Waveform inversion using one-way wave extrapolation operators

Rustam Akhmadiev, Biondo Biondi and Robert G. Clapp

ABSTRACT

We continue investigating the problem of waveform inversion using one-way wave extrapolation operators. The nonlinear modeling operator is generalized to account for lateral velocity variations. Born scattering operator is formulated via a linearization of the one-way wave extrapolation and its forward and adjoint operators are derived. Finally, the validity this approach for velocity model reconstruction is demonstrated on a synthetic example.

INTRODUCTION

The full-waveform inversion problem (FWI) has gained considerable attention within the seismic community over the past years as a way of recovering low- and high-wavenumber components of the velocity model (Tarantola, 1984; Mora, 1989). Each method that currently addresses the FWI problem is based on solving the full wave equation. Solutions in the time domain primarily rely on the finite-difference approximations of the derivatives and therefore, have intrinsic limitations on the frequency content of the data and computational cost due to the numerical dispersion caused by the scheme (Courant et al., 1967). Methods based on solving full wave equation in the space-frequency domain (Helmholtz equation) are not affected by this limitation but are expensive to compute because of the required matrix inversion involved in the calculation (Pratt, 1999).

On the other hand, even though seismic imaging methods based on the one-way wave equation have been extensively studied for decades, the problem of waveform inversion using this technique has not been broadly considered. However, some of the attempts have been performed before. For example, the one-way wave equation is used in a deformed coordinate system that allows modeling transmitted waves, therefore, being the main target for waveform inversion (Shragge, 2007). Moreover, Guerra and Cunha (2013) demonstrate some successful examples of using one-way wave extrapolation for full-waveform inversion. The problem is studied as well by Davydenko and Verschuur (2018), where explicit decomposition of the propagator matrix into square root operators, which propagate energy in one direction, is used.

Being just an approximation to the full wave equation, one-way wave equation still gained its popularity in seismic migration due to the accurate imaging in complex models and relatively low computational cost (compared to reverse-time migration). Moreover, even though the wave propagation is inherently angle-limited, this constraint can be circumvented using, for example, one-way wave propagation in the tilted coordinate system oriented along different plane waves (Shan et al., 2007).

Akhmadiev et al. (2018) introduce the methodology of inversion using one-way wave extrapolation and necessary main operators. Here we continue investigating the problem in further detail, constructing more general nonlinear modeling and linearized operators using phase-shift plus interpolation (Gazdag and Sguazzero, 1984), verifying the accuracy of their adjoints. Finally, we validate the methodology with waveform inversion results.

THEORY

In this section we briefly summarize the main equations of one-way wave extrapolation operators and demonstrate forward and adjoint operators, required for a waveform inversion in laterally varying velocity models.

Nonlinear modeling operator

The modeling operator $\mathbf{f(s)}$ nonlinear with respect to slowness using one-way wave extrapolation can be shown to take the following form (Berkhout, 1982):

$$\mathbf{f(s)} = \mathbf{P} = \mathbf{Up(s)R(s)Down(s)}\ \mathbf{w}, \tag{1}$$

where $\mathbf{P}$ is the observed wavefield in the space-frequency domain, $\mathbf{s}$ is the slowness model generaly varying laterally and in depth, $\mathbf{Up(s)}$ is the upward extrapolation, $\mathbf{R(s)}$ is the reflectivity and $\mathbf{Down(s)}$ is the downward extrapolation operator and $\mathbf{w}$ is the spectrum of the source wavelet.

Note, that the reflectivity operator $\mathbf{R(s)}$ is nonlinear with respect to the slowness model and can be approximated as a weighting operator with diagonal entries equal to normal incidence reflectivity:

$$\mathbf{R(s)} = \mathbf{diag}\left[\frac{s_i - s_{i+1}}{s_i + s_{i+1}}\right]. \tag{2}$$

Its role is to scale the downward propagated wavefield that is later propagated upwards. Clearly, the amplitudes of the reflected waves obtained using this approach are accurate just at the normal incidence angle. However, this method is sufficient for modeling the kinematics of reflected events, that are of main interest at this point of the study.

