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Elastic Wave Propagation and Attenuation in a Double-Porosity Dual-Permeability Medium

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ABSTRACT

To account for large-volume low-permeability storage porosity and low-volume high-permeability fracture/crack porosity in oil and gas reservoirs, phenomenological equations for the poroelastic behavior of a double porosity medium have been formulated and the coefficients in these linear equations identified. The generalization from a single porosity model increases the number of independent inertial coefficients from three to six, the number of independent drag coefficients from three to six, and the number of independent stress-strain coefficients from three to six for an isotropic applied stress and assumed isotropy of the medium. The analysis leading to physical interpretations of the inertial and drag coefficients is relatively straightforward, whereas that for the stress-strain coefficients is more tedious. In a quasistatic analysis, the physical interpretations are based upon considerations of extremes in both spatial and temporal scales. The limit of very short times is the one most relevant for wave propagation, and in this case both matrix porosity and fractures are expected to behave in an undrained fashion, although our analysis makes no assumptions in this regard. For the very long times more relevant for reservoir drawdown, the double porosity medium behaves as an equivalent single porosity medium. At the macroscopic spatial level, the pertinent parameters (such as the total compressibility) may be determined by appropriate field tests. At the mesoscopic scale pertinent parameters of the rock matrix can be determined directly through laboratory measurements on core, and the compressiblity can be measured for a single fracture. We show explicitly how to generalize the quasistatic results to incorporate wave propagation effects and how effects that are usually attributed to squirt flow under partially saturated conditions can be explained alternatively in terms of the double-porosity model. The result is therefore a theory that generalizes, but is completely consistent with, Biot's theory of poroelasticity and is valid for analysis of elastic wave data from highly fractured reservoirs.

1 INTRODUCTION

It is well-known in the phenomenology of earth materials that rocks are generally heterogeneous, porous, and often fractured or cracked. In situ, rock pores and cracks/fractures can contain oil, gas, or water. These fluids are all of great practical interest to us. Distinguishing these fluids by their seismic signatures is a key issue in seismic exploration and reservoir monitoring. Understanding their flow characteristics is typically the responsibility of the reservoir engineer.

Traditional approaches to seismic exploration have often made use of Biot's theory of poroelasticity [Biot, 1941; 1956a,b; 1962; Gassmann, 1951]. Many of the predictions of this theory, including the observation of the slow compressional wave, have been confirmed by both laboratory and field experiment [Berryman, 1980; Plona, 1980; Johnson *et al.*, 1982; Chin *et al.*, 1985; Winkler, 1985; Pride and Morgan, 1991; Thompson and Gist, 1993; Pride, 1994]. Nevertheless, this theory has always been limited by an explicit assumption that the porosity itself is homogeneous. Although this assumption is known to be adequate for acoustic studies of many rock core samples in a laboratory setting, it is probably not a very good assumption for applications to realistic heterogeneous reservoirs. One approach to dealing with the heterogeneity is to construct a model that is locally homogeneous, *i.e.*, a sort of finite element approach in which each block of the model satisfies Biot-Gassmann equations. This approach may be adequate in some applications, and is certainly amenable to study with large computers. However, such models avoid the question of how we are to deal with heterogeneity on the local scale, *i.e.*, much smaller than the size of blocks typically used in the codes.

Although it is clear that porosity in the earth can and does come in virtually all shapes and sizes, it is also clear that often just two types of porosity are most important: (1) Matrix porosity occupies a finite and substantial fraction of the volume of the reservoir. This porosity is often called the storage porosity, because this is the volume that stores the fluids of interest to us. (2) Fracture or crack porosity may occupy very little volume, but nevertheless has two very important effects on the reservoir properties. The first effect is that fractures/cracks drastically weaken the rock elastically, and at very low effective stress levels introduce nonlinear behavior since very small changes in stress can lead to large changes in the fracture/crack apertures (and at the same time change the fracture strength for future changes). The second effect is that the fractures/cracks often introduce a high permeability pathway for the fluid to escape from the reservoir. This effect is obviously key to reservoir analysis and the economics of fluid withdrawal.

