Stanford Exploration Project, Report 117, October 23, 2004, pages 121–129

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Short Note

Analytical traveltimes for arbitrary multiples in constant velocity

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INTRODUCTION

Levin and Shah (1977) compute analytical traveltimes for internal multiples generated by a single-CMP seismic survey over a 2-D, two-layer, constant-velocity Earth. Their expression treats the specific case of a single reflection from the bottom of the second layer, preceded and followed by a number of bounces inside the first layer. To obtain the traveltimes, they use the method of images, computing successive images of the source through successive reflections towards the receivers, then computing an image of the receiver through the last reflector. The traveltime is obtained by dividing the distance between the two images to the wavespeed. This way they obtain an analytical traveltime along the pegleg ray that joins the given source and receiver positions only as a function of the respective positions, without the need to take the ray parameter into account.

We extend this procedure to an Earth model with an arbitrary number of layers and an arbitrary sequence of internal bounces between the respective layers. The Earth model is still 2-D, constant-velocity, and with linear interfaces defining constant-density layers. The computations are also done individually for each CMP. The ultimate goal of this study is to assist in the computation of amplitudes for pegleg multiples. This will be used in further studies to isolate the geologic settings in which pegleg multiples are strong enough to cause errors in the interpretation.

PROBLEM SETUP

Let us assume that reflecting interface i is given through two points belonging to it, A and B. Since traveltimes are computed independently for each CMP, we use a coordinate system with the origin in the midpoint between source and receiver (both located at the surface). In this CMP-centric coordinate system, interface i can be expressed as

$$z = x \tan\theta_i + \frac{d_i}{\cos\theta_i}, \tag{1}$$

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where θ_i is the dip of the interface:

$$\tan\theta_i = \frac{z_b - z_a}{x_b - x_a} \tag{2}$$

and d_i is the distance from the CMP point to the interface:

$$d_i = z_a \cos\theta - x_a \sin\theta. \tag{3}$$

Figure 1 shows a three-interface example. Similar to Levin and Shah (1977), we use the method of images to compute traveltimes. Let us denote with $Q\left(x_q, z_q\right)$ the image of point $P\left(x_p, z_p\right)$ through reflector i (see Figure 2). Through simple analytical geometry we find that

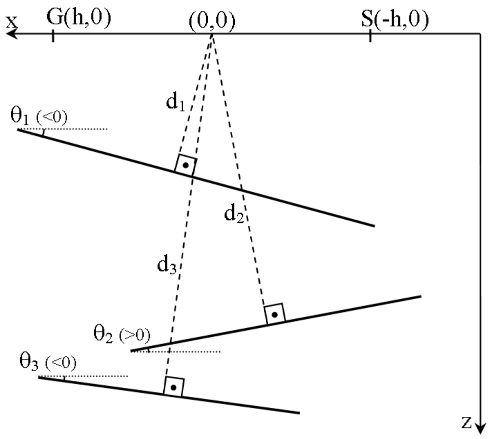

Figure 1: Three interface reflectivity model for illustrating the meaning of the notations d_i and θ_i. Notice the sign convention for angles. $\boxed{\text{nick2-peglegmodel}}$ [NR]

$$\underbrace{\begin{bmatrix} x_q \\ z_q \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} \cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & -\cos 2\theta_i \end{bmatrix}}_{\mathbf{A}(2\theta_i)} \underbrace{\begin{bmatrix} x_p \\ z_p \end{bmatrix}}_{\mathbf{p}} + \underbrace{\begin{bmatrix} -2d_i \sin\theta_i \\ 2d_i \cos\theta_i \end{bmatrix}}_{\mathbf{b}(\theta_i)}. \tag{4}$$

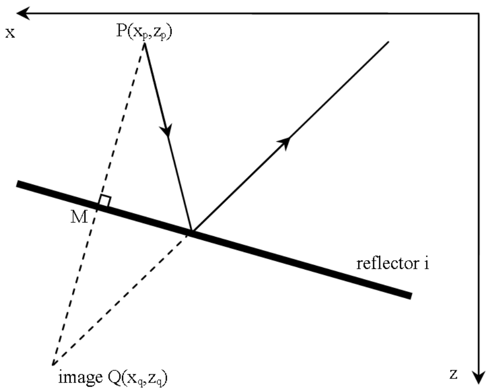

Figure 2: Image point concept illustration. $\boxed{\text{nick2-imex}}$ [NR]