The downward and upward extrapolation operators are propagating waves in one direction stepping down and up in depth, respectively. The engine of both extrapolation processes is the phase-shift operator **PS** (Claerbout, 2010):

$$\mathbf{P}_{j+1}(\omega, \mathbf{k}, z = \Delta z) = \exp\left[-i\Delta z\sqrt{\omega^2 s_j^2 - \mathbf{k}^2}\right]\mathbf{P}_j(\omega, \mathbf{k}, z = 0)$$
$$= \mathbf{PS}_j(s_j)\mathbf{P}_j(\omega, \mathbf{k}, z = 0),$$

where ω is the angular frequency, $\mathbf{k}$ is the horizontal wavenumber, $\mathbf{P}_j$ is the wavefield and s_j is the slowness at the $j-$th depth level. The adjoint of this operator is constructed simply by switching the sign in the complex exponent (equivalent to propagating the energy backward in time).

It is easy to notice, that the square root in the complex exponent becomes imaginary when the expression under the square root becomes negative. This corresponds to evanescent, or inhomogeneous, waves (Aki and Richards, 2002), which mathematically demonstrate exponential growth that should be avoided. Therefore, the evanescent region must be truncated in $(w-k)$ domain. To avoid truncation artifacts (Gibbs phenomenon) and speedup the computation, the values of the square root can be precomputed with proper tapering in the evanescent region and stored in the lookup table.

The downward extrapolation process can be described in the operator form as follows (Biondi, 2006):

$$\begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_{\mathrm{nz}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathbf{PS}_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \mathbf{PS}_{\mathrm{nz}-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_{\mathrm{nz}} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_{\mathrm{nz}} \end{bmatrix}, \tag{3}$$

where $\mathbf{P}_i$ is the wavefield at the $i-$th depth level in the $(\omega - k)-$domain and $\mathbf{w}_i$ is the general source term, assuming that sources can in theory exist at all depth levels. This expression can be rewritten in the short form:

$$\mathbf{P} = \mathbf{PS(s)P} + \mathbf{w},$$

or equivalently

$$\mathbf{P} = [\mathbf{1\text{-}PS(s)}]^{-1}\,\mathbf{w} = \mathbf{Down(s)}\ \mathbf{w}. \tag{4}$$

Therefore, the downward extrapolation computed using an operator **Down** that is the inverse of the lower triangular matrix, can be performed via forward substitution, or stepping down in depth. That is why the adjoint of this operator is easily seen to be the inverse of the upper triangular matrix. Hence, in this case the computation is done as a backward substitution, or stepping up in depth and using the adjoint of a phase-shift operator.

The upward extrapolation operator has a similar form as Equation 3, except that the wavefield vector is stored in the reverse order. Therefore, the forward of upward extrapolation is done stepping up in depth and using the forward phase-shift operator. This difference distinguishes forward of upward extrapolation from the adjoint of the downward extrapolation. Similarly, the adjoint of the upward extrapolation is done stepping down in depth using adjoint of the phase-shift operator.

One of the main difficulties in the one-way wave extrapolation using phase-shift operator is handling lateral velocity variations. In fact, if the velocity is varying laterally, the domain of the phase-shift operator turns into mixed space-wavenumber domain, which is challenging to deal with directly. In this case the computation of the phase-shift has to be performed in the dual (space-wavenumber) domain, which is not trivial. Some of the methods are based on utilizing the phase-shift operator in a nonstationary manner (Margrave and Ferguson, 1999), whereas others on expanding the complex exponent in Taylor series around some reference velocity and differ depending on the amount of terms used in the expansion (Stoffa et al., 1990, Ristow and Rühl, 1994).

One of the methods that attempts to account for lateral velocity variation is phase-shift plus interpolation (PSPI) (Gazdag and Sguazzero, 1984). The idea behind it is that the velocity model is approximated by a given number of reference velocities at every depth level. The wavefield is then extrapolated using each reference velocity. Finally, the resulting wavefield is obtained by interpolating all the reference wavefields depending on the difference between the true and corresponding reference velocity. This process can be written in the operator form as follows:

$$\mathbf{P}_{i+1} = \begin{bmatrix} \mathbf{M}_i^1 & \mathbf{M}_i^2 & \dots & \mathbf{M}_i^{\text{nref}} \end{bmatrix} \begin{bmatrix} \mathbf{F}^*\mathbf{PS}_i^1 \\ \mathbf{F}^*\mathbf{PS}_i^2 \\ \vdots \\ \mathbf{F}^*\mathbf{PS}_i^{\text{nref}} \end{bmatrix} \mathbf{F}\mathbf{P}_i = \mathbf{PSPI}_i\mathbf{P}_i. \tag{5}$$

Here to compute wavefield $\mathbf{P}_{i+1}$ from the previous depth level $\mathbf{P}_i$ first, it transformed to wavenumber domain using forward Fourier transform $\mathbf{F}$, then for every reference velocity it is extrapolated using phase-shift operators $\mathbf{PS}_i^1, \dots, \mathbf{PS}_i^{\text{nref}}$ and mapped back to space domain using inverse Fourier transform $\mathbf{F}^*$. Finally, the resulting wavefield is calculated using interpolation operators $\mathbf{M}_i^1, \dots, \mathbf{M}_i^{\text{nref}}$, where operator $\mathbf{M}_i^j$ is constructed based on the difference between $j-$th reference velocity and the true velocity at the $i-$th depth level.

The adjoint of $\mathbf{PSPI}_i$ operator is therefore:

$$\mathbf{P}_i^* = \mathbf{F}^* \begin{bmatrix} \mathbf{PS}_i^{*1}\mathbf{F} & \mathbf{PS}_i^{*2}\mathbf{F} & \dots & \mathbf{PS}_i^{*\text{nref}}\mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{M}_i^1 \\ \mathbf{M}_i^2 \\ \vdots \\ \mathbf{M}_i^{\text{nref}} \end{bmatrix} \mathbf{P}_{i+1} = \mathbf{PSPI}_i^*\mathbf{P}_{i+1}. \tag{6}$$

Consequently, in order to account for lateral velocity variation in downward and upward extrapolation operators, we need to replace the phase-shift operators $\mathbf{PS}_i$ in Equation 3 by $\mathbf{PSPI}_i$.

Born scattering operator

Another crucial component of the inversion process is the Jacobian matrix, or, in the case of waveform inversion problem, the Born scattering operator. First, because there are three terms nonlinear with respect to slowness in the wave propagation (Equation 1), each will contribute to the full Born operator. Taking the derivative with respect to the slowness we can write the following expression:

$$\frac{\mathbf{df}}{\mathbf{ds}} = \frac{\mathbf{dUp}}{\mathbf{ds}}\mathbf{R(s)Down(s)\ w} + \mathbf{Up(s)}\frac{\mathbf{dR}}{\mathbf{ds}}\mathbf{Down(s)\ w} + \mathbf{Up(s)R(s)}\frac{\mathbf{dDown}}{\mathbf{ds}}\mathbf{w}. \tag{7}$$

Akhmadiev et al. (2018) show that the expressions for each separate linearized operator can be computed using the perturbation theory (see Appendix). For example, the forward linearized operator of the downward extrapolation (the third term in the Equation 7) takes the following form:

$$\begin{cases} \mathbf{P}_{+}^{0} = \mathbf{Down(s)\ w} \\ \delta\mathbf{P} = \mathbf{Up(s)R(s)Down(s)PSPI(s)G(s)P}_{+}^{0}\mathbf{C}\delta\mathbf{s}. \end{cases} \tag{8}$$

Here to calculate the scattered wavefield $\delta\mathbf{P}$, the downward propagated background wavefield $\mathbf{P}_{+}^{0}$ is scaled by the slowness perturbation $\delta\mathbf{s}$ ($\mathbf{C}$ is an operator that spreads the perturbation over the entire frequency range), after that it is scattered using operator $\mathbf{G(s)}$ (see Appendix), phase-shifted using $\mathbf{PSPI(s)}$, propagated downward, reflected off the reflectors present in the background model and propagated upward to the surface. Therefore, this part of the Born operator considers forward downscattering.

Similarly, the expression for the forward up-scattering (the first term in Equation 7) takes the following form:

$$\begin{cases} \mathbf{P}_{-}^{0} = \mathbf{Up(s)\ R(s)\ P}_{+}^{0} \\ \delta\mathbf{P} = \mathbf{Up(s)PSPI(s)G(s)P}_{-}^{0}\mathbf{C}\delta\mathbf{s}. \end{cases} \tag{9}$$

First, we reflect the background wavefield $\mathbf{P}_{+}^{0}$ off the reflectors present in the background velocity model, then propagate the wavefield upwards to get $\mathbf{P}_{-}^{0}$, which is then scaled by the perturbation of the slowness $\delta\mathbf{s}$, scattered, propagated upwards and recorded at the surface.

Finally, the third part of the Born scattering operator (the second term in Equation 7) is based on the linearization of the reflectivity operator:

$$\delta\mathbf{P} = \mathbf{Up(s)P}_{+}^{0}\mathbf{L_R C}\delta\mathbf{s}, \tag{10}$$

where $\mathbf{L_R}$ is the linearization of the reflectivity operator that maps the slowness perturbation into reflectivity perturbation. This part of the full Born operator controls the backward scattering of the background wavefield and it is therefore sensitive to high-wavenumber component of the slowness model (Mora, 1989). Note, that the down-propagating background wavefield $\mathbf{P}_+^0$ generates upward-propagating scattered wavefield.