It is therefore not surprising that there have been many attempts to incorporate fractures into rock models, and especially models that try to account for partial saturation effects and the possibility that fluid moves (or squirts) during the passage of seismic waves [Budiansky and O'Connell, 1975; O'Connell and Budiansky, 1977; Mavko and Nur, 1979; Mavko and Jizba, 1991; Dvorkin and Nur, 1993]. Previous attempts to incorporate have generally been rather ad hoc in their approach to the introduction of the fractures into Biot's theory, if Biot's theory is used at all. The present authors have recently started an effort to make a rigorous extension of Biot's poroelasticity to include fractures/cracks by making a generalization to double-porosity/dual-permeability modeling [Berryman and Wang, 1995]. The previously published work concentrated on the fluid flow aspects of this problem in order to deal with the interactions between fluid withdrawal and the elastic behavior (closure) of fractures during reservoir drawdown. The resulting equations have been applied recently to the reservoir consolidation problem by Lewallen and Wang [1998].

It is the purpose of the present work to point out that a similar analysis applies to the wave propagation problem. Just as Biot's early work on poroelasticity for consolidation [Biot, 1941] led to his later work on wave propagation [Biot, 1956; 1962], the present work follows our own work on consolidation [Berryman and Wang, 1995; Lewallen and Wang, 1998] with its extension to wave propagation. We expect it will be possible to incorporate all of the important physical effects in a very natural way into this double-porosity extension of poroelasticity for seismic wave propagation. The price we pay for this rigor is that we must solve a larger set of coupled equations of motion locally. Within traditional poroelasticity, there are two types of equations that are coupled. These are the equations for the elastic behavior of the solid rock and the equations for elastic and fluid flow behavior of the pore fluid. In the double-porosity extension of poroelasticity, we have not two types of equations but three. The equations for the elastic behavior of the solid rock will be unchanged except for the addition of a new coupling term, while there will be two types of pore-fluid equations (even if there is only one fluid present) depending on the environment of the fluid. Pore fluid in the matrix (storage) porosity will have one set of equations with coupling to fracture fluid and solid; while fluid in the fractures/cracks will have another set of equations with coupling to storage fluid and solid. Although solution of these equations is no doubt more difficult than for simple acoustics/elasticity, it is not significantly more difficult than traditional single-porosity poroelasticity. We will solve these equations in the present paper. We will first derive them and then show that the various coefficients in these equations can be readily identified with measurable quantities. Then we develop and solve (numerically) the dispersion relation.

2 EQUATIONS OF MOTION

The seismic equations of motion for a double-porosity medium have been derived recently by Tuncay and Corapcioglu [1996] using a volume averaging approach. (These authors also provide a thorough review of the prior literature on this topic.) We will present instead a quick derivation based on ideas similar to those of Biot's original papers [Biot, 1956; 1962], wherein a Lagrangian formulation is presented and the phenomenological equations derived.

Physically what we need is quite simple — just equations embodying the concepts of *force = mass×acceleration* and dissipation due to viscous loss mechanisms. The forces are determined by taking a derivative of an energy storage functional. The appropriate energies are discussed at length later in this paper, so for our purposes in this section it will suffice to assume that the constitutive laws relating stress and strain are known, and so the pertinent forces are the divergence of the solid stress field $\tau_{ij,j}$ and the gradients of the two fluid pressures $p_{,i}^{(1)}$ and $p_{,i}^{(2)}$ for the matrix and fracture fluids, respectively. (In this notation, i, j index the three Cartesian coordinates x_1, x_2, x_3 and a comma preceding a subscript indicates a derivative with respect to the specified coordinate direction.) Then, the only work we need to do to establish the equations of motion for dynamical double-porosity systems concerns the inertial terms arising from the kinetic energy of the system.

Generalizing Biot's approach [Biot, 1956] to the formulation of the kinetic energy terms, we find that for a system with two fluids the kinetic energy T is determined by

$$2T = \rho_{11}\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \rho_{22}\dot{\mathbf{U}}^{(1)}\cdot\dot{\mathbf{U}}^{(1)} + \rho_{33}\dot{\mathbf{U}}^{(2)}\cdot\dot{\mathbf{U}}^{(2)} + 2\rho_{12}\dot{\mathbf{u}}\cdot\dot{\mathbf{U}}^{(1)} + 2\rho_{13}\dot{\mathbf{u}}\cdot\dot{\mathbf{U}}^{(2)} + 2\rho_{23}\dot{\mathbf{U}}^{(1)}\cdot\dot{\mathbf{U}}^{(2)}, \quad (1)$$

where $\mathbf{u}$ is the displacement of the solid, $\mathbf{U}^{(k)}$ is the displacement of the kth fluid which occupies volume fraction $\phi^{(k)}$, and the various coefficients ρ_{11}, ρ_{12}, etc., are mass coefficients that take into account the fact that the relative flow of fluid through the pores is not uniform, and that oscillations of solid mass in the presence of fluid leads to induced mass effects. Clarifying the precise meaning of these displacements is beyond our current scope, but recent publications help with these interpretations [Pride and Berryman,1998].

