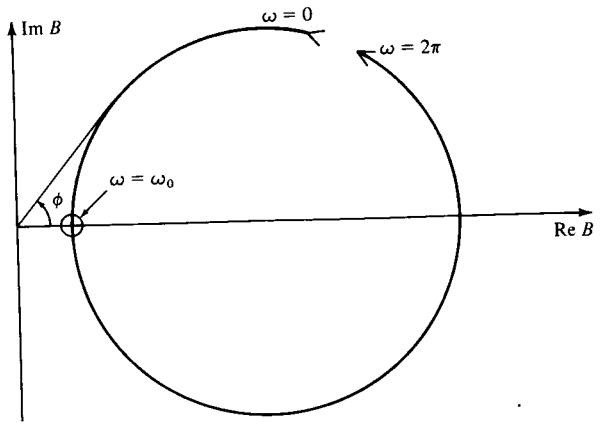


FIGURE 2-5
Phase of the two-term filter of Fig. 2-4.

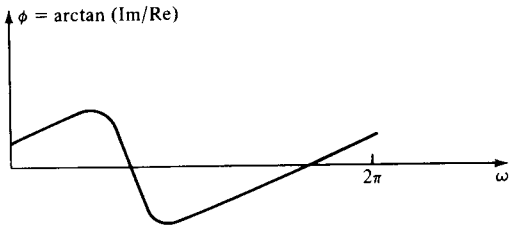


FIGURE 2-6
The phase of a two-term minimum-phase filter.

The phase is the arctangent of Im B/Re B. The easiest way to keep track of the phase is in the complex B plane. This is shown in Fig. 2-5.

Thus phase as a function of frequency is shown in Fig. 2-6. Notice that the phase ϕ at $\omega = 0$ is the same as the phase at $\omega = 2\pi$. This follows because the real and imaginary parts are periodic with 2π. The situation will be different when there is a zero inside the unit circle; that is, $\rho < 1$. The real and imaginary parts are shown in Fig. 2-7 and the complex plane in Fig. 2-8.

The phase ϕ increases by 2π as ω goes from zero to 2π because the circular path surrounds the origin. The phase curve is shown in Fig. 2-9. The case $\rho > 1$ where $\phi(\omega) = \phi(\omega + 2\pi)$ has come to be called *minimum phase* or *minimum delay*.

Now we are ready to consider a complicated filter like

$$B(Z) = \frac{(Z - c_1)(Z - c_2)\cdots}{(Z - a_1)(Z - a_2)\cdots} \qquad (2\text{-}2\text{-}1)$$

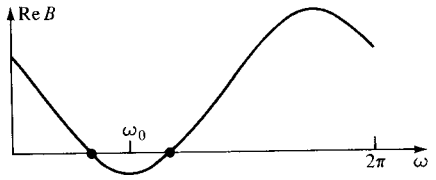
By the rules of complex-number multiplication the phase of $B(Z)$ is the sum of the phases in the numerator minus the sum of the phases in the denominator. Since we are discussing realizable filters the denominator factors must all be minimum phase, and so the denominator phase curve is a sum of curves like Fig. 2-6. The numerator factors may or may not be minimum phase. Thus the numerator phase curve is a sum of curves like either Fig. 2-6 or Fig. 2-9. If any factors at all are like Fig. 2-9, then the total phase will resemble Fig. 2-9 in that the phase at $\omega = 2\pi$ will be greater than the phase at $\omega = 0$. Then the filter will be nonminimum phase.

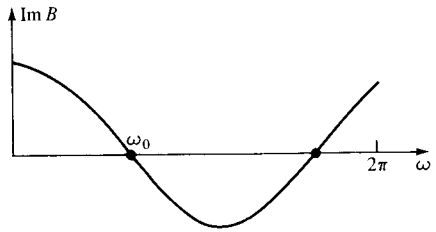
2-3 FILTERS IN PARALLEL

We have seen that in a cascade of filters the filter polynomials are multiplied together. One might conceive of adding two polynomials $A(Z)$ and $G(Z)$ when they correspond to filters which operate in parallel. See Fig. 2-10.

When filters operate in parallel their Z transforms add together. We have seen that a cascade of filters is minimum phase if, and only if, each element of the product is minimum phase. Now we will see a sufficient (but not necessary) condition that the sum $A(Z) + G(Z)$ be minimum phase. First of all, let us assume that $A(Z)$ is minimum phase. Then we may write

$$A(Z) + G(Z) = A(Z)\left[1 + \frac{G(Z)}{A(Z)}\right]$$





GURE 2-7
al and imaginary parts of the two-
m nonminimum-phase filter,
$1.25\,Z\,e^{-i2\pi/3}$.

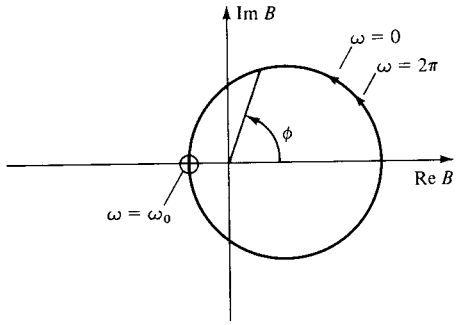


FIGURE 2-8
Phase in complex plane.

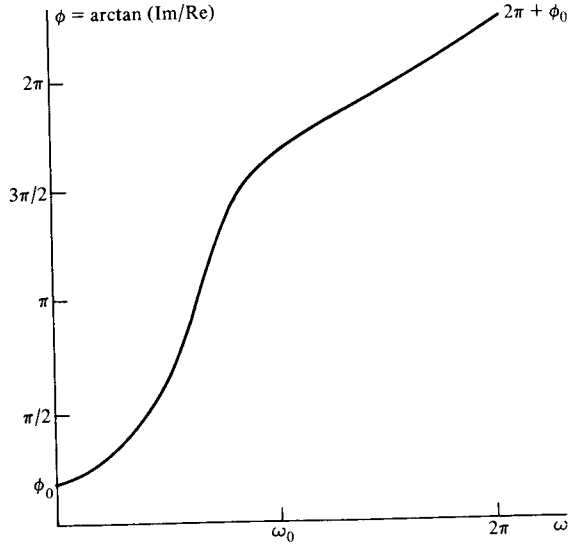


FIGURE 2-9
The phase of a two-term nonminimum-phase filter.

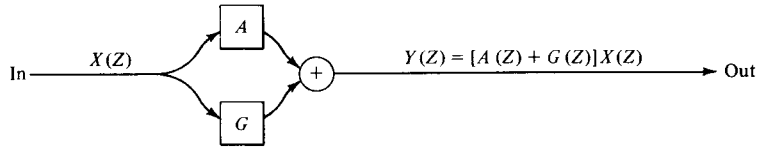


FIGURE 2-10
Filters operating in parallel.