CASCADING IMAGE-CONSTRUCTION OPERATIONS

In order to compute the position of the image after a cascade of several image-construction operations, we first need to define the cascade sequence c as the ordered sequence of interface numbers at which we will consider that a reflection occurs. We define the source as $S(-h,0)$ and the receiver as $G(0,h)$, as shown in Figure 1. Because both S and G are at the surface, any sort of multiple event will be reflected more than once by the same interface. Therefore, the mapping of the counting index i of the cascading sequence onto the values c_i of the cascading sequence is therefore surjective, but not injective. To be able to work with indices in an efficient manner, we describe the geometry of the problem through the sequences

$$\phi_i = \theta_{c_i} \tag{5}$$

and

$$l_i = d_{c_i} \tag{6}$$

which incorporate information both on the geometry of the interfaces and on the order of the cascade, and for which the index numbering starts with the value 1. The subscripts for q will also denote the counting index for the image reflection cascade. The first reflection operation can be written as

$$\mathbf{q}_1 = \mathbf{A}(2\phi_i)\,p + \mathbf{b}(\phi_i)\,. \tag{7}$$

Then,

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{A}\left(2\phi_j\right)\mathbf{q}_1 + \mathbf{b}\left(\phi_j\right) & (8)\\ &= \mathbf{A}\left(2\phi_j\right)\mathbf{A}(2\phi_i)\,\mathbf{p} + \mathbf{A}\left(2\phi_j\right)\mathbf{b}(\phi_i) + \mathbf{b}\left(\phi_j\right), & (9)\end{aligned}$$

$$\begin{aligned} \mathbf{q}_3 &= \mathbf{A}(2\phi_k)\,\mathbf{q}_2 + \mathbf{b}(\phi_k) & (10)\\ &= \mathbf{A}(2\phi_k)\mathbf{A}\left(2\phi_j\right)\mathbf{A}(2\phi_i)\,p + \mathbf{A}(2\phi_k)\mathbf{A}\left(2\phi_j\right)\mathbf{b}(\phi_i) + \mathbf{A}(2\phi_k)\mathbf{b}\left(\phi_j\right) + \mathbf{b}(\phi_k), & (11)\end{aligned}$$

and so on. Let us denote the counterclockwise rotation matrix with

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{12}$$

Both $\mathbf{A}$ and $\mathbf{R}$ are involutory matrices. It can be easily verified that:

$$\mathbf{A}(\alpha)\mathbf{A}(\beta) = \mathbf{R}(\alpha-\beta) \tag{13}$$

$$\mathbf{R}(\alpha)\mathbf{R}(\beta) = \mathbf{R}(\alpha+\beta) \tag{14}$$

$$\mathbf{A}(\alpha)\mathbf{R}(\beta) = \mathbf{A}(\alpha-\beta) \tag{15}$$

$$\mathbf{R}(\alpha)\mathbf{A}(\beta) = \mathbf{A}(\alpha+\beta) \tag{16}$$

Chains of $\mathbf{A}$ operators can be written as a single operator:

$$\mathbf{A}(\beta)\mathbf{A}(\gamma)\mathbf{A}(\delta) = \mathbf{A}(\beta)\mathbf{R}(\gamma-\delta) = \mathbf{A}(\beta-\gamma+\delta), \tag{17}$$

$$\mathbf{A}(\alpha)\mathbf{A}(\beta)\mathbf{A}(\gamma)\mathbf{A}(\delta) = \mathbf{A}(\alpha)\mathbf{A}(\beta-\gamma+\delta) = \mathbf{R}(\alpha-\beta+\gamma-\delta), \tag{18}$$

According to (15), we can write any $\mathbf{A}$ as

$$\mathbf{A}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{R}(-\alpha), \tag{19}$$

so the product of any number k of $\mathbf{A}$ operators can be written as

$$\prod_{i=1}^{k} \mathbf{A}(\alpha_i) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^k \end{bmatrix} \mathbf{R}\left(\sum_{i=1}^{k} (-1)^{i+k-1} \alpha_i \right). \tag{20}$$