Dissipation plays a crucial role in the motion of the fluids and so cannot be neglected in this context. The appropriate dissipation functional will take the form

$$2D = b_{12}(\dot{\mathbf{u}} - \dot{\mathbf{U}}^{(1)})\cdot(\dot{\mathbf{u}} - \dot{\mathbf{U}}^{(1)}) + b_{13}(\dot{\mathbf{u}} - \dot{\mathbf{U}}^{(2)})\cdot(\dot{\mathbf{u}} - \dot{\mathbf{U}}^{(2)}) + b_{23}(\dot{\mathbf{U}}^{(1)} - \dot{\mathbf{U}}^{(2)})\cdot(\dot{\mathbf{U}}^{(1)} - \dot{\mathbf{U}}^{(2)}). \quad (2)$$

This formula assumes that all dissipation is caused by motion of the fluids either relative to the solid, or relative to each other. (Other potential sources of attenuation, especially for partially saturated porous media [Stoll, 1985; Miksis, 1988], should also be considered, but will not be discussed here.) We expect the coefficient b_{23} will generally be small and probably negligible, whenever the double-porosity model is appropriate for the system under study.

Lagrange's equations then show easily that

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_i}\right) + \frac{\partial D}{\partial \dot{u}_i} = \tau_{ij,j}, \quad \text{for} \quad i = 1, 2, 3, \quad (3)$$

and that

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{U}_i^{(k)}}\right)+\frac{\partial D}{\partial \dot{U}_i^{(k)}}=-\bar{p}_i^{(k)}, \quad \text{for} \quad i=1,2,3; k=1,2, \tag{4}$$

where the pressures $\bar{p}^k$ are the macroscopic fluid pressures across interfaces and are related to the internal pore pressures by factors of the porosity so that $\bar{p}^{(1)}=(1-v^{(2)})\phi^{(1)}$ and $\bar{p}^{(2)}=v^{(2)}\phi^{(2)}$, with $v^{(2)}$ being the total volume fraction of the fracture porosity. These equations now account properly for inertia and elastic energy, strain, and stress, as well as for the specified types of dissipation mechanisms, and are in complete agreement with those developed by Tuncay and Corapcioglu [1996] using a different approach. In (4), the parts of the equation not involving the kinetic energy can be shown to be equivalent to a two-fluid Darcy's law in this context, so b_{12} and b_{13} are related to Darcy's constants for two single phase flow and b_{23} is the small coupling coefficient. Explicit relations between the b's and the appropriate permeabilities (see Eqs. (53) and (54) of Berryman and Wang [1995]) are not difficult to establish. The harder part of the analysis concerns the constitutive equations required for the right hand side of (3). After the section on inertia and drag, the remainder of the paper will necessarily be devoted to addressing some of these issues concerning stress-strain relations.

In summary, equations (3) and (4) can be combined into

$$\begin{pmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ \rho_{12} & \rho_{22} & \rho_{23}\\ \rho_{13} & \rho_{23} & \rho_{33}\end{pmatrix}\begin{pmatrix}\ddot{u}_i\\ \ddot{U}_i^{(1)}\\ \ddot{U}_i^{(2)}\end{pmatrix}+\begin{pmatrix}b_{12}+b_{13} & -b_{12} & -b_{13}\\ -b_{12} & b_{12}+b_{23} & -b_{23}\\ -b_{13} & -b_{23} & b_{13}+b_{23}\end{pmatrix}\begin{pmatrix}\dot{u}_i\\ \dot{U}_i^{(1)}\\ \dot{U}_i^{(2)}\end{pmatrix}=\begin{pmatrix}\tau_{ij,j}\\ -\bar{p}_{,i}^{(1)}\\ -\bar{p}_{,i}^{(2)}\end{pmatrix}, \tag{5}$$

showing the coupling between the solid and both types of fluid components.