It is now straightforward to derive the adjoint Born operator that will be the sum of the following three operators:

$$\delta\mathbf{s}^* = \mathbf{C}^*\overline{\mathbf{P}_+^0}\mathbf{G(s)}^*\mathbf{PSPI(s)}^*\mathbf{Down(s)}^*\mathbf{R(s)}^*\mathbf{Up(s)}^*\delta\mathbf{P} \tag{11}$$

$$\delta\mathbf{s}^* = \mathbf{C}^*\overline{\mathbf{P}_-^0}\mathbf{G(s)}^*\mathbf{PSPI(s)}^*\mathbf{Up(s)}^*\delta\mathbf{P} \tag{12}$$

$$\delta\mathbf{s}^* = \mathbf{C}^*\mathbf{L}_\mathbf{R}^*\overline{\mathbf{P}_+^0}\mathbf{Up(s)}^*\delta\mathbf{P} \tag{13}$$

Remembering that the adjoint of downward extrapolation propagates the energy up and the adjoint of upward extrapolation – down the surface, to compute the adjoint of down-scattering (Equation 11), first, we propagate scattered wavefield down, reflect it off the reflectors present in the background slowness model, propagate it upwards, cross-correlate with the background wavefield and sum all the frequencies. The adjoint of up-scattering (Equation 12), in turn, uses the upward propagating background wavefield and downward propagated scattered wavefield and applies imaging condition. Finally, the adjoint of back-scattering (Equation 13) cross-correlates the scattered wavefield extrapolated downward back in time (because of the complex conjugation) and background wavefield propagated downwards. Then all the frequencies are summed and the adjoint of the linearized reflectivity operator maps the result to slowness perturbation.

NUMERICAL EXAMPLE

In order to test the aforementioned waveform inversion method, the synthetic seismic data was generated using a minimum-phase analogue of Ricker wavelet with central frequency of 5 Hz (Figure 1). The receivers are located at the surface with 10 m spacing interval along with 51 shots with 50 m spacing.

Forward modeling is performed in the slowness model shown in the Figure 2. It consists of two layers with constant velocities of 2500 and 2300 m/s respectively and a sharp velocity anomaly with a maximum amplitude of 2250 m/s. Ten reference slowness values were used. Obviously, the choice of reference velocities is crucial for modeling and inversion. Here we use uniform sampling, however there are other methods that allow more accurate representation of the actual velocity model (Clapp, 2004) and as a result lessen the distortion of propagating wavefields. Examples of common shot gathers are shown in Figure 3.

In the waveform inversion process the constant velocity background of 2500 m/s was used as a starting model and all the shots were used with the correct wavelet. The final model after 325 iterations of nonlinear conjugate gradient and the model after 50 iterations of LBFGS is shown in the Figure 4 (Biondi et al., 2019). The objective function values over the iterations are shown in the Figure 5.

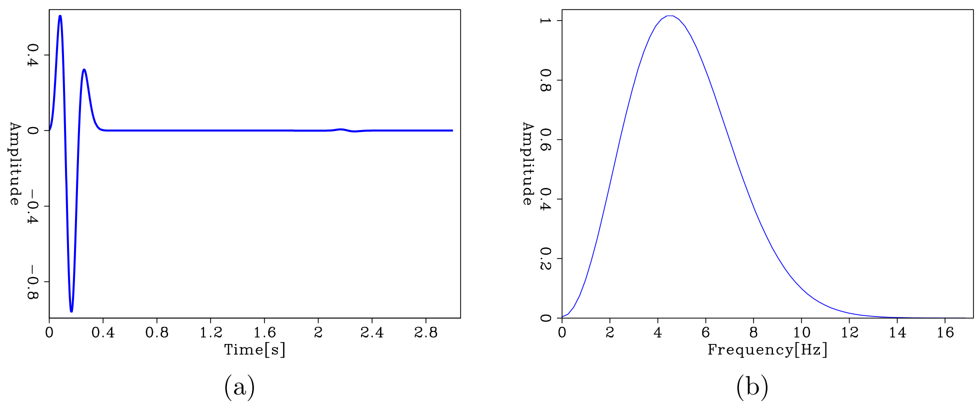


Figure 1: Minimum-phase Ricker wavelet used for modeling and inversion: a - time-domain, b - frequency domain. [**ER**]