The question of whether $A(Z) + G(Z)$ is minimum phase is now reduced to determining whether $A(Z)$ and $1 + G(Z)/A(Z)$ are both minimum phase. We have assumed that $A(Z)$ is minimum phase. Before we ask whether $1 + G(Z)/A(Z)$ is minimum phase we need to be sure that it's causal. Since $1/A(Z)$ is expandable in positive powers of Z only, then $G(Z)/A(Z)$ is also causal. We will next see that a sufficient condition for $1 + G(Z)/A(Z)$ to be minimum phase is that the spectrum of A exceeds that of G at all frequencies. In other words, for any real ω, $|A| > |G|$. Thus, if we plot the curve of $G(Z)/A(Z)$ in the complex plane, for real $0 \le \omega \le 2\pi$ it lies everywhere inside the unit circle. Now if we add unity—getting $1 + G(Z)/A(Z)$, the curve will always have a positive real part. See Fig. 2-11. Since the curve cannot enclose the origin, the phase must be that of a minimum-phase function. In words, "You can add garbage to a minimum-phase wavelet if you do not add too much." This somewhat abstract theorem has an immediate physical consequence. Suppose a wave characterized by a minimum phase $A(Z)$ is emitted from a source and detected at a receiver some time later. At a still later time an echo bounces off a nearby object and is also detected at the receiver. The receiver sees the signal $Y(Z) = A(Z) + Z^n\alpha A(Z)$ where n measures the delay from the first arrival to the echo and α represents the amplitude attenuation of the echo. To see that $Y(Z)$ is minimum phase, we note that the magnitude of Z^n is unity and that the reflection coefficient α must be less than unity (to avoid perpetual motion) so that $Z^n\alpha A(Z)$ takes the role of $G(Z)$. Thus a minimum-phase wave along with its echo is minimum phase. We will later consider wave propagation situations with echoes of the echoes ad infinitum.

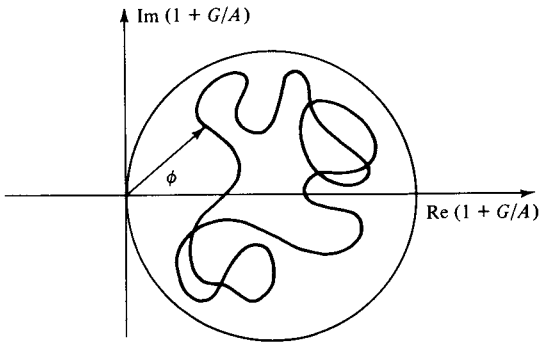


FIGURE 2-11
Phase of a positive real function lies between $\pm\pi/2$.

nminimum-phase wavelets whose sum is minimum phase.
a minimum-phase polynomial of degree N. Let $A'(Z) = Z^N\bar{A}(1/Z)$. Locate
ex Z plane the roots of $A'(Z)$. $A'(Z)$ is called *maximum phase*. [HINT: Work
ase $A(Z) = a_0 + a_1 Z$ first.]
Z) is maximum phase and that the degree of $G(Z)$ is less than or equal to
f $A(Z)$. Assume $|A| > |G|$. Show that $A(Z) + G(Z)$ is maximum phase.
minimum phase. Where are the roots of $A(Z) + cZ^N\bar{A}(1/Z)$ in the three
1, $|c| > 1$, $|c| = 1$? (HINT: The roots of a polynomial are continuous
the polynomial coefficients.)

IVE REAL FUNCTIONS

ypes of functions called *admittance functions* $Y(Z)$ and *impedance*
occur in many physical problems. In electronics, they are ratios of
tage and of voltage to current; in acoustics, impedance is the ratio of
locity. When the appropriate electrical network or acoustical region
ources of energy, then these ratios have the positive real property. To
echanical example, we may imagine applying a known force $F(Z)$ and
resulting velocity $V(Z)$. In filter theory, it is like considering that
to a filter $Y(Z)$ giving output $V(Z)$. We have

$$V(Z) = Y(Z)F(Z) \qquad (2\text{-}4\text{-}1)$$

Z) is obviously causal. Since we believe we can do it the other way
is, prescribe the velocity and observe the force, there must exist a
usal $I(Z)$ such that

$$F(Z) = I(Z)V(Z) \qquad (2\text{-}4\text{-}2)$$

are inverses of one another and since they are both presumed bounded
hen they both must be minimum phase.
efore we consider any physics, note that if the complex number $a + ib$
real part a, then the real part of $(a + ib)^{-1}$ namely $a/(a^2 + b^2)$ is also
ing $a + ib$ to represent a value of $Y(Z)$ or $I(Z)$ on the unit circle, we
us fact that if either Y or I has the positive real property, then the
o.
dissipated is the product of force times velocity, that is

$$\text{Power} = \cdots + f_0 v_0 + f_1 v_1 + f_2 v_2 + \cdots \qquad (2\text{-}4\text{-}3)$$

expressed in terms of Z transforms as

$$\begin{aligned}\text{Power} &= \frac{1}{2}\,\text{coeff of } Z^0 \text{ of } V\left(\frac{1}{Z}\right)F(Z) + F\left(\frac{1}{Z}\right)V(Z) \\ &= \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[V\left(\frac{1}{Z}\right)F(Z) + F\left(\frac{1}{Z}\right)V(Z)\right]d\omega\end{aligned} \qquad (2\text{-}4\text{-}4)$$

Using (2-4-1) to eliminate $V(Z)$ we get

$$\text{Power} = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi}F\left(\frac{1}{Z}\right)\left[Y\left(\frac{1}{Z}\right) + Y(Z)\right]F(Z)\,d\omega$$

We note that $Y(Z) + Y(1/Z)$ looks superficially like a spectrum because the coefficient of Z^k equals that of Z^{-k}, which shows the symmetry of an autocorrelation function. Defining

$$R(Z) = Y(Z) + Y\left(\frac{1}{Z}\right) \qquad (2\text{-}4\text{-}5)$$

(2-4-4) becomes

$$\text{Power} = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi}R(Z)F\left(\frac{1}{Z}\right)F(Z)\,d\omega \qquad (2\text{-}4\text{-}6)$$

The integrand is the product of the arbitrary positive input force spectrum and $R(Z)$. If the power dissipation is expected to be positive at all frequencies (for all $\bar{F}F$), then obviously $R(Z)$ must be positive at all frequencies; thus R is indeed a spectrum. Since we have now discovered that $Y(Z) + Y(1/Z)$ must be positive for all frequencies, we have discovered that $Y(Z)$ is not an arbitrary minimum-phase filter. The real part of both $Y(Z)$ and $Y(1/Z)$ is

$$\text{Re}[Y(Z)] = \text{Re}\left[Y\left(\frac{1}{Z}\right)\right] = y_0 + y_1\cos\omega + y_2\cos 2\omega + \cdots$$

Since the real part of the sum must be positive, then obviously the real part of each of the equal parts must be positive.

Now if the material or mechanism being studied is passive (contains no energy sources) then we must have positive dissipation over a time gate from minus infinity up to any time t. Let us find an expression for dissipation in such a time gate. For simplicity take both the force and velocity vanishing before $t = 0$. Let the end of the time gate include the point $t = 2$ but not $t = 3$.