In the next section we show how to identify the inertial and drag coefficients with physically measureable quantities.

3 INERTIAL AND DRAG COEFFICIENTS

3.1 Inertial coefficients

It is easy to understand that the inertial coefficients appearing in the kinetic energy T must depend on the densities of solid and fluid constituents ρ_s and ρ_f, and also on the volume fractions $\phi^{(1)}$ and $\phi^{(2)}$ of the matrix and fracture porosities. The total porosity is given by $\phi=\phi^{(1)}+\phi^{(2)}$ and the volume fraction occupied by the solid material is therefore $1-\phi$.

For a single porosity material, there are only three inertial coefficients and the kinetic energy can be written as

$$2T=\begin{pmatrix}\dot{\mathbf{u}} & \dot{\mathbf{U}}\end{pmatrix}\begin{pmatrix}\bar{\rho}_{11} & \bar{\rho}_{12}\\ \bar{\rho}_{12} & \bar{\rho}_{22}\end{pmatrix}\begin{pmatrix}\dot{\mathbf{u}}\\ \dot{\mathbf{U}}\end{pmatrix}, \tag{6}$$

where $\dot{\mathbf{U}}$ is the velocity of the only fluid present. Then, it is easy to see that, if $\dot{\mathbf{u}}=\dot{\mathbf{U}}$, the total inertia $\bar{\rho}_{11}+2\bar{\rho}_{12}+\bar{\rho}_{22}$ must equal the total inertia present in the system $(1-\phi)\rho_s+\phi\rho_f$. Furthermore, Biot [1956] has shown that $\bar{\rho}_{11}+\bar{\rho}_{12}=(1-\phi)\rho_s$, and that $\bar{\rho}_{22}+\bar{\rho}_{12}=\phi\rho_f$. These three equations are not linearly independent and therefore do not determine the three coefficients. So we make the additional assumption that $\bar{\rho}_{22}=\tau\phi\rho_f$, where τ (Note: This τ without subscripts should not be confused with the stress tensor introduced earlier in the paper.) was termed the structure factor by Biot [1956], but has more recently been termed the electrical tortuosity [Brown, 1980; Johnson *et al.*, 1982], since $\tau = \phi F$ where F is the electrical formation factor. Berryman [1980] has shown that

$$\tau = 1 + r\left(\frac{1}{\phi} - 1\right), \tag{7}$$

follows from interpreting the coefficient $\bar{\rho}_{11}$ as resulting from the solid density plus the induced mass due to the oscillation of the solid in the surrounding fluid. Then, $\bar{\rho}_{11} = (1-\phi)(\rho_s + r\rho_f)$, where r is a factor dependent on microgeometry that is expected to lie in the range $0 \le r \le 1$, with $r = \frac{1}{2}$ for spherical grains. For example, if $\phi = 0.2$ and $r = 0.5$, equation (7) implies $\tau = 3.0$, which is a typical value for tortuosity of sandstones.

For double porosity, the kinetic energy may be written as

$$2T = \begin{pmatrix} \dot{\mathbf{u}} & \dot{\mathbf{U}}^{(1)} & \dot{\mathbf{U}}^{(2)} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{U}}^{(1)} \\ \dot{\mathbf{U}}^{(2)} \end{pmatrix}. \tag{8}$$

We now consider some limiting cases. First, suppose that all the solid and fluid material moves in unison. Then, we again have the result $\rho_{11} + \rho_{22} + \rho_{33} + 2\rho_{12} + 2\rho_{13} + 2\rho_{23}$ must equal the total inertia of the system $(1-\phi)\rho_s + \phi\rho_f$. Next, if we suppose that the two fluids can be made to move in unison, but independently of the solid, then we can take $\dot{\mathbf{U}} = \dot{\mathbf{U}}^{(1)} = \dot{\mathbf{U}}^{(2)}$, and telescope the expression for the kinetic energy to

$$2T = \begin{pmatrix} \dot{\mathbf{u}} & \dot{\mathbf{U}} \end{pmatrix} \begin{pmatrix} \rho_{11} & (\rho_{12} + \rho_{13}) \\ (\rho_{12} + \rho_{13}) & (\rho_{22} + 2\rho_{23} + \rho_{33}) \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{U}} \end{pmatrix}. \tag{9}$$

We can now relate the matrix elements in (9) directly to the barred matrix elements appearing in (6), which then gives us three equations for our six unknowns. Again these equations are not linearly independent, so we still need four more equations.

