Full-waveform inversion problem using one-way wave extrapolation operators

Rustam Akhmadiev, Biondo Biondi, and Robert G. Clapp

ABSTRACT

We pose the problem of waveform inversion using one-way wave extrapolation and derive the modeling operator nonlinear with respect to slowness model and its linearized operator. We show that linearized operator is composed of three parts corresponding to upward and downward scattering (low-wavenumber components of slowness perturbation) and perturbation in reflectivity (high-wavenumber components). Using the phase-shift method we simulate wave propagation using full nonlinear and linearized operators in simple 2D acoustic slowness models and discuss future work.

INTRODUCTION

The one-way wave equation has been widely and successfully used for decades in the seismological community. The methods used for solving it have proven to be very efficient and accurate for seismic modeling and imaging. However, with the development of fast computers and rising demand in accuracy of wave simulation in complex geological areas, the methods based on one-way wave equation gave way to more accurate methods operating in time domain and solving full wave equation such as finite-difference modeling and reverse-time migration. Currently, most of the methods for solving full-waveform inversion are based on these time-domain techniques.

The presence of low frequencies and transmitted waves in the data (long offsets in the acquisition) is crucial for recovering all the range of wavenumbers of the velocity model (Mora, 1989). However, historically most of the seismic observations were focused on reflections and even up to this date they prevail in the majority of the seismic data. Using the observations dominated by reflected energy still presents challenges for successful application of FWI (Gauthier et al., 1986).

Modeling and migration methods based on the one-way wave equation are not capable of handling overturned events and have limited angle range compared to full wave equation solutions (Stolt and Weglein, 2012). However, within the range of angles typically observed in the land and streamer seismic data, they are comparable with time-domain methods for accurately modeling the reflected waves. At the same time frequency-domain methods are computationally less expensive (Biondi, 2018), which oftentimes may be a bottleneck for iterative solutions. Moreover, the linearized one-way wave extrapolation operators provide natural scale separation of low- and high-wavenumber components of the slowness model that may also be used when solving waveform inversion problem.

Following the recipe of posing the nonlinear waveform inversion problem, here we show all the required ingredients necessary for finding its solution. First, we present the nonlinear modeling operator in the matrix form. Then we find its linearization with respect to slowness and derive the operators necessary for solving FWI problem. Finally, we show examples of data modeling in simple slowness models using full nonlinear and linearized operators based on phase-shift extrapolation.

THEORY

The full-waveform inversion problem is generally posed as minimization of an objective function:

$$\mathbf{J_{FWI}} = \frac{1}{2}||\mathbf{f}(\mathbf{m}) - \mathbf{d_{obs}}||^{\mathbf{2}}.$$

There are several constituent parts needed for solving this nonlinear inverse problem. One of the most important is knowledge of forward modeling operator $\mathbf{f}(\mathbf{m})$ that describes the process (the wave propagation in case of FWI). Another crucial part for solving optimization problem is finding the linearization of the modeling operator with respect to the model $\mathbf{m}$ (e.g., velocity or slowness in case of FWI). The first one allows us to reproduce the process, while the second one (namely its adjoint) is used for calculating the gradient of the objective function, which is needed for updating the model using gradient-based methods.

Nonlinear modeling operator

When using the full wave equation, the typical way of solving it is in the time domain using finite-difference scheme and updating the wavefields as they progress in time. Therefore, all the interactions of the wavefields with the media are evolving also with time.

If, however, we are to use the one-way wave equation to model the wave propagation, we will observe waves advancing sequentially with depth. Henceforth, in this case the interaction of the wavefields with underlying media evolve with depth rather than time. Consequently, the wavefields $\mathbf{P}_i$ are computed recursively at every i-th depth level and can be described in the matrix form (Biondi, 2006):

$$\begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \\ \vdots \\ \mathbf{P_{z_{max}}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathbf{E_0}(\mathbf{s}) & 0 & 0 & \dots & 0 \\ 0 & \mathbf{E_1}(\mathbf{s}) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \\ \vdots \\ \mathbf{P_{z_{max}}} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_\omega \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\mathbf{E}_i$ is the propagation operator at the $i-$th depth level nonlinearly depending on the slowness $\mathbf{s}$ and $\mathbf{W}_\omega$ stands for the source wavelet injected in the propagation domain at the angular frequency ω.