Define

$$f'_t = \begin{cases} f_t & t \leqslant 2 \\ 0 & t > 2 \end{cases} \qquad (2\text{-}4\text{-}7)$$

To find the work done over all time we may integrate (2-4-6) over all frequencies. To find the work done in the selected gate we may replace F by F' and integrate over all frequencies, namely

$$W_2 = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi}F'\left(\frac{1}{Z}\right)R(Z)F'(Z)\,d\omega \qquad (2\text{-}4\text{-}8)$$

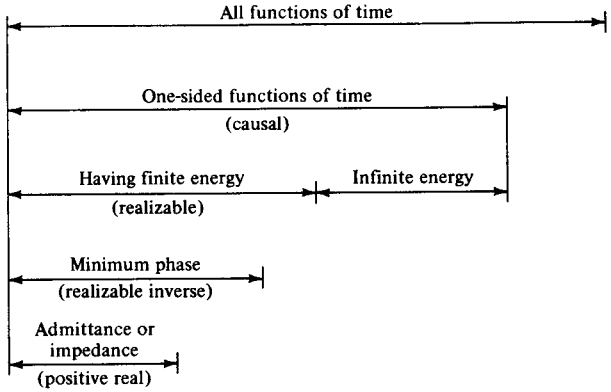


FIGURE 2-12
Important classes of time functions.

en, this integral merely selects the coefficient of Z^0 of the integrand. his out. First, collect coefficients of powers of Z in $R(Z)F'(Z)$. We

$$Z^0: \quad r_0 f_0' + r_{-1} f_1' + r_{-2} f_2'$$
$$Z^1: \quad r_1 f_0' + r_0 f_1' + r_{-1} f_2'$$
$$Z^2: \quad r_2 f_0' + r_1 f_1' + r_0 f_2'$$

coefficient of Z^0 in $F'(1/Z)[R(Z)\,F'(Z)]$ we must multiply the top row the second row by f_1' and the third row by f_2'. The result can be very orderly fashion by

$$W_2 = \tfrac{1}{2}[f_0 \quad f_1 \quad f_2]\begin{bmatrix} r_0 & r_{-1} & r_{-2} \\ r_1 & r_0 & r_{-1} \\ r_2 & r_1 & r_0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$= \tfrac{1}{2}[f_0 \quad f_1 \quad f_2]\begin{bmatrix} 2y_0 & y_1 & y_2 \\ y_1 & 2y_0 & y_1 \\ y_2 & y_1 & 2y_0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \qquad (2\text{-}4\text{-}9)$$

st the 3×3 quadratic form (2-4-9) be positive (i.e., $W_2 \geqslant 0$ for ut all $t \times t$ similar quadratic forms W_t must be positive.
lusion, the positive real property in the frequency domain means that Z) is positive for any real ω and the positive real property in the time ns that all $t \times t$ matrices like that of (2-4-9) are positive definite. ummarizes the function types which we have considered.

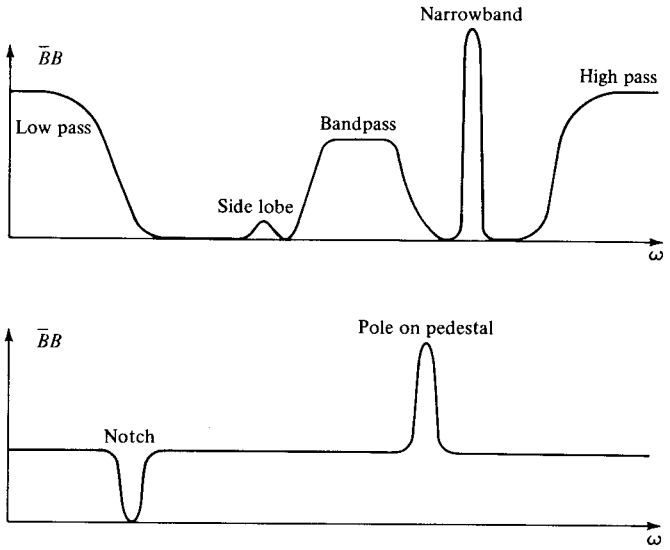


FIGURE 2-13
Spectra of various filters.

EXERCISES

1 In mechanics we have force and velocity of a free unit mass related by $dv/dt = f$ or $v = \int_{-\infty} f\,dt$. Compute the power dissipated as a function of frequency if integration is approximated by convolution with $(.5, 1., 1., 1., \ldots)$. [HINT: Expand $(1+Z)/2(1-Z)$ in positive powers of Z.]
2 Construct an example of a simple function which is minimum phase but not positive real.

2-5 NARROW-BAND FILTERS

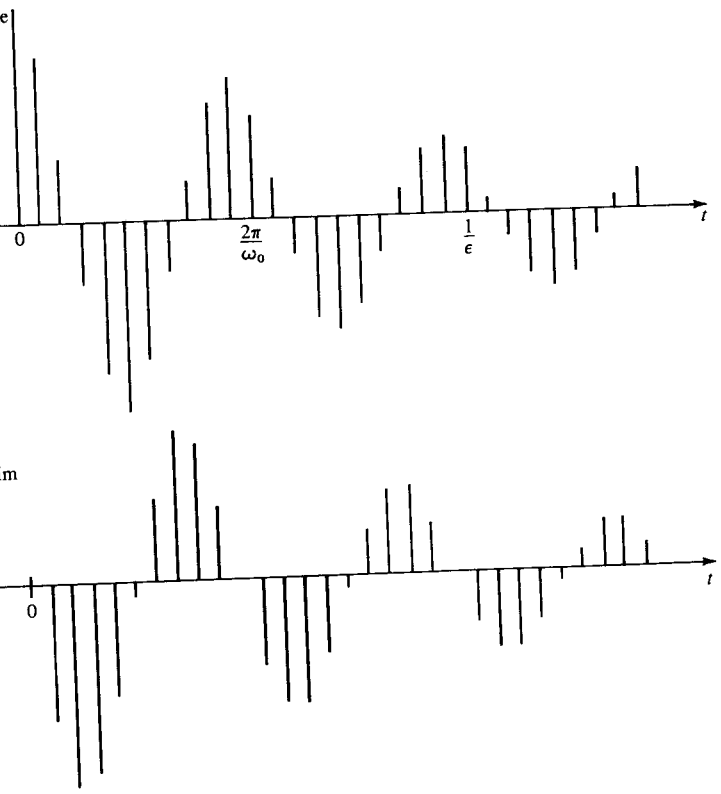
Filters are often used to modify the spectrum of given data. With input $X(Z)$, filter $B(Z)$, and output $Y(Z)$ we have $Y(Z) = B(Z)X(Z)$ and the Fourier conjugate $\overline{Y}(1/Z) = \overline{B}(1/Z)\overline{X}(1/Z)$. Multiplying these two relations together we get

$$\overline{Y}Y = (\overline{B}B)(\overline{X}X)$$

which says that the spectrum of the input times the spectrum of the filter equals the spectrum of the output. Filters are often characterized by the shape of their spectra. Some examples are shown in Fig. 2-13.

