Chebyshev Polynomials (Mathematica notebook: http://math.lbl.gov/~fomel/128A/Chebyshev.nb)

Polynomial Shape

Chebyshev polynomials can be defined by the explicit formula

\[ T_n(x) = \cos(n \arccos x) \]

or by the recursive relationship

\[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) . \]

The first three polynomials are

\[
\begin{align*}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1
\end{align*}
\]
The next six polynomials are

\[
\begin{align*}
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{align*}
\]

The extremum points of every Chebyshev polynomial alternate between -1 and 1.
Zeros of the Chebyshev polynomials

The zeros of $T_n(x)$ are distributed denser near the ends of the interval and sparser in the middle.

The explicit formula for $k$-th zero is

$$\hat{x}_k = \cos \left( \frac{2k - 1}{2n} \pi \right), \quad k = 1, 2, \ldots, n.$$
Interpolation

In 1901, Runge demonstrated the pitfalls of equidistant polynomial interpolation using the function

$$f(x) = \frac{1}{1 + 25x^2}.$$

Interpolation with 21 equidistant (regularly spaced) nodes:

Interpolation with 21 Chebyshev nodes: