PLATE WAVE NORMAL MODES

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Impromptu seminar Thursday, May 22, 1:15pm, room 452, all invited

We have here some shallow water marine data. We have also some mathematical theory for waves in a plate. Qualitatively, the plate theory fits the marine data pretty well. We should examine the theory and the marine data more carefully to see how good a fit we actually have. Hopefully we can get a decent estimate of the water depth. If we can, the model may have applicability to land data. It could provide us with a locally measured effective layer thickness. Fancy theories might seem nicer, but they tend to require a laterally constant effective layer thickness, while lateral variation is just what people want to map.

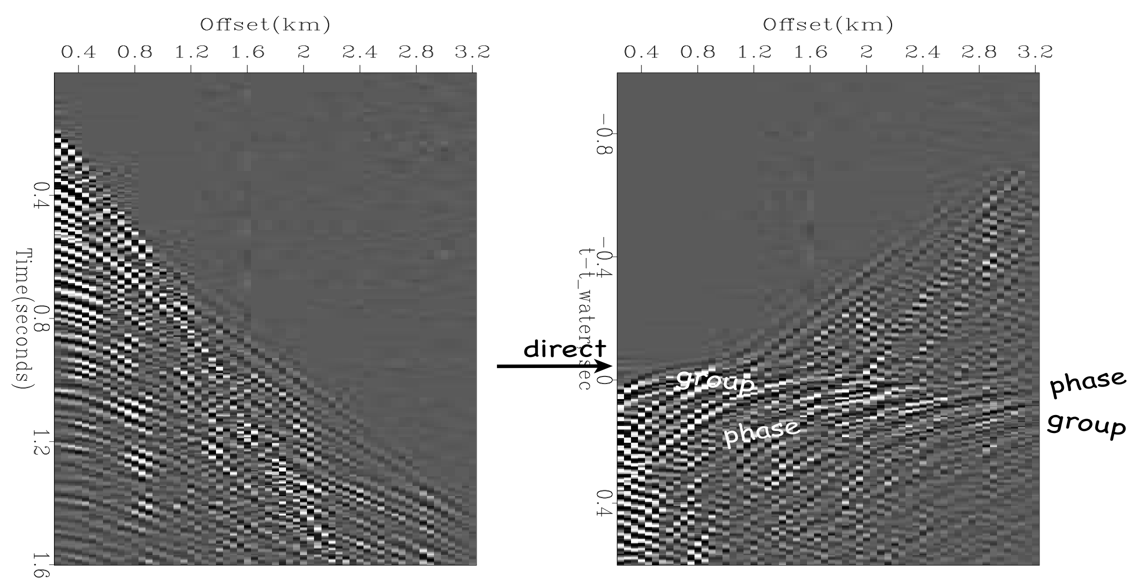

Figure 1: Shallow water marine data (left) and moved out linearly at water velocity (right) shows the fundamental mode with group velocity a bit slower than water velocity and phase velocity a bit higher. Data at vostok:/book/bei/Data/wz.34.H

Figure 1 shows data before and after linear moveout (LMO) with water velocity. To recognize that the LMO is done at water velocity, look carefully at the direct arrival. It is horizontal. It is weak and barely visible only in the first few traces. Also, there are a huge number of multiples that would asymptote to horizontal if the data included more offsets. Observe a clustering of energy at an apparant velocity a little slower than water velocity. This velocity is called the group velocity. In the middle of the group packet observe the phase packets moving upwards with offset.

Next look at Figure 2. I believe there are two clusters of energy at two different group velocities. I'll claim these are the "fundamental mode" and the "first higher