We will have frequent occasion to deal with sinusoidal time functions. A simple way to represent a sinusoid by Z transforms is

$$\frac{1}{1 - Ze^{i\omega_0}} = 1 + Ze^{i\omega_0} + Z^2 e^{i2\omega_0} + \cdots \qquad (2\text{-}5\text{-}1)$$



URE 2-14
time function associated with a simple pole just outside the unit circle at $1.1\,e^{i\pi/5}$.

n associated with this Z transform is $e^{i\omega_0 t}$, but it is "turned on" at the left-hand side of (2-5-1) contains a pole exactly on the unit series sits on the borderline between convergence and divergence. aradoxical situations [you could expand (2-5-1) so that the sinusoid] which we will avoid by pushing the pole from the unit circle to a outside the unit circle. Let $Z_0 = (1 + \varepsilon)e^{i\omega_0}$. Then define

$$B(Z) = \frac{1}{A(Z)} = \frac{1}{1 - Z/Z_0}$$
$$= 1 + \frac{Z}{Z_0} + \left(\frac{Z}{Z_0}\right)^2 + \cdots \qquad (2\text{-}5\text{-}2)$$

on corresponding to $B(Z)$ is zero before $t = 0$ and is $e^{-i\omega_0 t}/(1 + \varepsilon)^t$ a sinusoidal function which decreases gradually with time accord- The coefficients are shown in Fig. 2-14.

It is intuitively obvious, although we will prove it later, that convolution with the coefficients of (2-5-2), which are sketched in Fig. 2-14, is a narrow-banded filtering operation. If the pole is chosen very close to the unit circle, the filter bandpass becomes narrower and narrower and the coefficients of $B(Z)$ drop off more and more slowly. To actually perform the convolution it is necessary to truncate, that is, to drop powers of Z beyond a certain practical limit. It turns out that there is a very much cheaper method of narrow-band filtering than convolution with the coefficients of $B(Z)$. This method is polynomial division by $A(Z)$. We have for the output $Y(Z)$

$$Y(Z) = B(Z)X(Z) \qquad (2\text{-}5\text{-}3)$$

$$Y(Z) = \frac{X(Z)}{A(Z)} \qquad (2\text{-}5\text{-}4)$$

Multiply both sides of (2-5-4) by $A(Z)$

$$Y(Z)A(Z) = X(Z) \qquad (2\text{-}5\text{-}5)$$

For definiteness, let us suppose the x_t and y_t vanish before $t = 0$. Now identify coefficients of successive powers of Z. We get

$$\begin{aligned} y_0 a_0 &= x_0 \\ y_1 a_0 + y_0 a_1 &= x_1 \\ y_2 a_0 + y_1 a_1 + y_0 a_2 &= x_2 \\ y_3 a_0 + y_2 a_1 + y_1 a_2 + y_0 a_3 &= x_3 \\ &\text{etc.} \end{aligned} \qquad (2\text{-}5\text{-}6)$$

A general equation is

$$y_k a_0 + \sum_{i=1}^{\infty} y_{k-i} a_i = x_k \qquad (2\text{-}5\text{-}7)$$

Solving for y_k we get

$$y_k = \frac{x_k - \sum_{i=1}^{\infty} y_{k-i} a_i}{a_0} \qquad (2\text{-}5\text{-}8)$$

Equation (2-5-8) may be used to solve for y_k once $y_{k-1}, y_{k-2}, \ldots$ are known. Thus the solution is recursive, and it will not diverge if the a_i are coefficients of a minimum-phase polynomial. In practice the infinite limit on the sum is truncated whenever you run out of coefficients of either $A(Z)$ or $Y(Z)$. For the example we have been considering, $B(Z) = 1/A(Z) = 1/(1 - Z/Z_0)$, there will be only one term in the sum. Filtering in this way is called *feedback filtering*, and for narrowband filtering it will be vastly more economical than filtering by convolution, since there

coefficients in $A(Z)$ than $B(Z) = 1/A(Z)$. Finally, let us examine the Z). We have

$$A(Z) = 1 - \frac{Z}{Z_0}$$

$$= 1 - \frac{e^{i\omega}}{(1+\varepsilon)e^{i\omega_0}}$$

$$= 1 - \frac{e^{i(\omega-\omega_0)}}{(1+\varepsilon)}$$

$$\bar{A}\left(\frac{1}{Z}\right) = 1 - \frac{e^{-i(\omega-\omega_0)}}{1+\varepsilon}$$

$$\left.\right) A(Z) = \left(1 - \frac{e^{-i(\omega-\omega_0)}}{1+\varepsilon}\right)\left(1 - \frac{e^{i(\omega-\omega_0)}}{1+\varepsilon}\right)$$

$$= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{1}{1+\varepsilon}\left(e^{-i(\omega-\omega_0)} + e^{i(\omega-\omega_0)}\right)$$

$$= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{2\cos(\omega-\omega_0)}{1+\varepsilon}$$

$$= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{2}{1+\varepsilon} + \frac{2}{1+\varepsilon}[1 - \cos(\omega-\omega_0)]$$

$$= \left(1 - \frac{1}{1+\varepsilon}\right)^2 + \frac{4}{1+\varepsilon}\sin^2\frac{\omega-\omega_0}{2}$$

$$\left.\frac{1}{Z}\right) B(Z) = \frac{(1+\varepsilon)^2}{\varepsilon^2 + 4(1+\varepsilon)\sin^2\left(\dfrac{\omega-\omega_0}{2}\right)} \qquad (2\text{-}5\text{-}9)$$

pproximation this function may be thought of as $1/[\varepsilon^2 + (\omega - \omega_0)^2]$. 5-9) is shown in Fig. 2-15.

should be apparent why this is called a narrowband filter. It amplifies w band of frequencies and attenuates all others. The frequency window s said to be $\Delta\omega \approx 2\varepsilon$ in width. The time window is $\Delta t = 1/\varepsilon$, the damp- stant of the damped sinusoid b_t.

ractical disadvantage of the filter under discussion is that although its e a real time series its output will be a complex time series. For many a filter with real coefficients may be preferred.

pproach is to follow the filter $[1, e^{i\omega_0}/(1+\varepsilon)]$ by the time-domain, njugate filter $[1, e^{-i\omega_0}/(1+\varepsilon)]$. The composite time-domain operator is

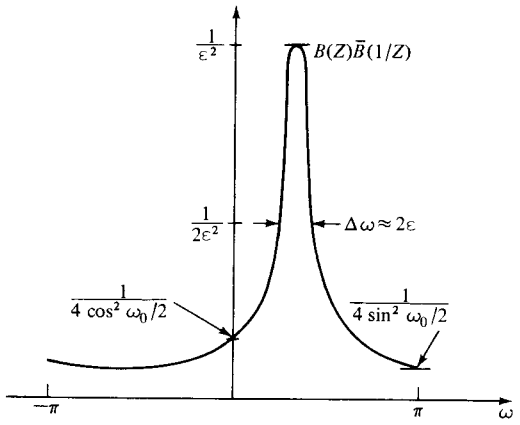


FIGURE 2-15
Spectrum associated with a single pole at $Z_0 = (1+\varepsilon)e^{i\omega_0}$.