To use these properties for constructing cascades of image reflections, we must replace the set α_i with the *inverse* succession of the dips of the reflecting interfaces, multiplied by two according to the definition of $\mathbf{A}$ in (4):

$$\alpha_i = 2\phi_{k-i+1}, \tag{21}$$

where $i = 1 \ldots k$. The reverse "chronological" order is a consequence of the operators in the chain being matrices that multiply the previous image coordinate vector from the left, as exemplified by (8) and (10). The result of the succession of image-building operations can be written as

$$\mathbf{q}_n = \sum_{j=0}^{n} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^j \end{bmatrix} \mathbf{R}\left(2\sum_{i=1}^{j} (-1)^{i+j-1} \phi_{j-i+1} \right) \mathbf{b}\left(\phi_{n-j}\right), \tag{22}$$

where we define a nonphysical quantity $\phi_0 = -\frac{\pi}{2}$ and we also define $\mathbf{b}(\phi_0)$ as the coordinates vector of the initial point in the cascade of reflections. We also consider that the summation index increases in increments of 1 and that summation operators return zero when the upper summation limit is smaller than the lower summation limit. Under the assumption that the starting point of the cascade is at the surface, and by denoting *half* of its x coordinate with l_0, we can write all $\mathbf{b}$ vectors using rotations:

$$\mathbf{b}(\phi_i) = 2l_i \begin{bmatrix} -\sin\phi_i \\ \cos\phi_i \end{bmatrix} = 2l_i \mathbf{R}(\phi_i) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2l_i \mathbf{R}\left(\phi_i + \frac{\pi}{2}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{23}$$

Substituting this into (22),

$$\mathbf{q}_n = 2\sum_{j=0}^{n} l_{n-j} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^j \end{bmatrix} \mathbf{R}\left(\phi_{n-j} + \frac{\pi}{2} + 2\sum_{i=1}^{j} (-1)^{i+j-1} \phi_{j-i+1} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{24}$$

These particular choices of l_0 and ϕ_0, together with the assumption of a surface starting point, ensure that (23) is consistent for the starting point of the cascading operations too. After a few algebraic manipulations, we obtain

$$\mathbf{q}_n = 2\sum_{j=0}^{n} l_{n-j} \begin{bmatrix} -\sin\beta_j \\ (-1)^j \cos\beta_j \end{bmatrix}, \tag{25}$$

where

$$\beta_j = \phi_{n-j} + 2\sum_{i=1}^{j} (-1)^{i+j-1} \phi_{j-i+1}. \tag{26}$$

COMPUTING THE TRAVELTIME: THE THEORY

Let us denote by n a value smaller 1 less than the total number of bounces of the wave in the earth. Let us pretend "not to know" that the horizontal coordinates of S and G are $-h$ and h, respectively, and denote them with s and g instead, since we will later need a more general expression that can be differentiated with respect to these variables. We start by generating the sequences ϕ_i and l_i, according to (5) and (6), and keeping in mind the nonphysical prependix $\phi_0 = -\frac{\pi}{2}$.

Using the fact that n is always even because the total number of bounces inside the earth is always odd, and substituting into (25), we find the image cascaded through n reflection operations from the source to be

$$\mathbf{q}_n^S = s \begin{bmatrix} -\sin\beta_n \\ \cos\beta_n \end{bmatrix} + 2\sum_{j=0}^{n-1} l_{n-j} \begin{bmatrix} -\sin\beta_j \\ (-1)^j \cos\beta_j \end{bmatrix}. \tag{27}$$

The receiver image is obtained from a single reflection operation, through the last reflecting interface:

$$\mathbf{q}_1^G = g \begin{bmatrix} \cos 2\phi_{n+1} \\ \sin 2\phi_{n+1} \end{bmatrix} + 2l_{n+1} \begin{bmatrix} -\sin\phi_{n+1} \\ \cos\phi_{n+1} \end{bmatrix}. \tag{28}$$

The traveltime is the distance between $\mathbf{q}_n^S$ and $\mathbf{q}_1^G$ divided by the velocity. This distance will be computed as the magnitude of the vector $\mathbf{q}_n^S - \mathbf{q}_1^G$. By making the notations

$$\mathbf{u}_1 = \frac{2}{v} \left\{ \sum_{j=0}^{n-1} l_{n-j} \begin{bmatrix} -\sin\beta_j \\ (-1)^j \cos\beta_j \end{bmatrix} + l_{n+1} \begin{bmatrix} \sin\phi_{n+1} \\ -\cos\phi_{n+1} \end{bmatrix} \right\}, \tag{29}$$

$$\mathbf{u}_2 = \frac{1}{v} \begin{bmatrix} -\sin\beta_n \\ \cos\beta_n \end{bmatrix}, \tag{30}$$