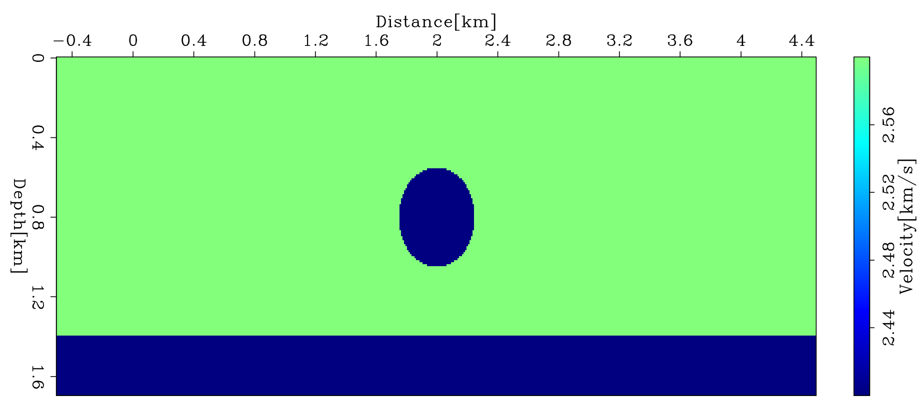


Figure 2: True velocity model used for modeling. [**ER**]

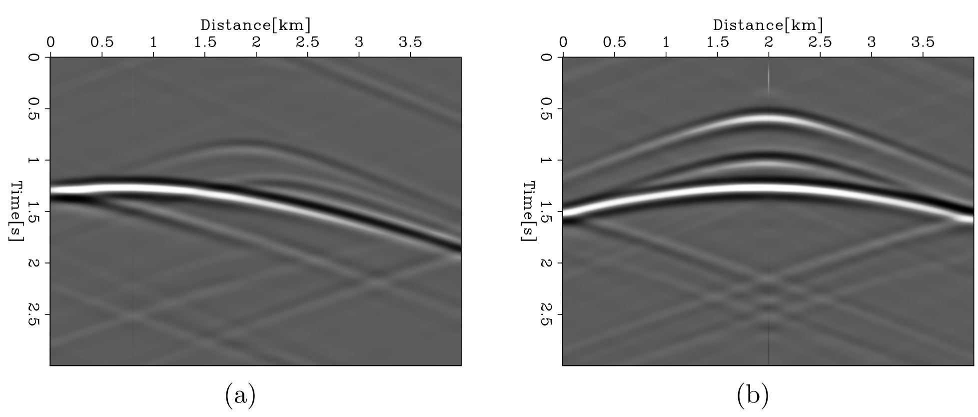
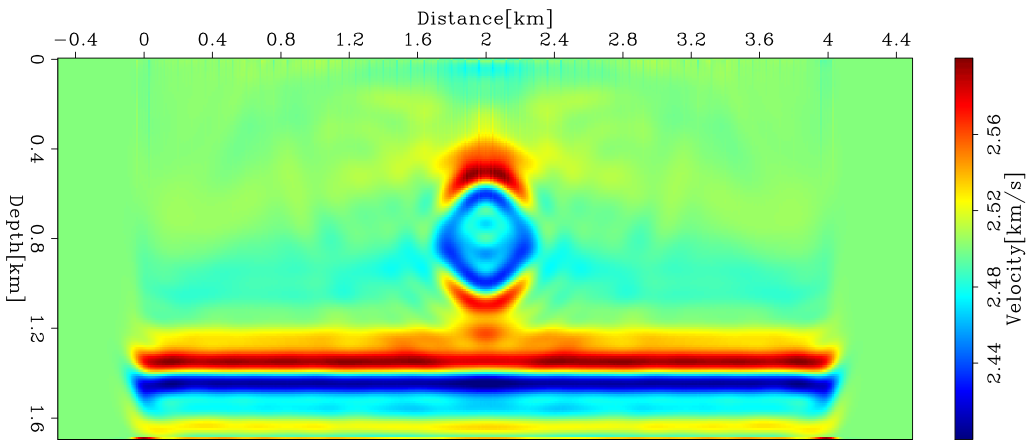
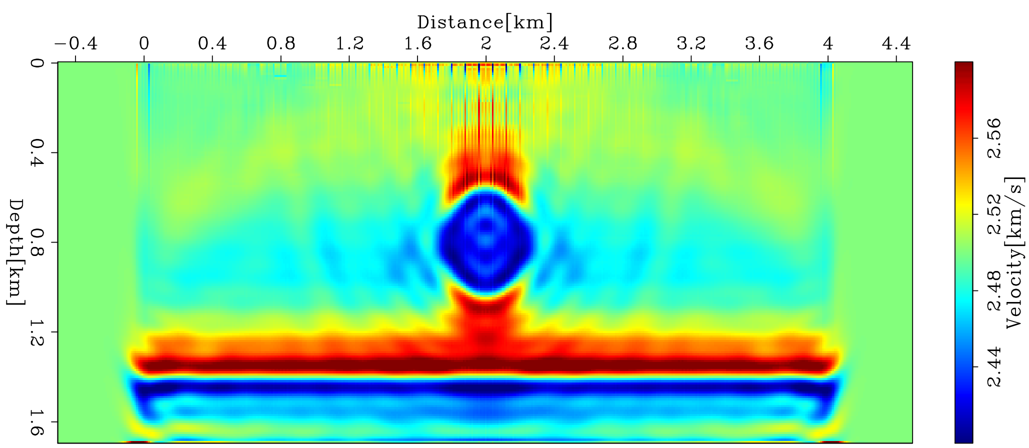