now $[1, (2\cos\omega_0)/(1+\varepsilon), 1/(1+\varepsilon)^2]$ which is real. [Note that the complex conjugate in the frequency domain is $\bar{B}(1/Z)$ but in the time domain it is $\bar{B}(Z) = \bar{b}_0 + \bar{b}_1 Z + \cdots$]. The composite filter may be denoted by $B(Z)\bar{B}(Z)$. The spectrum of this filter is $[B(Z)\bar{B}(1/Z)][\bar{B}(Z)B(1/Z)]$. One may quickly verify that the spectrum of $\bar{B}(Z)$ is like that of $B(Z)$, but the peak is at $-\omega_0$ instead of $+\omega_0$. Thus, the composite spectrum is the product of Fig. 2-15 with itself reversed along the frequency axis. This is shown in Fig. 2-16.

EXERCISES

1 A simple feedback operation is $y_t = (1-\varepsilon)y_{t-1} + x_t$. This operation is called leaky integration. Give a closed form expression for the output y_t if x_t is an impulse. What is the decay time τ of your solution (the time it takes for y_t to drop to $e^{-1}y_0$)? For small ε, say $= 0.1$, .001, or 0.0001, what is τ?

2 How far from the unit circle are the poles of $1/(1 - .1Z + .9Z^2)$? What is the decay time of the filter and its resonant frequency?

3 Find a three-term real feedback filter to pass 59–61 Hz on data which are sampled at 500 points/sec. Where are the poles? What is the decay time of the filter?

2-6 ALL-PASS FILTERS

In this section we consider filters with constant unit spectra, that is, $B(Z)\bar{B}(1/Z) = 1$. In other words, in the frequency domain $B(Z)$ takes the form $e^{i\phi(\omega)}$ where ϕ is real and is called the *phase shift*. Clearly $B\bar{B} = 1$ for all real ϕ. It is an easy matter to construct a filter with any desired phase shift; one merely Fourier transforms $e^{i\phi(\omega)}$ into the time domain. If $\phi(\omega)$ is arbitrary, the resulting time function is likely to be two-sided. Since we are interested in physical processes which are

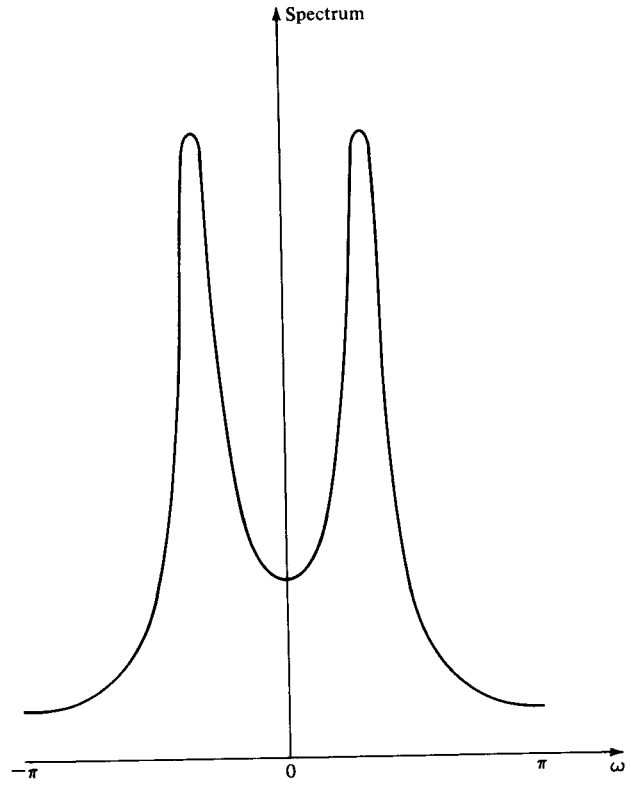


FIGURE 2-16
Spectrum of a two-pole filter where one pole is like Fig. 2-15 and the other is at the conjugate position.

ay wonder what class of functions $\phi(\omega)$ corresponds to one-sided time he easiest way to proceed is to begin with a simple case of a single-ero all-pass filter. Then more elaborate all-pass filters can be made up these simple filters. Consider the filter

$$P(Z) = \frac{Z - 1/\bar{Z}_0}{1 - Z/Z_0} \qquad (2\text{-}6\text{-}1)$$

is is a simple case of functions of the form $Z^N\bar{A}(1/Z)/A(Z)$, where $A(Z)$ nial of degree N or less. Now observe that the spectrum of the filter p_t requency-independent constant. The spectrum is

$$\bar{P}\left(\frac{1}{Z}\right)P(Z) = \frac{1/Z - 1/Z_0}{1 - 1/(Z\bar{Z}_0)}\frac{Z - 1/\bar{Z}_0}{Z - 1/Z_0} \qquad (2\text{-}6\text{-}2)$$

p and bottom on the left by Z. We now have

$$\bar{P}\left(\frac{1}{Z}\right)P(Z) = \frac{1 - Z/Z_0}{Z - 1/\bar{Z}_0}\frac{Z - 1/\bar{Z}_0}{1 - Z/Z_0} = 1 \qquad (2\text{-}6\text{-}3)$$

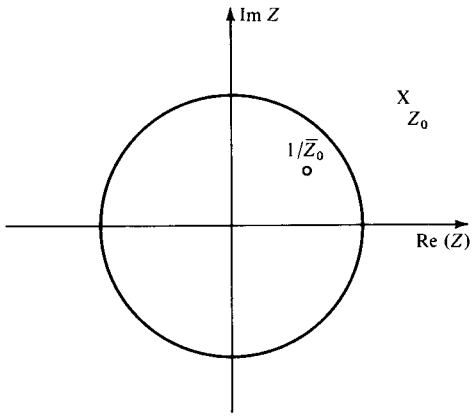


FIGURE 2-17
The pole of the all-pass filter lies outside the unit circle and the zero is inside. They lie on the same radius line.

It is easy to show that $\bar{P}(1/Z)P(Z) = 1$ for the general form $P(Z) = Z^N\bar{A}(1/Z)/A(Z)$. If Z_0 is chosen outside the unit circle, then the denominator of (2-6-1) can be expanded in positive powers of Z and the expansion is convergent on the unit circle. This means that causality is equivalent to Z_0 outside the unit circle. Setting the numerator of $P(Z)$ equal to zero, we discover that the zero $Z = 1/\bar{Z}_0$ is then inside the unit circle. The situation is depicted in Fig. 2-17. To see that the pole and zero are on the same radius line, express Z_0 in polar form $r_0 e^{i\phi_0}$.