Next we consider the possibility that the fracture fluid can oscillate independently of the solid and the matrix fluid, and furthermore that the matrix fluid velocity is locked to that of the solid so that $\dot{\mathbf{u}} = \dot{\mathbf{U}}^{(1)}$. For this case, the kinetic energy telescopes in a different way to

$$2T = \begin{pmatrix} \dot{\mathbf{u}} & \dot{\mathbf{U}}^{(2)} \end{pmatrix} \begin{pmatrix} (\rho_{11} + 2\rho_{12} + \rho_{22}) & (\rho_{13} + \rho_{23}) \\ (\rho_{13} + \rho_{23}) & \rho_{33} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{U}}^{(2)} \end{pmatrix}. \tag{10}$$

This equation is also of the form (6), but we must be careful to account properly for the parts of the system included in the matrix elements. Now we treat the solid and matrix fluid as a single unit, so

$$\rho_{11} + 2\rho_{12} + \rho_{22} = (1-\phi)\rho_s + (1 - v^{(2)})\phi^{(1)}\rho_f + (\tau^{(2)} - 1)v^{(2)}\phi^{(2)}\rho_f, \tag{11}$$

$$\rho_{13} + \rho_{23} = -(\tau^{(2)} - 1)v^{(2)}\phi^{(2)}\rho_f, \tag{12}$$

and

$$\rho_{33} = \tau^{(2)}v^{(2)}\phi^{(2)}\rho_f, \tag{13}$$

where $\tau^{(2)}$ is the tortuosity of fracture porosity alone and $v^{(2)}$ is the volume fraction of the fractures in the system.

Finally, we consider the possibility that the matrix fluid can oscillate independently of the solid and the fracture fluid, and furthermore that the fracture fluid velocity is locked to that of the solid so that $\dot{\mathbf{u}} = \dot{\mathbf{U}}^{(2)}$. The kinetic energy telescopes in a very similar way to the previous case with the result

$$2T = \begin{pmatrix} \dot{\mathbf{u}} & \dot{\mathbf{U}}^{(1)} \end{pmatrix} \begin{pmatrix} (\rho_{11} + 2\rho_{13} + \rho_{33}) & (\rho_{12} + \rho_{23}) \\ (\rho_{12} + \rho_{23}) & \rho_{22} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{U}}^{(1)} \end{pmatrix}. \tag{14}$$

We imagine that this thought experiment amounts to analyzing the matrix material alone without fractures being present. The equations resulting from this identification are completely analogous to those in (11)-(13), so we will not show them explicitly here.

We now have nine equations in the six unknowns and six of these are linearly independent, so the system can be solved. The result of this analysis is that the off-diagonal terms are given by

$$2\rho_{12}/\rho_f = (\tau^{(2)} - 1)v^{(2)}\phi^{(2)} - (\tau^{(1)} - 1)(1 - v^{(2)})\phi^{(1)} - (\tau - 1)\phi, \tag{15}$$

$$2\rho_{13}/\rho_f = (\tau^{(1)} - 1)(1 - v^{(2)})\phi^{(1)} - (\tau^{(2)} - 1)v^{(2)}\phi^{(2)} - (\tau - 1)\phi, \tag{16}$$

and

$$2\rho_{23}/\rho_f = (\tau - 1)\phi - (\tau^{(1)} - 1)(1 - v^{(2)})\phi^{(1)} - (\tau^{(2)} - 1)v^{(2)}\phi^{(2)}. \tag{17}$$

The diagonal terms are given by

$$\rho_{11} = (1 - \phi)\rho_s + (\tau - 1)\phi\rho_f, \tag{18}$$

$$\rho_{22} = \tau^{(1)}(1 - v^{(2)})\phi^{(1)}\rho_f, \tag{19}$$

and ρ_{33} is given by (13).

Estimates of the three tortuosities τ, $\tau^{(1)}$, and $\tau^{(2)}$ may be obtained using (7), or direct measurements may be made using electrical methods as advocated by Brown [1980] and Johnson *et al.* [1982]. Appendix A explains one method of estimating τ for the whole medium when the constituent tortuosities and volume fractions are known.