Figure 3: Common shot gathers modeled using PSPI: a - at the surface location 800 m, b - at the surface location 2000 m. [**CR**]



(a)



(b)

Figure 4: Reconstructed velocity model: a – after 325 iterations of noninear conjugate gradient, b – after 50 iterations of LBFGS. [**CR**]

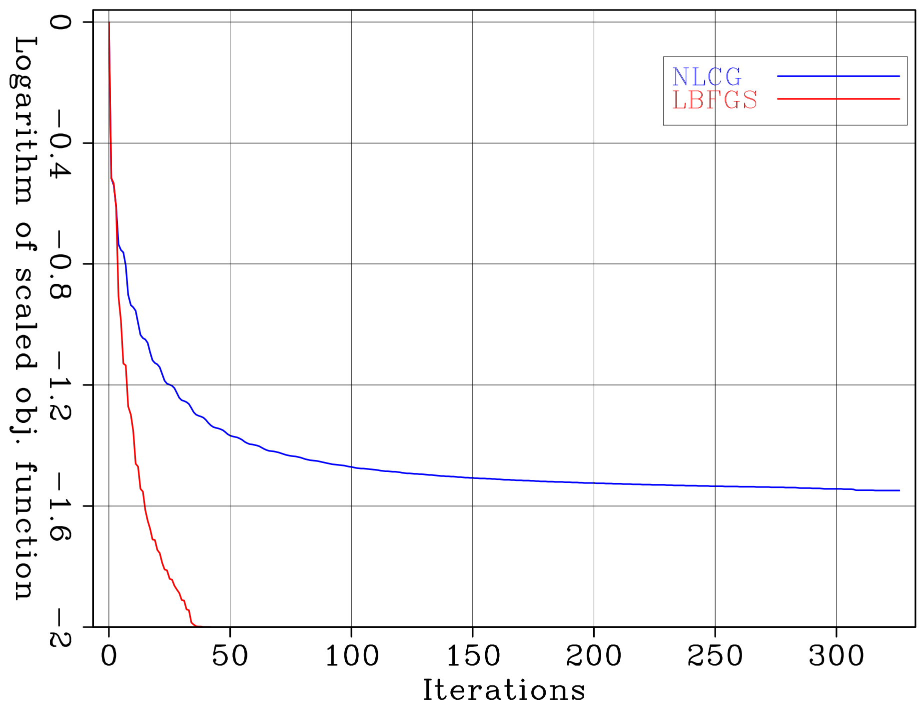


Figure 5: Logarithm of the objective function values over the iterations. [**CR**]

CONCLUSION

We demonstrate how the one-way wave extrapolation operators can be used for the waveform inversion problem. However, it is well known, that the reflection-based full-waveform inversion is a challenging problem in general. That is why even though the example above has not converged to the true solution, this issue, probably comes from the limitations of the full-waveform inversion problem itself, rather than the method presented here (i.e. frequency content of the source wavelet and surface coverage of the receivers).

One of the attractive features of one-way wave extrapolation is that the propagation is performed in the frequency domain. Because this does not cause any issues (compared to the time-domain finite-difference propagation), in theory, the full frequency bandwidth of the data can be used in the inversion process. Moreover, the technique of tomographic full-waveform inversion (Almomin and Biondi, 2013) with time-lag model extension, can also be reformulated in terms of one-way extrapolation operators. In this case, the time convolution of the source wavefield with extended model perturbations, involved in the extended Born operator, turns into a multiplication in the frequency domain. Therefore, it will, potentially, have the same computational cost as a non-extended modeling or inversion. Nevertheless, this methodology has to be investigated in more details in the future.