From Sec. 2-2 (on minimum phase) we see that the numerator of P is not minimum phase and its phase is augmented by 2π as ω goes from 0 to 2π. Thus the average group delay $d\phi/d\omega$ is positive. Not only is the average positive but, in fact, the group delay turns out to be positive at every frequency. To see this, first note that

$$Z = e^{i\omega}$$

$$\frac{dZ}{d\omega} = ie^{i\omega} = iZ$$

$$\frac{d}{d\omega} = \frac{dZ}{d\omega}\frac{d}{dZ} = iZ\frac{d}{dZ} \qquad (2\text{-}6\text{-}4)$$

The phase of the all-pass filter (or any complex number) may be written as

$$\phi = \operatorname{Im} \ln P(Z) \qquad (2\text{-}6\text{-}5)$$

Since $|P| = 1$ the real part of the log vanishes; and so, for the all-pass filter (only) we may specialize (2-6-5) to

$$\phi = \frac{1}{i}\ln P(Z) = \frac{1}{i}\ln\frac{Z - 1/\bar{Z}_0}{1 - Z/Z_0}$$

$$= \frac{1}{i}\left[\ln\left(Z - \frac{1}{\bar{Z}_0}\right) - \ln\left(1 - \frac{Z}{Z_0}\right)\right] \qquad (2\text{-}6\text{-}6)$$

the group delay is now found to be

$$\frac{d\phi}{d\omega} = iZ\frac{d\phi}{dZ} = Z\left(\frac{1}{Z - 1/\bar{Z}_0} + \frac{1/Z_0}{1 - Z/Z_0}\right)$$

$$\frac{1}{1 - 1/\bar{Z}_0 Z} + \frac{Z/Z_0}{1 - Z/Z_0}$$

$$\frac{1 - Z/Z_0 + (1 - 1/\bar{Z}_0 Z)(Z/Z_0)}{(1 - 1/\bar{Z}_0 Z)(1 - Z/Z_0)} = \frac{1 - 1/Z_0\bar{Z}_0}{(1 - 1/\bar{Z}_0 Z)(1 - Z/Z_0)} \qquad (2\text{-}6\text{-}7)$$

or of (2-6-7) is a positive real number (since $|Z_0| > 1$), and the de- of the form $\bar{A}(1/Z)A(Z)$, which is a spectrum and also positive. Thus vn that the group delay of this causal all-pass filter is always positive. we take a filter and follow it with an all-pass filter, the phases add and lay of the composite filter must necessarily be greater than the group original filter. By the same reasoning the minimum-phase filter must up delay than any other filter with the same spectrum.

mary, a single-pole, single-zero all-pass filter passes all frequency with constant gain and a phase shift which may be adjusted by the a pole. Taking Z_0 near the unit circle causes most of the phase shift trated near the frequency where the pole is located. Taking the pole causes the delay to be spread over more frequencies. Complicated or group delays may be built up by cascading several single-pole filters.

ES

le of an all-pass filter is the time function $p_t = (\frac{1}{2}, -\frac{3}{4}, -\frac{3}{8}, -\frac{3}{16} \cdots)$. Calcu- lags of its autocorrelation by summing some infinite series.

e amplitude, phase, and group delay of the all-pass filter $(1 - \bar{Z}_0 Z)/(Z_0 - Z)$ $= (1 + \varepsilon)e^{i\omega_0}$ and ε is small. Indicate important parameters on the curve.

t the coefficients of an all-pass, phase-shifting filter made by cascading $)/(Z_0 - Z)$ with $(1 - Z_0 Z)/(\bar{Z}_0 - Z)$ are real.

uous time function is the impulse response of a continuous-time, all-pass scribe the function in both time domain and frequency domain. Interchange s *time* and *frequency* in your description of the function. What is a physical of such a function? What happens to the statement: "The group delay of an lter is positive."?

of the group delay $\tau_g(\omega)$ in equation (2-6-7) shows τ_g to be positive for all ω. he area under τ_g in the range $0 < \omega < 2\pi$. (HINT: This is a trick question you e in your head.)

CH FILTER AND POLE ON PEDESTAL

plications it is desired to reject a very narrow frequency band leaving the spectrum little changed. The most common example is 60-Hz noise r lines. Such a filter can easily be made with a slight variation on the

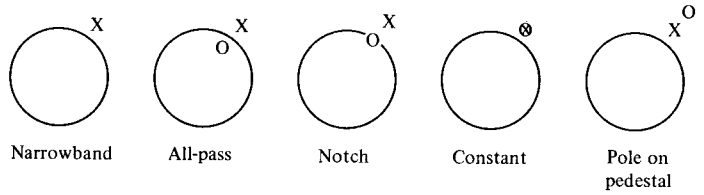


FIGURE 2-18
Pole and zero locations for some simple filters. Circles are unit circles in the Z plane. Poles are marked by X and zeros by 0.

all-pass filter. In the all-pass filter the pole and zero have an equal (logarithmic) relative distance from the unit circle. All we need to do is to put the zero closer to the circle. In fact, there is no reason why we should not put the zero right on the circle. Then the frequency at which the zero is located is exactly canceled from the spectrum of input data. If the undesired frequency need not be completely rejected, then the zero can be left just inside or outside the circle. As the zero is moved farther away from the circle, the notch becomes less deep until finally the zero is farther from the circle than the pole and the notch has become a hump. The resulting filter which will be called *pole on pedestal* is in many respects like the narrowband filter discussed earlier. Some of these filters are illustrated in Figs. 2-18 and 2-19. The difference between the pole-on-pedestal and the narrowband filters is in the asymptotic behavior away from ω_0. The former is flat, while the latter continues to decay with increasing $|\omega - \omega_0|$. This makes the pole on pedestal more convenient for creating complicated filter shapes by cascades of single-pole filters.

Narrowband filters and sharp cutoff filters should be used with caution. An ever-present penalty for such filters is that they do not decay rapidly in time. Although this may not present problems in some applications, it will do so in others. Obviously, if the data collection duration is shorter or comparable to the impulse response of the narrowband filter, then the transient effects of starting up the experiment will not have time to die out. Likewise, the notch should not be too narrow in a 60-Hz rejection filter. Even a bandpass filter (easier to implement with fast Fourier transform than with a few poles) has a certain decay rate in the time domain which may be too slow for some experiments. In radar and in reflection seismology the importance of a signal is not related to its strength. Late-arriving echoes may be very weak, but they contain information not found in earlier echoes. If too sharp a frequency characteristic is used, then filter resonance from early strong arrivals may not have decayed sufficiently by the time that the weak late echoes arrive.

EXERCISES

1 Consider a symmetric (nonrealizable) filter which passes all frequencies less than ω_0 with unit gain. Frequencies above ω_0 are completely attenuated. What is the rate of decay of amplitude with time for this filter?