The forward modeling of the wavefields observed at the surface using one-way wave extrapolation can be represented by the sequence of three linear operators (Berkhout, 1982): propagator "surface-reflector", the reflection operator and propagator "reflector-surface". Using this logic and representing all the operators $\mathbf{E}_i$ and wavefields $\mathbf{P}_i$ as one operator $\mathbf{E}(\mathbf{s})$ and wavefield vector $\mathbf{P}$ respectively, we can write the forward modeling process in the following form:

$$\begin{cases} \mathbf{P_{down}} & = \mathbf{E}_{+}(\mathbf{s})\mathbf{P_{down}} + \mathbf{I}\mathbf{W}_\omega \\ \mathbf{P_{up}} & = \mathbf{E}_{-}(\mathbf{s})\mathbf{P_{up}} + \mathbf{C}\mathbf{R}(\mathbf{s})\mathbf{P_{down}}. \end{cases} \tag{1}$$

Propagation using system of equations 1 happens in two steps. First, we inject the source $\mathbf{W}_\omega$ (namely its Fourier spectrum) into the media using operator $\mathbf{I}$ and propagate the waves downwards using propagator $\mathbf{E}_{+}(\mathbf{s})$ that nonlinearly depends on the slowness model $\mathbf{s}$. Then we use reflection operator $\mathbf{R}(\mathbf{s})$, that is approximated by normal-incidence reflectivity, to weight the downgoing wavefield according to the reflectivity in the model. After that we inject this modified wavefield as the source for upgoing waves $\mathbf{P_{up}}$ and propagate them upwards using propagator $\mathbf{E}_{-}$. The operator $\mathbf{C}$ is a spreading operator that is needed to match the dimensions of the operators and is extending the reflectivity over all the frequencies.

However, for the purpose of full-waveform inversion we need a single expression that models the data in the form of $\mathbf{f}(\mathbf{s}) = \mathbf{d}$. Reorganizing the equation 1 it is straightforward to get the modeling operator $\mathbf{f}(\mathbf{s})$ in the form:

$$\begin{aligned} \mathbf{d} = \mathbf{K}\mathbf{P_{up}} &= \mathbf{K}[\mathbf{1} - \mathbf{E}_{-}(\mathbf{s})]^{-\mathbf{1}}\mathbf{C}\mathbf{R}(\mathbf{s})[\mathbf{1} - \mathbf{E}_{+}(\mathbf{s})]^{-\mathbf{1}}\mathbf{I}\mathbf{W}_\omega \\ &= \mathbf{K}\mathbf{Up}(\mathbf{s})\mathbf{C}\mathbf{R}(\mathbf{s})\mathbf{Down}(\mathbf{s})\mathbf{I}\mathbf{W}_\omega = \mathbf{f}(\mathbf{s}), \end{aligned} \tag{2}$$

where $\mathbf{K}$ is an operator sampling wavefield at the receiver locations and operators $\mathbf{Down}(\mathbf{s})$ and $\mathbf{Up}(\mathbf{s})$ are propagating a given source downward and upward respectively.

This expression splits the modeling operator into its constituent parts that is convenient for the modular implementation. Moreover, now it is somewhat straightforward to find its linearization with respect to a slowness perturbation.

Linearized forward operator

An important ingredient of any optimization problem using gradient-based methods relies on the accurate estimation of the gradient of the objective function

$$\nabla \mathbf{J_{FWI}} = \left(\frac{\mathbf{df}}{\mathbf{dm}}\right)^{*} [\mathbf{f}(\mathbf{m}) - \mathbf{d}],$$

$\frac{\mathbf{df}}{\mathbf{dm}}$ is a Frechet derivative of a nonlinear operator $\mathbf{f}(\mathbf{m})$ and can be found using the perturbation analysis.

In the case of operator $\mathbf{f}(\mathbf{s})$ represented by equation 2 we have:

$$\begin{aligned}\frac{\mathbf{df}}{\mathbf{ds}}(\mathbf{s_0}) = \mathbf{K}\frac{\mathbf{dUp}}{\mathbf{ds}}(\mathbf{s_0})\mathbf{R}(\mathbf{s_0})\mathbf{Down}(\mathbf{s_0})\mathbf{IW}_\omega + \\ + \mathbf{KUp}(\mathbf{s_0})\mathbf{R}(\mathbf{s_0})\frac{\mathbf{dDown}}{\mathbf{ds}}(\mathbf{s_0})\mathbf{IW}_\omega + \\ + \mathbf{KUp}(\mathbf{s_0})\frac{\mathbf{dR}}{\mathbf{ds}}(\mathbf{s_0})\mathbf{Down}(\mathbf{s_0})\mathbf{IW}_\omega\end{aligned} \tag{3}$$

Since there are three nonlinear operators with respect to slowness, we are going to have three linearized operators, which constitute the full expression for $\frac{\mathbf{df}}{\mathbf{ds}}$.