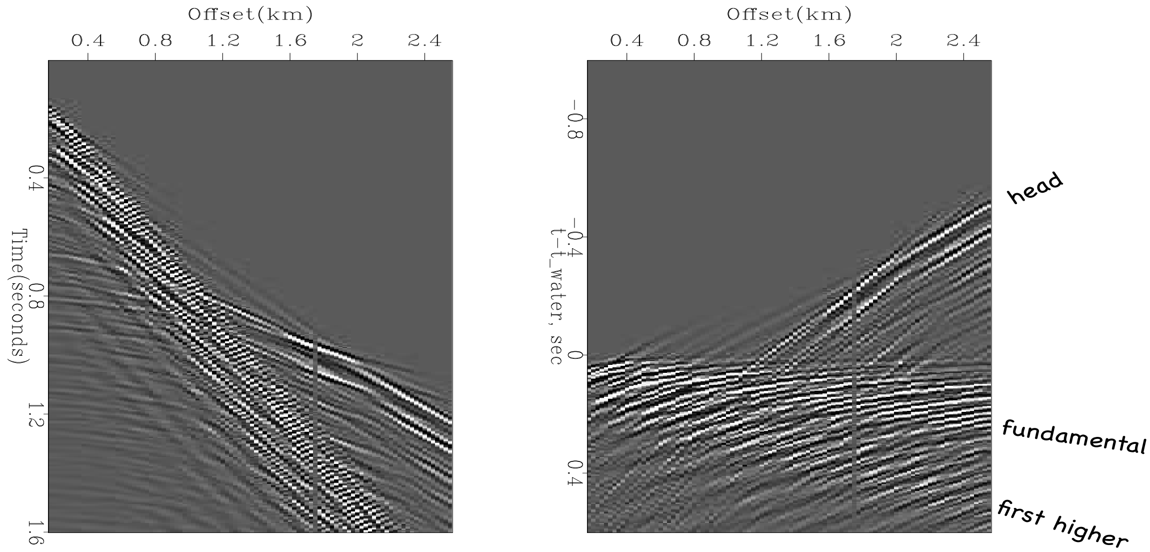

Figure 2: Shallow water marine, LMOed at water velocity, shows fundamental and one higher mode. Data at vostok:/book/bei/Data/wz.32.H

mode". Let us examine some theory to see how this data relates to the water depth and see if my modal interpretation is quantitatively plausible and if we can deduce the water depth for each of the two figures.

Consider the water layer to be a plate with a free surface on top and a rigid surface on the bottom. Mathematically we will have a zero value boundary on one surface and a zero slope boundary on the other. The longest vertical wavelength λ_z that will fit a quarter wave in the plate is four times the plate thickness H. Thus

$$\lambda_z = 4H$$

and

$$k_z = \frac{2\pi}{\lambda_z} = \frac{2\pi}{4H} \tag{1}$$

The dispersion relation of the scalar wave equation is

$$\frac{\omega^2}{v^2} = k_x^2 + k_z^2 \tag{2}$$

$$\frac{\omega^2}{v^2} = k_x^2 + \left(\frac{2\pi}{4H}\right)^2 \tag{3}$$

$$1 = \left(\frac{vk_x}{\omega}\right)^2 + \left(\frac{2\pi v}{4H\omega}\right)^2 \tag{4}$$

$$1 = \sin^2\theta + \left(\frac{v}{4Hf}\right)^2 \tag{5}$$

where we have used $\omega = 2\pi f$. If we imagine the last term should be a cosine, then we have a problem when $4Hf < v$. Thus there is a low frequency cutoff.

If the plate had identical boundary conditions on top and bottom it could contain a ray going exactly horizontally. Since the two boundary conditions differ, there must be a slightly upgoing ray and another slightly downgoing. At very short wavelengths these two rays would depart very slightly from the horizontal. At lower and lower frequencies the ray angle gets steeper and steeper until it is going straight up and down where we encounter the low frequency cutoff when the wavelength gets so long it fails to fit vertically in the plate.

The equation for a monochromatic plane wave is $\exp(-i\omega t + ik_x x)$. To see a wavefront, set the phase equal a constant, const $= -i\omega t + ik_x x$, and take the derivative of that constant by x getting $0 = -\omega dt/dx + k_x$ or $dt/dx = k_x/\omega$.

Using the definition of group velocity $v_g = \partial\omega/\partial\vec{k}$ (see Google) and taking the implicit derivative of (3) we have

$$\frac{2\omega}{v^2}\frac{\partial\omega}{\partial k_x} = 2k_x \tag{6}$$

$$\frac{\omega}{k_x}\frac{\partial\omega}{\partial k_x} = v^2 \tag{7}$$

$$\frac{dx}{dt}\frac{\partial\omega}{\partial k_x} = v^2 \tag{8}$$

which shows the product of group and phase velocities is the water velocity squared. Thus one must be faster than v while the other is slower. We saw this in both data sets.

With $dt/dx = k_x/\omega$ equation (4) becomes

$$\frac{1}{v^2} = \left(\frac{dt}{dx}\right)^2 + \left(\frac{1}{4Hf}\right)^2 \tag{9}$$

The expression $dt/dx = k_x/\omega$ assumes that we are dealing with a single plane wave. Real data always has a mess of plane waves. To deal with this reality you need to average over time and space to get the best fitting plane wave. This is a least squares problem with a classic solution. Looking at the field data you solved this problem with your eyeballs. Such averaging can also be done by computer. See for example my current book http://sep.stanford.edu/sep/prof/gee/book-sep.pdf , page 35, section 2.1.4 "The plane-wave destructor."

Exercise

I have not yet done these exercises. You will be the first. Hooray! Sjoerd observed a nice sharp frequency cutoff. I tried (unsuccessfully) to convince him to use it to measure the effective layer thickness H.

1. Do we need to have a frequency cutoff to measure depth H?
2. Using your eyeballs and a ruler on the enlarged data copies below what do you measure for dt/dx? Give also a standard error $\pm$ or a range.
3. Using your eyeballs and a ruler what do you estimate for frequency f and its standard error or range?
4. Using equation () what do you get for $H(x)$ including its standard error or range?
5. How do the equations change for the first higher mode? Measure the depth H from the higher mode.
6. Goal: Find numerator and denominator that can be smoothed locally whose ratio estimates $H(x)$.
7. Cook up a question for the group.

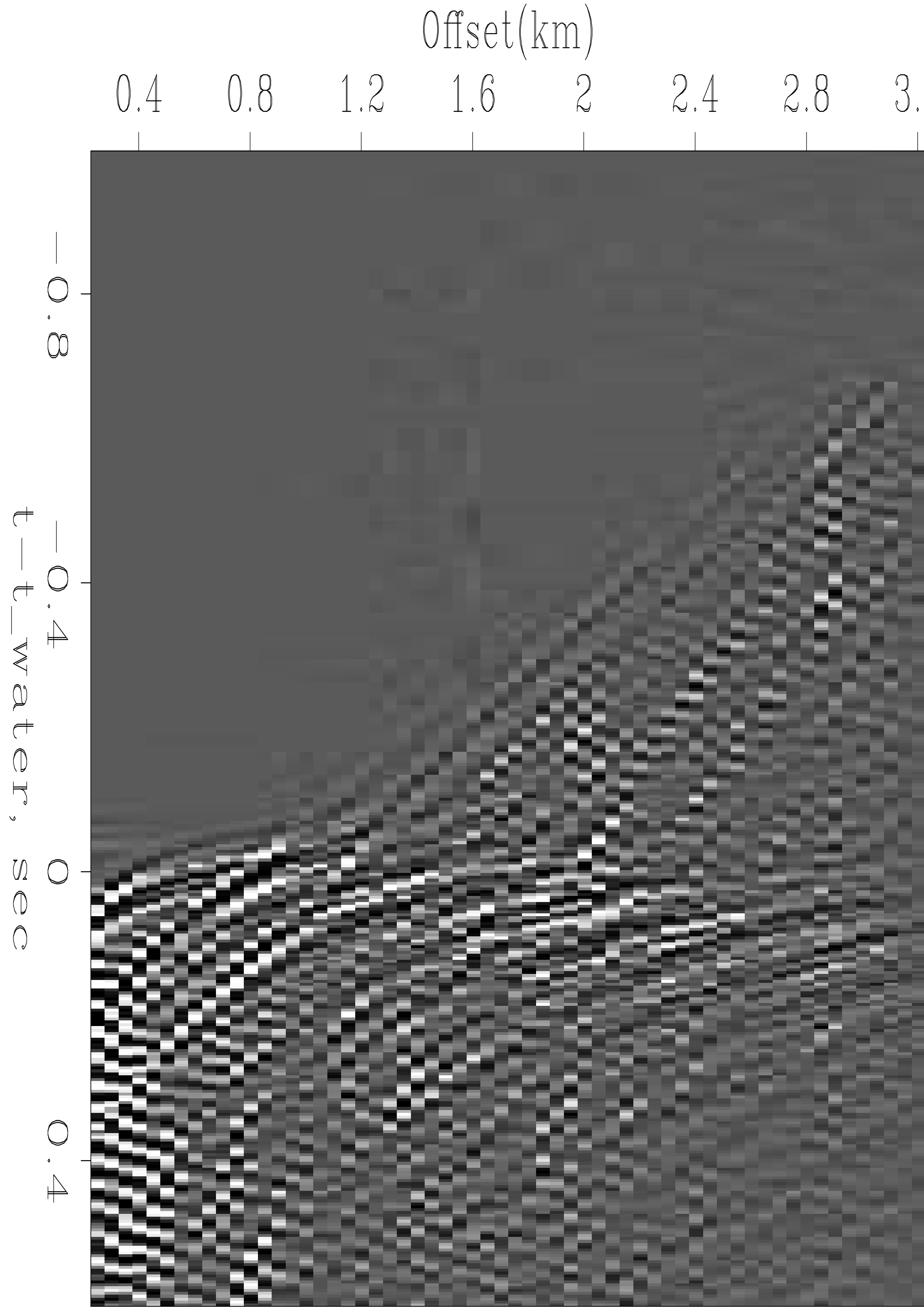

Figure 3: Enlargement of Fig 3

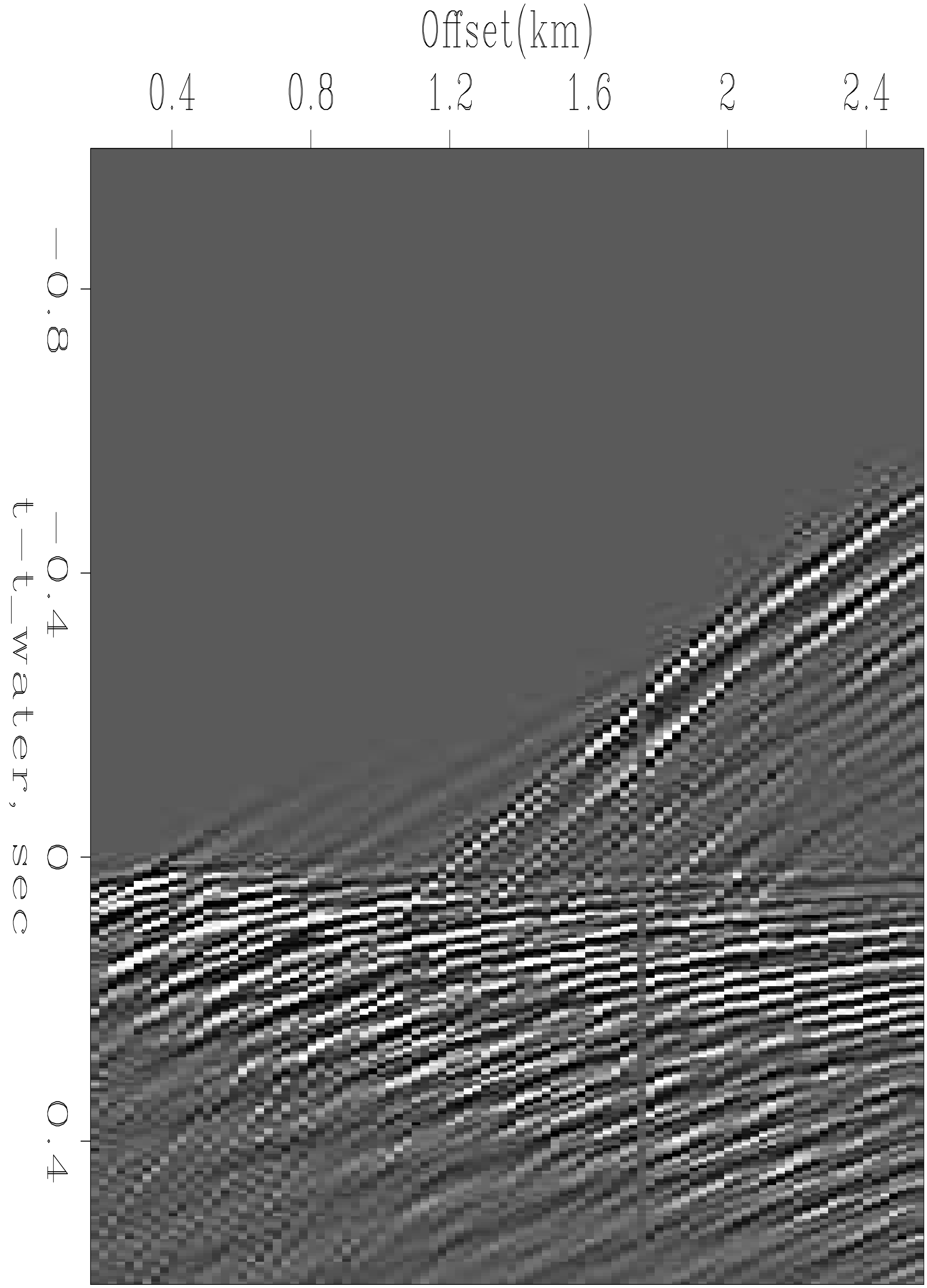

Figure 4: Enlargement of Fig 4