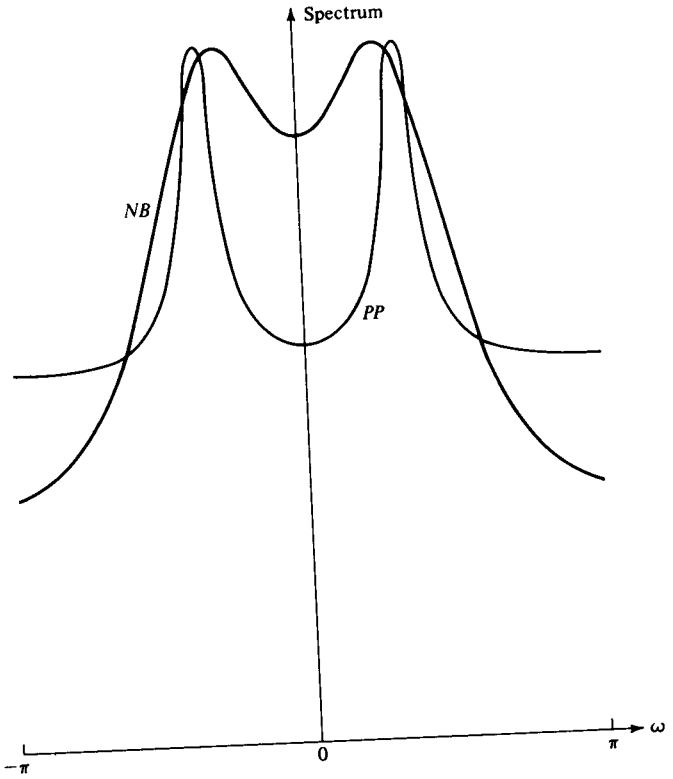


FIGURE 2-19
Amplitude vs. frequency for narrowband filter (NB) and pole-on-pedestal filter (PP). Each has one pole at $Z_0 = 1.2\,e^{i\pi/3}$. A second pole at $Z_0 = 1.2\,e^{-i\pi/3}$ enables the filters to be real in the time domain.

ading from a point source decay in energy as the area on a sphere. The lecays as the square root of the energy. This implies a certain decay in time-decay rate is the same if the waves reflect from planar interfaces. wer of time t do the signal amplitudes decay? For waves backscattered to rom point reflectors, energy decays as distance to the minus fourth power. associated decay with time?
use of the filter of Exercise 1 on the data of Exercise 2.
ngle-pole, single-zero notch filter to reject 59 to 61 Hz on data which are 500 points per second.

BILINEAR TRANSFORM

s and Fourier transforms are related by the relations $Z = e^{i\omega}$ and problem with these relations is that simple ratios of polynomials in Z ate to ratios of polynomials in ω and vice versa. The approximation

$$-i\hat{\omega} = 2\,\frac{1 - Z}{1 + Z} \qquad (2\text{-}8\text{-}1)$$

is easily solved for Z as

$$Z = \frac{1 + i\hat{\omega}/2}{1 - i\hat{\omega}/2} \qquad (2\text{-}8\text{-}2)$$

These approximations are often useful. They are truncations of the exact power series expansions

$$-i\omega = -\ln e^{i\omega} = -\ln Z = 2\left[\frac{1 - Z}{1 + Z} + \frac{1}{3}\frac{(1 - Z)^3}{(1 + Z)^3} + \frac{1}{5}\cdots\right] \qquad (2\text{-}8\text{-}3)$$

and

$$Z = e^{i\omega} = \frac{e^{i\omega/2}}{e^{-i\omega/2}} = \frac{1 + i\omega/2 + (i\omega/2)^2/2! + \cdots}{1 - i\omega/2 + (i\omega/2)^2/2! + \cdots} \qquad (2\text{-}8\text{-}4)$$

For a Z transform $B(Z)$ to be minimum phase, any root Z_0 of $0 = B(Z_0)$ should be outside the unit circle. Since $Z_0 = \exp\{i[\mathrm{Re}\,(\omega_0) + i\,\mathrm{Im}\,(\omega_0)]\}$ and $|Z_0| = e^{-\mathrm{Im}(\omega_0)}$, it means that for minimum phase $\mathrm{Im}\,(\omega_0)$ should be negative. (In other words, ω_0 is in the lower half-plane.) Thus it may be said that $Z = e^{i\omega}$ maps the exterior of the unit circle to the lower half-plane. By inspection of Figs. 2-20 and 2-21, it is found that the bilinear approximation (2-8-1) or (2-8-2) also maps the exterior of the unit circle into the lower half-plane.

Thus, although the bilinear approximation is an approximation, it turns out to exactly preserve the minimum-phase property. This is very fortunate because if a stable differential equation is converted to a difference equation via (2-8-1), the resulting difference equation will be stable. (Many cases may be found where the approximation of a time derivative by multiplication with $1 - Z$ would convert a stable differential equation into an unstable difference equation.)

A handy way to remember (2-8-1) is that $-i\omega$ corresponds to time differentiation of a Fourier transform and $(1 - Z)$ is the first differencing operator. The $(1 + Z)$ in the denominator gets things "centered" at $Z^{1/2}$.

To see that the bilinear approximation is a low-frequency approximation, multiply top and bottom of (2-8-1) by $Z^{-1/2}$

$$-i\hat{\omega} = 2\,\frac{Z^{-1/2} - Z^{1/2}}{Z^{-1/2} + Z^{1/2}}$$

$$= -2i\,\frac{\sin\omega/2}{\cos\omega/2}$$

$$\hat{\omega} = 2\tan\omega/2 \qquad (2\text{-}8\text{-}5)$$

Equation (2-8-5) implicitly refers to a sampling rate of one sample per second. Taking an arbitrary sampling rate Δt, the approximation (2-8-5) becomes

$$\omega\,\Delta t \approx 2\tan\omega\,\Delta t/2 \qquad (2\text{-}8\text{-}6)$$

This approximation is plotted in Fig. 2-22. Clearly, the error can be made as small as one wishes merely by sampling often enough; that is, taking Δt small enough.

	Z	$\omega = 2\pi n - i \ln Z$	$\hat{\omega} = 2i\frac{1-Z}{1+Z}$
A	1	$2\pi n + 0$	0
B	i	$2\pi n + \pi/2$	2
C	-1	$2\pi n + \pi$	$\pm\infty$
D	$-i$	$2\pi n - \pi/2$	-2
E	$\frac{1}{2}$	$2\pi n + .693i$	$i\frac{2}{3}$
F	2	$2\pi n - .693i$	$-i\frac{2}{3}$

E 2-20
pical points in the Z-plane, the
, and the $\hat{\omega}$-plane.