Downward extrapolation operator

The downward $\mathbf{E}_+$ and upward $\mathbf{E}_-$ extrapolation operators are exactly the same except that in the first we compute the wavefields starting from top down to the bottom of the model and in the latter the other way around – from bottom to the top. The nonlinearity of these operators is hidden in the complex exponentials that constitute the core of the extrapolation (Claerbout, 1985). At every j-th depth level:

$$\mathbf{E_{\mp j}} = \mathbf{diag}\left[\exp\left(\pm i\Delta z\sqrt{\omega^2 s_j^2 - |k|^2}\right)\right],$$

where i is the imaginary unit, Δz is the depth step used in extrapolation, ω is the angular frequency, s_j is the reference slowness at the j-th depth level and k is the horizontal wavenumber.

Hence, first of all we need to find the linear approximation of the complex exponential with respect to the slowness perturbation $\mathbf{ds} = [ds_0, \ldots, ds_{nz}]^T$. Using Taylor expansion around background slowness $\mathbf{s_0}$ and ignoring higher order terms we get

$$\begin{aligned}\mathbf{E_{\mp j}}(s_{0j} + ds_j) &= \mathbf{diag}\left[\exp\left(\pm i\Delta z\sqrt{\omega^2(s_{0j} + ds_j)^2 - |k|^2}\right)\right] \\ &\approx \mathbf{diag}\left[\exp\left(\pm i\Delta z\sqrt{\omega^2 s_{0j}^2 - |k|^2}\right)\right] \times \\ &\times \left(\mathbf{1} + \mathbf{diag}\left[\frac{\pm i\omega\Delta z}{\sqrt{1 - |k|^2/\omega^2 s_{0j}^2}} ds_j\right]\right).\end{aligned} \tag{4}$$

This additional term is easily recognized to be the correction used in split-step and Fourier finite-difference migration methods.

Consequently, the linearization of downward extrapolation operator $\mathbf{E}_+(\mathbf{s})$ is represented in the form

$$\mathbf{E}_+(\mathbf{s_0} + \mathbf{ds}) = \mathbf{E}_+(\mathbf{s_0}) + \mathbf{E}_+(\mathbf{s_0})\mathbf{G}_+(\mathbf{s_0})\mathbf{ds}, \tag{5}$$

where now operator $\mathbf{G}_+(\mathbf{s_0})$ is a forward scattering operator consisting of the aforementioned correction term and is nonlinear with respect to background slowness $\mathbf{s_0}$. Because the expression under the square root is in the mixed space-wavenumber domain, its numerical implementation in the laterally heterogeneous medium is challenging but can be done in the similar fashion with split-step and Fourier finite-difference migration algorithms.

Now using the perturbation analysis:

$$\mathbf{P^0_{down}} + \mathbf{dP_{down}} = [\mathbf{E}_+(\mathbf{s_0}) + \mathbf{E}_+(\mathbf{s_0})\mathbf{G}_+(\mathbf{s_0})\mathbf{ds}][\mathbf{P^0_{down}} + \mathbf{dP_{down}}] + \mathbf{IW}_\omega.$$

Therefore, the background $\mathbf{P^0_{down}}$ and perturbed wavefield $\mathbf{dP_{down}}$ can be computed using following system of equations (based on equation 3) :

$$\begin{cases} \mathbf{P^0_{down}} & = \mathbf{E}_+(\mathbf{s_0})\mathbf{P^0_{down}} + \mathbf{IS}_\omega \\ \mathbf{dP_{down}} & = \mathbf{E}_+(\mathbf{s_0})\mathbf{dP_{down}} + \mathbf{E}_+(\mathbf{s_0})\mathbf{G}_+(\mathbf{s_0})\mathbf{P^0_{down}}\mathbf{ds} \\ \mathbf{dP^{(1)}_{up}} & = \mathbf{E}_-(\mathbf{s_0})\mathbf{dP_{down}} + \mathbf{CR}(\mathbf{s_0})\mathbf{dP_{down}}. \end{cases} \tag{6}$$

First, we propagate the downgoing wavefield $\mathbf{P^0_{down}}$ in the background slowness model $\mathbf{s_0}$. Then, we scatter it off of slowness perturbation $\mathbf{ds}$ using operator $\mathbf{G}_+$ and propagate scattered wavefield downward. Finally, we reflect the scattered wavefield from the reflectors existing in the background model using operator $\mathbf{R}(\mathbf{s_0})$ and propagate the resulting wavefield up to the surface with $\mathbf{E}_-(\mathbf{s_0})$. Hence, this part of the linearized operator represents the downward scattering (Figure 1).

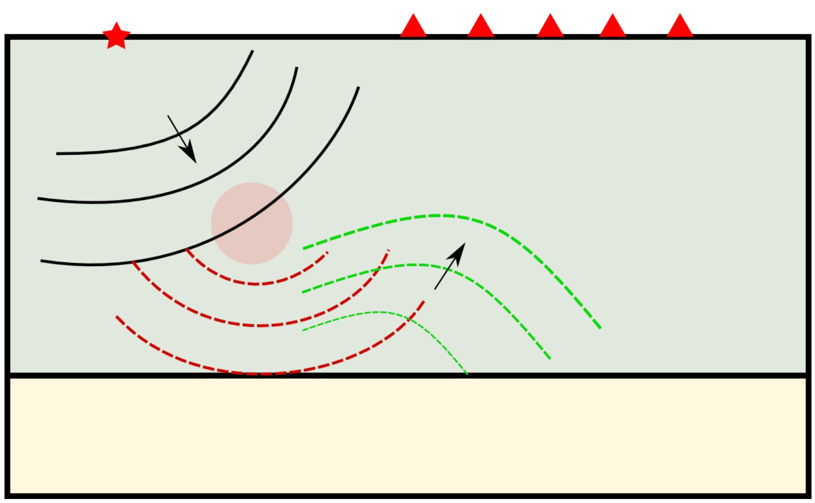

Figure 1: Downward scattering off the slowness perturbation shown in red. The wavefront shown in black corresponds to the first line, red – to the second line and green – to the third line of equation 6. [**NR**]

Upward extrapolation operator

In similar manner we can analyze the second part of the full linearized operator that represents upward scattering. The linearized expression comes from Taylor expansion of complex exponential, which is exactly the same as for the downward extrapolation operator (equation 4) except that it propagates the energy from bottom to the top of the model.

$$\mathbf{E}_{-}(\mathbf{s_0}+\mathbf{ds}) = \mathbf{E}_{-}(\mathbf{s_0}) + \mathbf{E}_{-}(\mathbf{s_0})\mathbf{G}_{-}(\mathbf{s_0})\mathbf{ds} \tag{7}$$

Again using the perturbation analysis and equation 3 we write

$$\begin{cases} \mathbf{P^0_{up}} & = \mathbf{E}_{-}(\mathbf{s_0})\mathbf{P^0_{up}} + \mathbf{CR}(\mathbf{s_0})\mathbf{P^0_{down}} \\ \mathbf{dP^{(2)}_{up}} & = \mathbf{E}_{-}(\mathbf{s_0})\mathbf{dP_{up}} + \mathbf{E}_{-}(\mathbf{s_0})\mathbf{G}_{-}(\mathbf{s_0})\mathbf{P^0_{up}}\mathbf{ds}. \end{cases} \tag{8}$$

Here we see that the upgoing background wavefield $\mathbf{P^0_{up}}$ is formed by reflecting of the downgoing background wavefield $\mathbf{P^0_{down}}$ obtained at the previous step and propagating the result upwards. This upgoing background wavefield is now scattered on its way up and contributes to the final scattered wavefield.