3.2 Drag coefficients

The drag coefficients may be determined by first noting that the equations presented here reduce to those of Berryman and Wang [1995] in the low frequency limit by merely neglecting the inertial terms. What is required to make the direct identification of the coefficients is a pair of coupled equations for the two increments of fluid content $\zeta^{(1)}$ and $\zeta^{(2)}$. These quantities are related to the displacements by $\zeta^{(1)} = -(1 - v^{(2)})\phi^{(1)}\nabla \cdot (\mathbf{U}^{(1)} - \mathbf{u})$ and $\zeta^{(2)} = -v^{(2)}\phi^{(2)}\nabla \cdot (\mathbf{U}^{(2)} - \mathbf{u})$.

The pertinent equations from Berryman and Wang [1995] are

$$\eta \begin{pmatrix} \dot{\zeta}^{(1)} \\ \dot{\zeta}^{(2)} \end{pmatrix} = \begin{pmatrix} k^{(11)} & k^{(12)} \\ k^{(21)} & k^{(22)} \end{pmatrix} \begin{pmatrix} p^{(1)}_{,ii} \\ p^{(2)}_{,ii} \end{pmatrix}, \tag{20}$$

where η is the shear viscosity of the fluid, the ks are permeabilities including possible cross-coupling terms, and the pressures appearing here are the actual pore pressures in the storage and fracture porosity. We can extract the terms we need from (5), and then take the divergence to obtain

$$\begin{pmatrix} 1/(1 - v^{(2)})\phi^{(1)} & 0 \\ 0 & 1/v^{(2)}\phi^{(2)} \end{pmatrix} \begin{pmatrix} b_{12} + b_{23} & -b_{23} \\ -b_{23} & b_{13} + b_{23} \end{pmatrix} \begin{pmatrix} \nabla \cdot (\dot{\mathbf{U}}^{(1)} - \dot{\mathbf{u}}) \\ \nabla \cdot (\dot{\mathbf{U}}^{(2)} - \dot{\mathbf{u}}) \end{pmatrix} = - \begin{pmatrix} p^{(1)}_{,ii} \\ p^{(2)}_{,ii} \end{pmatrix}. \tag{21}$$

Comparing these two sets of equations and solving for the b coefficients, we find

$$b_{12} = \frac{\eta(1-v^{(2)})\phi^{(1)}\left[(1-v^{(2)})\phi^{(1)}k^{(22)} - v^{(2)}\phi^{(2)}k^{(21)}\right]}{k^{(11)}k^{(22)} - k^{(12)}k^{(21)}}, \tag{22}$$

$$b_{13} = \frac{\eta v^{(2)}\phi^{(2)}\left[v^{(2)}\phi^{(2)}k^{(11)} - (1-v^{(2)})\phi^{(1)}k^{(12)}\right]}{k^{(11)}k^{(22)} - k^{(12)}k^{(21)}}, \tag{23}$$

and

$$b_{23} = \frac{\eta v^{(2)}(1-v^{(2)})\phi^{(1)}\phi^{(2)}k^{(21)}}{k^{(11)}k^{(22)} - k^{(12)}k^{(21)}} = \frac{\eta v^{(2)}(1-v^{(2)})\phi^{(1)}\phi^{(2)}k^{(12)}}{k^{(11)}k^{(22)} - k^{(12)}k^{(21)}}. \tag{24}$$

For many applications it will be adequate to assume that the cross-coupling vanishes. In this situation, $b_{23} = 0$,

$$b_{12} = \frac{\eta(1-v^{(2)})^2(\phi^{(1)})^2}{k^{(11)}} \tag{25}$$

and

$$b_{13} = \frac{\eta(v^{(2)}\phi^{(2)})^2}{k^{(22)}}, \tag{26}$$

which also provides a simple interpretation of these coefficients in terms of the porosities and diagonal permeabilities.

This completes the identification of the inertial and drag coefficients introduced in the previous section.

4 THE DISPERSION RELATION AND ITS SOLUTION

It is now possible to write down and solve the dispersion relation for waves propagating through the double-porosity medium that we have been developing in the previous sections.

4.1 Derivation of the dispersion relation

We will first take a Fourier transform of (5) in the time domain, equivalent to assuming a time dependence of the form $\exp(-i\omega t)$. Then, (5) becomes

$$-\omega^2 \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} \begin{pmatrix} u_i \\ U_i^{(1)} \\ U_i^{(2)} \end{pmatrix} = \begin{pmatrix} \tau_{ij,j} \\ -\bar{p}_{,i}^{(1)} \\ -\bar{p}_{,i}^{(2)} \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} q_{11} &= \rho_{11} + \frac{i}{\omega}(b_{12} + b_{13}), \\ q_{12} &= \rho_{12} - \frac{i}{\omega}b_{12}, \quad \text{etc.} \end{aligned} \tag{28}$$

It is also convenient to notice that

$$\frac{\partial}{\partial x_i}\begin{pmatrix} u_i \\ U_i^{(1)} \\ U_i^{(2)} \end{pmatrix} = \begin{pmatrix} e \\ U_{i,i}^{(1)} \\ U_{i,i}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & & \\ 1 & \frac{1}{(1-v^{(2)})\phi^{(1)}} & \\ 1 & & \frac{1}{v^{(2)}\phi^{(2)}} \end{pmatrix}\begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix} \equiv \mathbf{R}\begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix}, \qquad (29)$$

which will permit us to write the final equation in terms of the macroscopic strain and fluid contents e, $\zeta^{(1)}$, and $\zeta^{(2)}$. The final equality in (29) defines the matrix $\mathbf{R}$, which we need again later in the analysis.

TABLE 1. Stress-strain parameters in double-porosity modeling as derived by Berryman and Wang [1995].

Parameter	Formula	Berea Sandstone
a_{11} (GPa^{-1})	$1/K$	0.167
a_{12} (GPa^{-1})	$-\alpha^{(1)}K_s^{(1)}/K^{(1)}K_s$	-0.074
a_{13} (GPa^{-1})	$-\alpha/K - a_{12}$	-0.068
a_{22} (GPa^{-1})	$v^{(1)}\alpha^{(1)}/B^{(1)}K^{(1)}$	0.144
a_{23} (GPa^{-1})	$-v^{(1)}\alpha^{(1)}/K^{(1)} - a_{12}$	0.001
a_{33} (GPa^{-1})	$v^{(2)}/K_f + v^{(1)}/K^{(1)} - (1-2\alpha)/K + 2a_{12}$	0.075
$\overline{a}_{33}$ (GPa^{-1})	$a_{33} - v^{(2)}/K_f$	0.067

TABLE 2. Material Properties for Berea Sandstone and Water

Parameter	Berea Sandstone
K (GPa)	6.0[a]
K_s (GPa)	39.0[a]
α	0.85[a]
$K^{(1)}$ (GPa)	10.0[a]
$\nu^{(1)}$	0.15
$K_s^{(1)}$ (GPa)	39.0[a]
$\alpha^{(1)}$	0.74[a]
$\phi^{(1)}$	0.178[a]
$B^{(1)}$	0.600
$v^{(2)}$	0.0178
$k^{(11)}$ (m^2)	1.0×10^{-16}
$k^{(22)}$ (m^2)	1.0×10^{-12}
	Water
K_f (GPa)	2.3
ρ_f (Kgm/m^3)	1000.0
η (Pa·s)	0.001

[a]From Coyner [1984]

Recall from previous work [Berryman and Wang, 1995] that

$$\begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} -p_c \\ -p^{(1)} \\ -p^{(2)} \end{pmatrix}, \tag{30}$$

where $-p_c = \tau_{ii}/3$, and the a_{ij}'s are the double-porosity compliances relating pressures to macroscopic strain and fluid contents. Formulas relating these parameters to properties of the constituents are summarized in Table 1, in which values are used for Berea sandstone from the results tabulated in Table 2. The definitions of the input parameters are: K and $K^{(1)}$ are the (jacketed) frame bulk moduli of the whole and the matrix respectively, K_s and $K_s^{(1)}$ are the unjacketed bulk moduli for the whole and the matrix, $\alpha = 1 - K/K_s$ and $\alpha^{(1)} = 1 - K^{(1)}/K_s^{(1)}$ are the corresponding Biot-Willis parameters, K_f is the pore fluid bulk modulus, $v^{(2)} = 1 - v^{(1)}$ is the total volume fraction of the fractures in the whole, and $B^{(1)}$ is Skempton's pore-pressure buildup coefficient for the matrix, while $\nu^{(1)}$ and $\phi^{(1)}$ are Poisson's ratio and the porosity of the matrix. It was observed by Berryman and Wang [1995] that the fluid-fluid coupling term a_{23} was small or negligible for the examples considered, and that it is expected to be small or negligible in most situations in which it makes sense to use the double-porosity model. Therefore, we will make the approximation in the remainder of this paper that $a_{23} \equiv 0$, as this reasonable choice will also make the subsequent analysis somewhat less tedious.