ACKNOWLEDGEMENT

We would like to thank sponsors of Stanford Exploration Project for the financial support. Also, we thank Ettore Biondi and Guillaume Barnier for the inversion library that was used for this work.

REFERENCES

Akhmadiev, R., B. Biondi, and R. G. Clapp, 2018, Full-waveform inversion problem using one-way wave extrapolation operators.

Aki, K., and P. Richards, 2002, Quantitative seismology: University Science Books.

Almomin, A., and B. Biondi, 2013, Tomographic full waveform inversion (TFWI) by extending the velocity model along the time-lag axis: SEG Technical Program Expanded Abstracts 2013.

Berkhout, A. J., 1982, Seismic migration: Imaging of acoustic energy by wave field extrapolation: Elsevier scientific publishing company.

Biondi, B., 2006, 3D seismic imaging: Society of Exploration Geophysicists.

Biondi, E., R. G. Clapp, and G. Barnier, 2019, A flexible library for geophysical inverse problems – structure and usage.

Claerbout, J. F., 2010, Basic earth imaging.

Clapp, R. G., 2004, Reference velocity selection by a generalized lloyd method: SEG Technical Program Expanded Abstracts 2004.

Courant, R., K. Friedrichs, and H. Lewy, 1967, On the partial difference equations of mathematical physics: IBM Journal of Research and Development, **11**, 215–234.

Davydenko, M., and D. Verschuur, 2018, Frequency domain finite-difference one-way full wavefield modelling: SEG International Exposition and 88th Annual Meeting.

Gazdag, J., and P. Sguazzero, 1984, Migration of seismic data by phase shift plus interpolation: Geophysics, **49**, 124–131.

Guerra, C., and C. Cunha, 2013, Full-waveform inversion using the one-way wave equation: 13th International Congress of the Brazilian Geophysical Society EXPOGEF, Rio de Janeiro, Brazil.

Margrave, G. F., and R. J. Ferguson, 1999, Wavefield extrapolation by nonstationary phase shift: Geophysics, **64**, 1942–2156.

Mora, P., 1989, Inversion = migration + tomography: Geophysics, **54**, 1521–1663.

Pratt, R. G., 1999, Seismic waveform inversion in the frequency domain, part 1: Theory and verification in a physical scale model: Geophysics, **64**, 888–301.

Ristow, D., and T. Rühl, 1994, Fourier finitedifference migration: Geophysics, **59**, 1882–1893.

Shan, G., R. Clapp, and B. Biondi, 2007, 3D plane-wave migration in tilted coordinates: SEG Technical Program Expanded Abstracts 2007.

Shragge, J., 2007, Waveform inversion by one-way wavefield extrapolation: Geophysics, **72**, A47–A50.

Stoffa, P. L., J. T. Fokkema, R. M. de Luna Freire, and W. P. Kessinger, 1990, Splitstep fourier migration: Geophysics, **55**, 410–421.

Tarantola, A., 1984, Inversion of seismic reflection data in the acoustic approximation: Geophysics, **49**, 1259–1266.

APPENDIX

To derive the Born scattering operator, first, start with linearizing the complex exponent in the phase-shift operator. Perturbing the expression under the square root and keeping just the first-order terms:

$$\exp\left[-i\Delta z\sqrt{\omega^2(s_0+\delta s)^2-\mathbf{k}^2}\right] = \exp\left[-i\Delta z\sqrt{\omega^2 s_0^2-\mathbf{k}^2}\sqrt{1+\frac{2\omega^2 s_0}{\omega^2 s_0^2-\mathbf{k}^2}\delta s}\right]$$

Using Taylor expansion of the second square root and the exponent, we get:

$$\exp\left[-i\Delta z\sqrt{\omega^2 s_0^2-\mathbf{k}^2}\right]\left(1+\frac{i\omega^2 s_0\Delta z}{\sqrt{\omega^2 s_0^2-\mathbf{k}^2}}\delta s\right) =$$
$$\exp\left[-i\Delta z\sqrt{\omega^2 s_0^2-\mathbf{k}^2}\right]\left(1+\frac{i\omega\Delta z}{\sqrt{1-\frac{\mathbf{k}^2}{\omega^2 s_0^2}}}\delta s\right)$$

Note, that the equation above represents the linearization of the phase-shift operator that can be written in the following form:

$$\mathbf{PS}(\mathbf{s}_0+\delta\mathbf{s}) = \mathbf{PS} + \mathbf{PS}(\mathbf{s}_0)\mathbf{G}(\mathbf{s}_0)\delta\mathbf{s}.$$

where $\mathbf{G}(\mathbf{s}_0)$ is a scattering operator. In practice it is computed using a Taylor expansion of the square root in the denominator (Biondi, 2006).