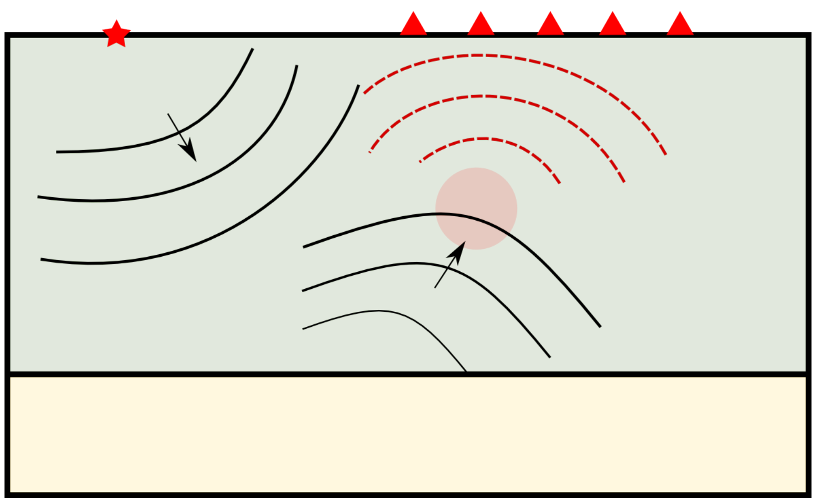

Figure 2: Upward scattering off the slowness perturbation shown in red. The upgoing wavefront shown in black corresponds to the first line, red – to the second line of equation 8. [**NR**]

Reflection operator

Correct representation of the reflection operator is important because it affects the amplitudes of the waves. The reflection coefficient varies depending on the P- and S-velocities, densities and angles of propagation. For this reason, theoretically, in order to model the amplitudes correctly, the reflection operator should be dependent on frequency and wavenumber (analogous to angle dependency). Moreover, it is not necessarily diagonal, because realistic wave interactions with the reflector are not local (Aki and Richards, 2002). As a result, it may potentially be applied as the weighting operator in the wavenumber domain, or equivalently, convolution in space.

Nevertheless, at this stage of our project as a first approximation we can construct reflection operator as a diagonal weighting operator with diagonal entries equal to normal incidence reflectivity. This means essentially that in forward modeling we are primarily aiming at reproducing the kinematics of wave propagation that have adequate but not necessarily exact amplitudes.

In this way, the reflection operator is equal to

$$\mathbf{R}(\mathbf{s}) = \mathbf{diag}\left[\frac{s_i - s_{i+1}}{s_i + s_{i+1}}\right] \tag{9}$$

To find its linearized version we again use perturbation theory. After several simplifications in the fraction and neglecting higher-order terms:

$$\begin{aligned}\mathbf{R}(\mathbf{s_0} + \mathbf{ds}) = \mathbf{diag}\left[\frac{s_i + ds_i - s_{i+1} - ds_{i+1}}{s_i + ds_i + s_{i+1} + ds_{i+1}}\right] \approx \\ \approx \mathbf{R}(\mathbf{s_0}) + \mathbf{diag}\left[\frac{2ds_i}{(s_i + s_{i+1})^2} + \frac{-2ds_{i+1}}{(s_i + s_{i+1})^2}\right] = \\ = \mathbf{R}(\mathbf{s_0}) + \mathbf{dR}(\mathbf{s_0})\end{aligned} \tag{10}$$

It is easy to see that linearized operator $\mathbf{dR}(\mathbf{s_0})$ corresponds to weighted backward difference along the depth axis with weights depending on local slowness values and consequently, its adjoint is a weighted forward difference.

Using this expression and equation 3 it is easy to see that slowness perturbation in the reflection operator gives rise to the scattered wavefield $\mathbf{dP_{up}^{(3)}}$ that can be computed as:

$$\mathbf{dP_{up}^{(3)}} = \mathbf{E}_{-}(\mathbf{s_0})\mathbf{dP_R} + \mathbf{CdR}(\mathbf{s_0})\mathbf{P_{down}^0} \tag{11}$$

This part of the linearized operator accounts for reflections of the background wavefield off the scatterers in contrast with scattering in the previous equations 6 and 8.

It is easy to show that its adjoint corresponds to the one-way wave equation migration that restores high-wavenumber component of the slowness model (reflectors). This fact is in concordance with the previously mentioned natural scale separation inherent to the waveform inversion using one-way wave extrapolation operators.

RESULTS

The full nonlinear modeling and linearized operators were implemented using phase-shift extrapolation (Gazdag, 1978) with one reference slowness equal to the average slowness at the current depth level. Due to the modular approach (following from equation 2), extending the propagator to more complicated wavefield extrapolation methods such as split-step or Fourier finite-difference and including multiple reference slownesses can be done in the same framework.

The wavefields are simulated in a simple model consisting of two layers (Figure 3a) of 500 and 1000 m/s with the reflector located at 50 m. The source located at the surface in the center and the receivers are uniformly distributed at the top of the model. The source wavelet is the minimum-phase analogue of the Ricker wavelet with the central frequency of 15 Hz. To test the linearized operator (equation 3), one scattering point with slowness perturbation equal to 10% of the background value was added right under the source location (Figure 3b) at the depth of 25 m.

