Hubbert equations

This short paper covers the math that was glossed over in Ken Deffeyes' book, "Beyond Oil: The view from Hubbert's Peak." We define:

t is time in years

$Q(t)$ is cumulative production in billion barrels at year t.

Q_∞ is final cumulative production.

$P(t) = dQ/dt$ is production in billion barrels/year at year t.

τ is the year at which production peaks.

ω is an inverse decay time (imaginary frequency).

We introduce the well-known "logistic" function.

$$Q(t) = \frac{Q_\infty}{1 + e^{\omega(t-\tau)}} \tag{1}$$

Using the rule from calculus that $d(1/v)/dt = -(dv/dt)/v^2$

$$\frac{dQ}{dt} = P(t) = -Q_\infty\,\omega\,\frac{e^{\omega(t-\tau)}}{(1+e^{\omega(t-\tau)})^2} \tag{2}$$

$$P(t) = -Q_\infty\,\omega\,\frac{1}{(e^{-(\omega/2)(t-\tau)} + e^{(\omega/2)(t-\tau)})^2} \tag{3}$$

Examining the form of this equation we see it is symmetric about the point $t = \tau$. Furthermore, asymptotically it decreases (or increases) exponentially towards its maximum value at $t = \tau$. The function resembles a gaussian but exponential decay is much weaker than gaussian decay. Exponential growth is common in ecological systems which may also decay exponentially as predator numbers grow exponentially or resources are depleted. This is the functional form chosen by M. King Hubbert to model world production of petroleum, coal, etc. All we need to do is find a way of choosing the three parameters Q_∞, ω, and τ.

Equation (1) allows us to eliminate the denominator in equation (2) getting

$$P/Q = (Q/Q_\infty)\,\omega\,e^{\omega(t-\tau)} \tag{4}$$

$$P/Q = (Q/Q_\infty)\,\omega\,((1+e^{\omega(t-\tau)})-1) \tag{5}$$

$$P/Q = (Q/Q_\infty)\,\omega\,(Q_\infty/Q-1) \tag{6}$$

$$P/Q = \omega\,(1-Q/Q_\infty) \tag{7}$$

Notice equation (7) is linear in Q and P/Q. Given we have defined this line by a scattering of observations of Q and P/Q we can read the axis intercepts. At $Q = 0$ we can read off the value of the growth/decay paramter $\omega = P/Q$. At $P/Q = 0$ we can read off the value of the ultimately produced volumn $Q_\infty = Q/\omega$.

All that remains is to figure out τ. The Hubbert curve is symmetrical and reaches its maximum when half the oil is gone. That happens when $Q = Q_\infty/2$. In the case of USA production which has passed its peak we can find the year that Q reached that value.