Riemannian wavefield extrapolation of seismic data
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SUMMARY
Images of Earth structure are created through seismic wavefield extrapolation, which extends surface-recorded data to depth through application of a wave-equation operator. The one-way extrapolation operators are derived from the acoustic wave-equation dispersion relation, and are usually defined in a Cartesian coordinate system with a vertical extrapolation axis. Riemannian wavefield extrapolation (RWE) (Sava and Fomel, 2004) is a generalization of the downward continuation concept to coordinate systems that closely conform to the orientation of extrapolated wavefields. Path propagation effects are consequently modeled directly in the coordinate system enabling accurate one-way wavefield extrapolation, even in situations where wavefields overturn.

THEORY
The acoustic propagation of a monochromatic wavefield is governed by the Helmholtz equation, \( \Delta U = -\omega^2 x U \), where \( \omega \) is temporal frequency, \( x \) is reciprocal of the wave propagation velocity, and \( U \) represents a propagating wave. Laplacian operator, \( \Delta \), can be defined in an arbitrary Riemannian space associated with coordinate system variables \( \xi = (x_1, x_2, x_3) \), and used to explicitly write the Helmholtz equation,

\[
\Delta U = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^i} \left( g^{ij} \sqrt{|g|} \frac{\partial U}{\partial \xi^j} \right) = -\omega^2 x U, \quad (1)
\]

where \( g^{ij} \) is a component of the conjugate metric tensor, \( |g| \) is its determinant (Synge and Schild, 1978), and Einstein summation notation is assumed. Importantly, the differential geometry of any coordinate system is implicitly represented in metric tensor, \( g^{ij} \).

One-way wave extrapolation in Riemannian spaces is greatly simplified where one coordinate is orthogonal to the other two. The metric tensor \( g_{ij} \) relates the geometry of vectors in Cartesian space \( \vec{x} = \vec{x}(x_1, x_2, x_3) \) with the geometry of vectors in general coordinate system \( \vec{\xi} = (\xi_1, \xi_2, \xi_3) \),

\[
[g_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{g_{11}}} & 0 & 0 \\ \frac{1}{\sqrt{g_{22}}} & \frac{1}{\sqrt{g_{22}}} & 0 \\ \frac{1}{\sqrt{g_{33}}} & 0 & \frac{1}{\sqrt{g_{33}}} \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & F \end{bmatrix},
\]

where Einstein summation notation over index \( k \) is assumed. In our treatment, coordinate \( \xi_3 = \zeta \) is defined as the extrapolation direction, and is orthogonal to the remaining two coordinates \( \xi_1 = \xi \), and \( \xi_2 = \eta \). This metric allows us to write a general version of the Helmholtz equation,

\[
\frac{1}{\alpha^2} \left[ \frac{\partial}{\partial \zeta} \left( E \frac{\partial U}{\partial \zeta} \right) + \frac{\partial}{\partial \xi} \left( F \frac{\partial U}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( G \frac{\partial U}{\partial \eta} \right) \right] = -\omega^2 x U, \quad (3)
\]

where coefficients, \( c_{ij} \) are given by, \( c_{\zeta \zeta} = 1/\alpha^2 \), \( c_{\xi \xi} = G/F^2 \), \( c_{\eta \eta} = E/F^2 \), \( c_{\xi \eta} = F/G \), \( c_{\zeta \eta} = 1/\alpha^2 \left[ \frac{E}{G} \left( \frac{G}{F} \right)^2 - \frac{E}{H} \right] \), \( c_{\zeta \xi} = 1/\alpha^2 \left[ \frac{E}{H} \left( \frac{G}{F} \right)^2 - \frac{E}{E} \right] \). (4)

This requires isolating the wavenumber in the extrapolation direction, \( k_e \), by selecting the solution with the positive radical,

\[
k_e = \frac{\alpha c_{\zeta \zeta}}{2c_{\zeta \zeta}} + \frac{\alpha \pi^2}{c_{\zeta \zeta}} - \frac{\left( \frac{c_{\zeta \zeta}}{c_{\zeta \zeta}} \right)^2}{2} - \sum_{j \neq \zeta} \frac{\left( c_{\eta \eta} k_j^2 - c_{\xi \xi} k_j^2 \right)}{c_{\zeta \zeta}} - \frac{c_{\xi \xi} k_{\zeta}^2 - c_{\xi \xi} k_{\eta}^2 + ic_{\eta \eta} k_{\zeta} + ic_{\xi \xi} k_{\xi} + ic_{\eta \eta} k_{\eta} - ic_{\xi \xi} k_{\xi} k_{\eta}} = -\omega^2 x^2, \quad (6)
\]

Individual differential operators in equation (3) may be replaced with corresponding Fourier domain wavenumber representations, which leads to the general wave-equation dispersion relation in a semi-orthogonal 3-D Riemannian space,