$$\mathbf{u}_3 = -\frac{1}{v} \begin{bmatrix} \cos 2\phi_{n+1} \\ \sin 2\phi_{n+1} \end{bmatrix}, \tag{31}$$

we can write:

$$t = |\mathbf{u}_1 + s\mathbf{u}_2 + g\mathbf{u}_3| . \tag{32}$$

In particular, for $s = -h$ and $g = h$ and

$$\mathbf{u}_4 = \mathbf{u}_2 - \mathbf{u}_3 = \frac{1}{v} \begin{bmatrix} \sin(-\beta_n) + \cos 2\phi_{n+1} \\ \cos(-\beta_n) + \sin 2\phi_{n+1} \end{bmatrix}, \tag{33}$$

the traveltime can be written as

$$t = |\mathbf{u}_1 - h\mathbf{u}_4| . \tag{34}$$

This vector magnitude can be computed using scalar products:

$$t = \sqrt{[\mathbf{u}_1 - h\mathbf{u}_4]\cdot[\mathbf{u}_1 - h\mathbf{u}_4]} \tag{35}$$

or it can be written as

$$t^2 = \mathbf{u}_4\cdot\mathbf{u}_4\left(h - \frac{\mathbf{u}_1\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\right)^2 + \mathbf{u}_1\cdot\mathbf{u}_1 - \frac{(\mathbf{u}_1\cdot\mathbf{u}_4)^2}{\mathbf{u}_4\cdot\mathbf{u}_4}, \tag{36}$$

which is the equation of a hyperbola with the apex at

$$h_{apex} = \frac{\mathbf{u}_1\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}, \tag{37}$$

$$t_{apex} = \sqrt{\mathbf{u}_1\cdot\mathbf{u}_1 - \frac{(\mathbf{u}_1\cdot\mathbf{u}_4)^2}{\mathbf{u}_4\cdot\mathbf{u}_4}}. \tag{38}$$

COMPUTING THE TRAVELTIME: AN EXAMPLE

We will illustrate the theory presented above using the multiple reflection event $S1010201G$ (the zeros denote the Earth surface). For this event, $n = 6$, 1 less than the total number of bounces in the earth. The first step is generating sequences ϕ_i and l_i, according to (5) and (6):

$$\{\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7\} = \{\theta_1,\theta_0,\theta_1,\theta_0,\theta_2,\theta_0,\theta_1\} \tag{39}$$

and

$$\{l_1,l_2,l_3,l_4,l_5,l_6,l_7\} = \{d_1,d_0,d_1,d_0,d_2,d_0,d_1\} \tag{40}$$

We prepend $\phi_0 = -\frac{\pi}{2}$ to the sequence of angles, then we compute the β sequence:

$$\begin{aligned}
\beta_0 &= \phi_6 && (41)\\
\beta_1 &= \phi_5 - 2\phi_1 && (42)\\
\beta_2 &= \phi_4 - 2\phi_1 + 2\phi_2 && (43)\\
\beta_3 &= \phi_3 - 2\phi_1 + 2\phi_2 - 2\phi_3 && (44)\\
\beta_4 &= \phi_2 - 2\phi_1 + 2\phi_2 - 2\phi_3 + 2\phi_4 && (45)\\
\beta_5 &= \phi_1 - 2\phi_1 + 2\phi_2 - 2\phi_3 + 2\phi_4 - 2\phi_5 && (46)\\
\beta_6 &= \phi_0 - 2\phi_1 + 2\phi_2 - 2\phi_3 + 2\phi_4 - 2\phi_5 + 2\phi_6 && (47)
\end{aligned}$$

It may be useful to notice the regularities in signs and indices. The summation and trigonometric operators in (25) and (26) can be written in matrix form to verify the correctness of their numerical implementation. We then compute the auxiliary vectors given by (29) and (33):

$$\begin{aligned}
\mathbf{u}_1 = \tfrac{2}{v}l_5\begin{bmatrix} -\sin\beta_1 \\ -\cos\beta_1\end{bmatrix} + \tfrac{2}{v}l_4\begin{bmatrix} -\sin\beta_2 \\ +\cos\beta_2\end{bmatrix} + \tfrac{2}{v}l_3\begin{bmatrix} -\sin\beta_3 \\ -\cos\beta_3\end{bmatrix} + \\
+\tfrac{2}{v}l_2\begin{bmatrix} -\sin\beta_4 \\ +\cos\beta_4\end{bmatrix} + \tfrac{2}{v}l_1\begin{bmatrix} -\sin\beta_5 \\ -\cos\beta_5\end{bmatrix} + \tfrac{2}{v}l_7\begin{bmatrix} \sin\phi_7 \\ -\cos\phi_7\end{bmatrix}
\end{aligned}, \tag{48}$$

$$\mathbf{u}_4 = \frac{1}{v}\left[\begin{array}{c} \sin(-\beta_6) + \cos 2\phi_7 \\ \cos(-\beta_6) + \sin 2\phi_7 \end{array}\right], \tag{49}$$

For our very particular case in which some of the bounces are with the surface ($d_0 = 0$, $\theta_0 = 0$),

$$\mathbf{u}_1 = \frac{2}{v} d_1 \left[\begin{array}{c} +\sin 3\theta_1 + \sin(3\theta_1 + 2\theta_2) + \sin\theta_1 \\ -\cos 3\theta_1 - \cos(3\theta_1 + 2\theta_2) - \cos\theta_1 \end{array}\right] + \frac{2}{v} d_2 \left[\begin{array}{c} +\sin(2\theta_1 - \theta_2) \\ -\cos(2\theta_1 - \theta_2) \end{array}\right], \tag{50}$$

$$\mathbf{u}_4 = \frac{2}{v}\cos(3\theta_1 + \theta_2)\left[\begin{array}{c} \cos(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) \end{array}\right], \tag{51}$$

and the traveltime for each offset h can now be computed by plugging these vectors directly into (35). By performing trigonometric operations, we may find that the expression for the distance is the same as that in Equation (A-14) of Levin and Shah (1977).

ANGLES OF DEPARTURE AND ARRIVAL

The previous sections presented a method to compute the traveltime from a given source point to a given receiver point. Using the image point reflection method has eschewed the need for traditional ray tracing. However, in order to estimate the effect of the acquisition arrays on the amplitudes, or to graphically display the raypaths, we need to compute the angles of departure of the rays from the source and of arrival to the receiver.

Shah (1973) shows that if we denote with α_s the smallest angle between the raypath departing from the source and the vertical, with α_g the similarly defined arrival angle, with s the coordinate of the source and with g the coordinate of the receiver, the two angles can be found from the relations:

$$\frac{\sin\alpha_s}{v} = \frac{\partial t}{\partial s}, \tag{52}$$

$$\frac{\sin\alpha_g}{v} = \frac{\partial t}{\partial g}. \tag{53}$$

Writing (32) as

$$t^2 = \mathbf{u}_1\cdot\mathbf{u}_1 + s^2\mathbf{u}_2\cdot\mathbf{u}_2 + g^2\mathbf{u}_3\cdot\mathbf{u}_3 + 2s\mathbf{u}_1\cdot\mathbf{u}_2 + 2g\mathbf{u}_1\cdot\mathbf{u}_3 + 2sg\mathbf{u}_2\cdot\mathbf{u}_3, \tag{54}$$

we obtain

$$\frac{\partial t}{\partial s} = \frac{1}{t}\mathbf{u}_2\cdot(\mathbf{u}_1 + \mathbf{u}_2 + g\mathbf{u}_3), \tag{55}$$

$$\frac{\partial t}{\partial g} = \frac{1}{t}\mathbf{u}_3\cdot(\mathbf{u}_1 + s\mathbf{u}_2 + \mathbf{u}_3). \tag{56}$$

Replacing now s with $-h$ and g with h, the angles are given by:

$$\sin\alpha_s = \frac{v}{t}\mathbf{u}_2\cdot(\mathbf{u}_1 + \mathbf{u}_2 + h\mathbf{u}_3), \tag{57}$$

$$\sin\alpha_g = \frac{v}{t}\mathbf{u}_3\cdot(\mathbf{u}_1 - h\mathbf{u}_2 + \mathbf{u}_3), \tag{58}$$

where t is computed as a function of h as given by (35).

CONCLUSION

We have derived a formula which describes the traveltimes of internal or surface-related multiples of any order, reflected between any number of layers in a constant-velocity medium. We have also derived an analytical formula for their angles of departure from the source and arrival for the receiver. The low computational cost of this algorithm makes it highly suitable for an analytical-stochastic estimation of the strength of internal multiples in various geological settings, with the ultimate purpose of identifying classes of settings in which internal multiples are likely to be a problem. We plan to perform this work in the near future.

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