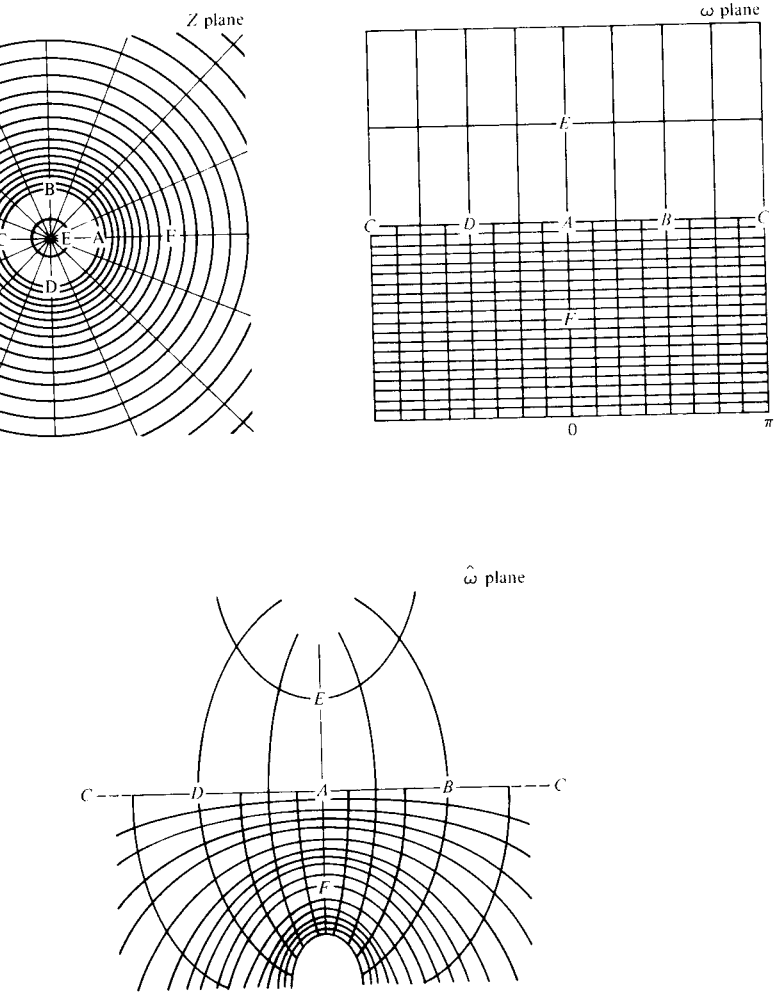


FIGURE 2-21
The points of Fig. 2-20 displayed in the Z plane, the ω plane, and the $\hat{\omega}$-plane.

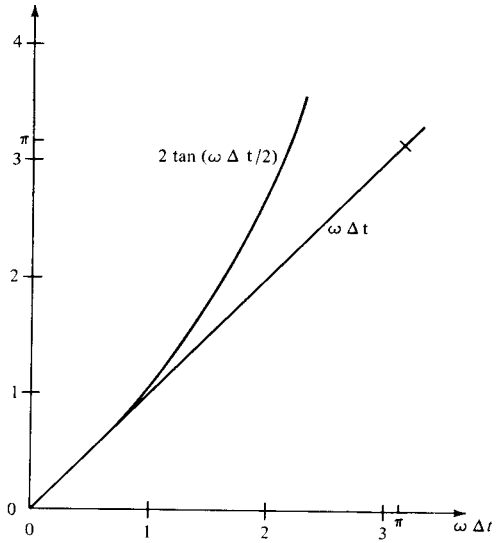


FIGURE 2-22
The accuracy of the bilinear transformation approximation.

From Fig. 2-22 we see that the error will be only a few percent if we choose Δt small enough so that $\omega_{max}\Delta t \leq 1$. Readers familiar with the folding theorem will recall that it gives the less severe restraint $\omega_{max}\Delta t < \pi$. Clearly, the folding theorem is too generous for applications involving the bilinear transform.

Now, by way of example, let us take up the case of a pole $1/-i\omega$ at zero frequency. This is integration. For reasons which will presently be clear, we will consider the slightly different pole

$$P = \frac{1}{-i\omega + \varepsilon} \qquad (2\text{-}8\text{-}7)$$

where ε is small. Inserting the bilinear transform, we get

$$P = \frac{1}{2[(1-Z)/(1+Z)] + \varepsilon} = \frac{0.5(1+Z)}{1 - Z + \varepsilon[(1+Z)/2]}$$

$$= \frac{0.5(1+Z)}{(1+\varepsilon/2) - Z(1-\varepsilon/2)} \qquad (2\text{-}8\text{-}8)$$

By inspection of (2-8-8) we see that the time-domain function is real, and as ε goes to zero it takes the form (.5, 1, 1, 1, . . .). (Taking ε positive forces the step to go out into positive time, whereas ε negative would cause the step to rise at negative time.) The properties of this function are summarized in Fig. 2-23. It is curious to note that if time domain and frequency domain are switched around, we have the quadrature filter described in Fig. 1–17.

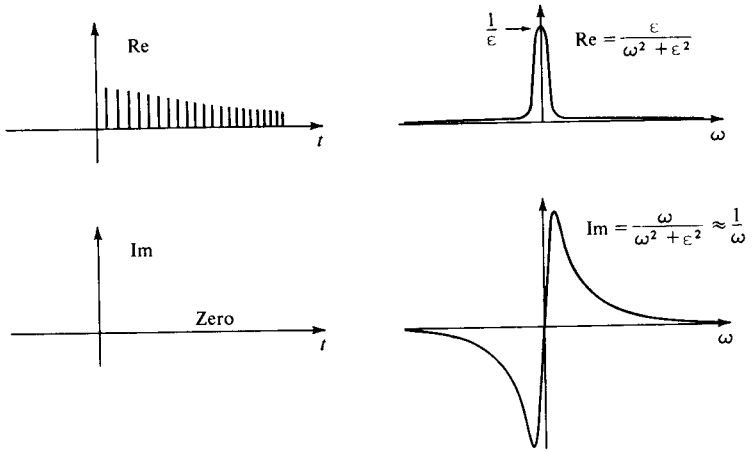


FIGURE 2-23
Properties of the integration operator.

on to diffusion problems, the factor $F(\omega) = 1/(-i\omega)^{1/2}$ often arises as a
o see the equivalent convolution operation, find a causal, sampled-time
on f_t of $F(\omega)$ by identification of powers of Z in

$$(f_0 + f_1 Z + f_2 Z^2 + \cdots)^2 = 1/(-i\omega) \simeq \tfrac{1}{2}(1 + Z)/(1 - Z)$$

ically for f_0 through f_7.

3

SPECTRAL FACTORIZATION

As we will see, there is an infinite number of time functions with any given spectrum. Spectral factorization is a method of finding the one time function which is also minimum phase. The minimum-phase function has many uses. It, and it alone, may be used for feedback filtering. It will arise frequently in wave propagation problems of later chapters. It arises in the theory of prediction and regulation for the given spectrum. We will further see that it has its energy squeezed up as close as possible to $t = 0$. It determines the minimum amount of dispersion in viscous wave propagation which is implied by causality. It finds application in two-dimensional potential theory where a vector field magnitude is observed and the components are to be inferred.

This chapter contains four computationally distinct methods of computing the minimum-phase wavelet from a given spectrum. Being distinct, they offer separate insights into the meaning of spectral factorization and minimum phase.

3-1 ROOT METHOD

The time function (2, 1) has the same spectrum as the time function (1, 2). The autocorrelation is (2, 5, 2). We may utilize this observation to explore the multiplicity of all time functions with the same autocorrelation and spectrum. It would