Reflected event has the expected hyperbolic moveout with correct zero-offset time of 0.2 s (Figure 4a). The artifacts are resulting from the wraparound effect inherent to the frequency-domain computations. They can be suppressed by adding the lateral tapering of the propagated wavefields at the model boundaries. The data simulated using linearized operator (equation 3) has three distinct events (Figure 4b). The first hyperbola corresponds to the reflection of the background wavefield off the scattering point (equation 11). The second event corresponds to the downward scattering of the background wavefield at the slowness perturbation (the hyperbola with faster aparent velocity) and results from equation 6. The third event (with slower aparent velocity) corresponds to the upward scattering and equation 8.

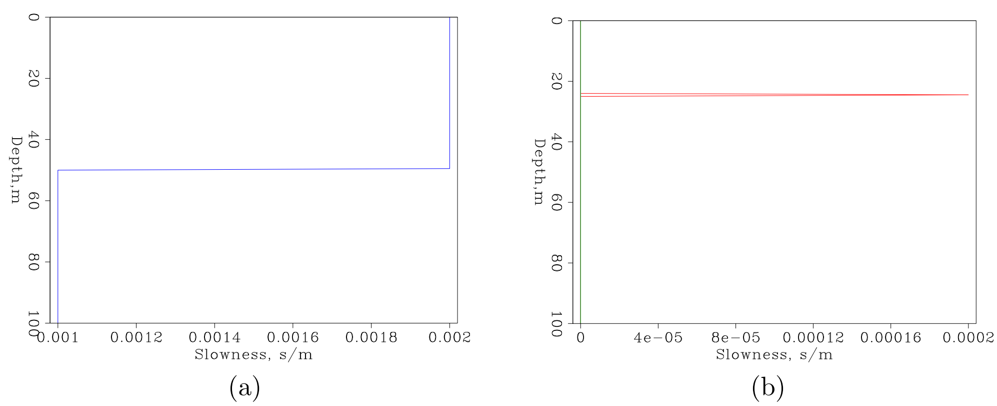

Figure 3: Models used for wave propagation: (a) – background slowness model, (b) – scattering point used as an input for linearized operator. [**ER**]

DISCUSSION AND FUTURE WORK

As we have seen the proposed method is able to model the wave propagation using the full nonlinear operator and its linearized version. The observed data behaves as expected and is in agreement with theoretical predictions. Easy parallelization of the algorithm over the frequency will allow faster computations than time-domain methods. Moreover, absence of limitations on the stability of the method will potentially

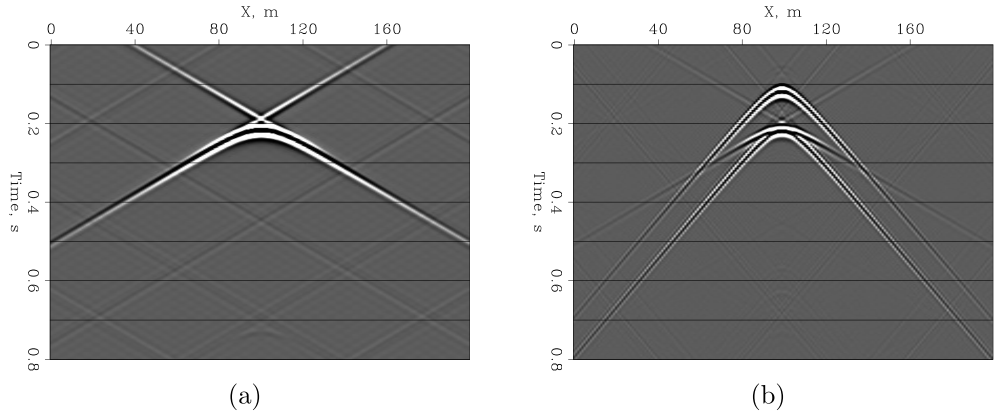

Figure 4: Wavefield modeling using phase-shift extrapolation: (a) – full nonlinear operator, (b) – linearized forward operator. [**ER**]

permit high-resolution waveform inversion. The future work will include extending the propagator to more complex extrapolation operators (e.g., split-step) and adding more reference slownesses. The following steps will involve implementing the adjoint operators and finally, running waveform inversion.

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