Now we can use the perturbation theory in order to find the linearization of the nonlinear modeling operator $\mathbf{f(s)}$. First, note, that the Equation 1 can be written as follows:

$$\begin{cases} [\mathbf{1} - \mathbf{PS}(\mathbf{s})]\,\mathbf{P}_+^0 = \mathbf{w} \\ [\mathbf{1} - \mathbf{PS}(\mathbf{s})]\,\mathbf{P}_-^0 = \mathbf{R}(\mathbf{s})\mathbf{P}_+^0 \end{cases}$$

In the presence of slowness perturbation $\delta\mathbf{s}$ the downward propagating background wavefield $\mathbf{P}_+^0$ generates downward scattered wavefield $\delta\mathbf{P}_+$:

$$\begin{cases} [\mathbf{1} - \mathbf{PS}(\mathbf{s}_0+\delta\mathbf{s})]\,(\mathbf{P}_+^0+\delta\mathbf{P}_+) = \mathbf{w} \\ [\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)]\,(\mathbf{P}_-^0+\delta\mathbf{P}_+) = \mathbf{R}(\mathbf{s}_0)(\mathbf{P}_+^0+\delta\mathbf{P}_+) \end{cases}$$

Using the aforementioned scattering operator $\mathbf{G}$, we can compute the downward scattered wavefield:

$$\begin{cases} [\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)]\,\mathbf{P}_+^0 = \mathbf{w} \\ [\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)]\,\delta\mathbf{P}_+ = \mathbf{PS}(\mathbf{s}_0)\mathbf{G}(\mathbf{s}_0)\mathbf{P}_+^0\delta\mathbf{s} \\ [\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)]\,\delta\mathbf{P}_+ = \mathbf{R}(\mathbf{s}_0)\delta\mathbf{P}_+ \end{cases}$$

This is equivalent to Equation 8.

In a similar way, the upward propagating background wavefield $\mathbf{P}_{-}^{0}$ generates upward scattering wavefield $\delta\mathbf{P}_{-}$. Follwing the same logic, it can be computed as:

$$\begin{cases} \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right]\mathbf{P}_{-}^{0} = \mathbf{R}(\mathbf{s}_0)\mathbf{P}_{+}^{0} \\ \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right]\delta\mathbf{P}_{-} = \mathbf{PS}(\mathbf{s}_0)\mathbf{G}(\mathbf{s}_0)\mathbf{P}_{-}^{0}\delta\mathbf{s} \end{cases}$$

Finally, the perturbation in the background slowness causes backward scattering. To compute it, we first linearize reflectivity operator. Since it is a diagonal operator with diagonal entries equal to normal incidence reflectivity:

$$\frac{s_i + \delta s_i - s_{i+1} - \delta s_{i+1}}{s_i + \delta s_i + s_{i+1} + \delta s_{i+1}} \approx \frac{s_i - s_{i+1}}{s_i + s_{i+1}} + \left[\frac{2s_i}{(s_i + s_{i+1})^2}\delta s_i - \frac{2s_{i+1}}{(s_i + s_{i+1})^2}\delta s_{i+1}\right]$$

The term in the brackets is a sought-for term $\mathbf{L_R}$ linear with respect to slowness perturbation. Using this operator, we again write:

$$\begin{cases} \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right]\mathbf{P}_{+}^{0} = \mathbf{w} \\ \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right](\mathbf{P}_{-}^{0} + \delta\mathbf{P_R}) = \mathbf{R}(\mathbf{s}_0 + \delta\mathbf{s})\mathbf{P}_{+}^{0} \end{cases}$$

Therefore, the backward scattered wavefield $\delta\mathbf{P_R}$ can be calculated using the following expression:

$$\begin{cases} \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right]\mathbf{P}_{+}^{0} = \mathbf{w} \\ \left[\mathbf{1} - \mathbf{PS}(\mathbf{s}_0)\right]\delta\mathbf{P_R} = \mathbf{P}_{+}^{0}\mathbf{L_R}\delta\mathbf{s} \end{cases}$$