We need to express the vector on the right hand side of (27) in terms of the macroscopic variables, using the constitutive relations in (30) plus the usual relations of linear elasticity. The basic set of equations for assumed isotropic media (analogous expressions for single-porosity with and without elastic anisotropy are given in Berryman [1998]) has the form

$$\begin{pmatrix} S_{11} & S_{12} & S_{12} & -\beta^{(1)} & -\beta^{(2)} & & & \\ S_{12} & S_{11} & S_{12} & -\beta^{(1)} & -\beta^{(2)} & & & \\ S_{12} & S_{12} & S_{11} & -\beta^{(1)} & -\beta^{(2)} & & & \\ -\beta^{(1)} & -\beta^{(1)} & -\beta^{(1)} & a_{22} & a_{23} & & & \\ -\beta^{(2)} & -\beta^{(2)} & -\beta^{(2)} & a_{23} & a_{33} & & & \\ & & & & & \frac{1}{2\mu} & & \\ & & & & & & \frac{1}{2\mu} & \\ & & & & & & & \frac{1}{2\mu} \end{pmatrix} \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ -p^{(1)} \\ -p^{(2)} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta^{(1)} \\ -\zeta^{(2)} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \tag{31}$$

where the S_{ij}'s are the usual drained elastic compliances, and the β's are analogous to the Biot-Willis parameter [Biot-Willis, 1957] for single-porosity within factors of 3 and K where K is the drained bulk modulus. We will not show our work here, but it is not hard to derive the following three relations:

$$\tau_{ij,j} = (\lambda + \mu)e_{,i} + \mu u_{i,jj} - 3K\left[\beta^{(1)}p^{(1)}_{,i} + \beta^{(2)}p^{(2)}_{,i}\right], \tag{32}$$

$$-3K\left[\beta^{(1)}p^{(1)}_{,i} + \beta^{(2)}p^{(2)}_{,i}\right] = -p_{c,i} - Ke_{,i}, \tag{33}$$

and

$$-3\left[\beta^{(1)}p^{(1)}_{,i} + \beta^{(2)}p^{(2)}_{,i}\right] = B^{(1)}\left[-\zeta^{(1)}_{,i} - 3\beta^{(1)}p_{c,i}\right] + B^{(2)}\left[-\zeta^{(2)}_{,i} - 3\beta^{(2)}p_{c,i}\right]. \tag{34}$$

The Lamé parameters for the drained medium are λ and μ. A linear combination of the last two equations can be found to eliminate the appearance of $p_{c,i}$, and then this result can be substituted into (7) to show that

$$\tau_{ij,j} = (K_u + \frac{1}{3}\mu)e_{,i} + \mu u_{i,jj} + K_u\left[B^{(1)}\zeta^{(1)}_{,i} + B^{(2)}\zeta^{(2)}_{,i}\right], \tag{35}$$

where

$$K_u = \frac{K}{1 - 3K\left[\beta^{(1)}B^{(1)} + \beta^{(2)}B^{(2)}\right]} \tag{36}$$

is the undrained bulk modulus for the double porosity medium. (This statement is consistent with our assumption that $a_{23} \equiv 0$, but needs some qualification if $a_{23} \neq 0$.)

Combining (35) with (30) and taking the divergence, we finally obtain the expression we need:

$$\begin{pmatrix} \tau_{ij,ji} \\ -p^{(1)}_{,ii} \\ -p^{(2)}_{,ii} \end{pmatrix} = \begin{pmatrix} K_u + \frac{4}{3}\mu & B^{(1)}K_u & B^{(2)}K_u \\ -a_{12}a_{33}/D & (a_{11}a_{33} - a_{13}^2)/D & a_{12}a_{13}/D \\ -a_{13}a_{22}/D & a_{12}a_{13}/D & (a_{11}a_{22} - a_{12}^2)/D \end{pmatrix} \begin{pmatrix} e_{,ii} \\ -\zeta^{(1)}_{,ii} \\ -\zeta^{(2)}_{,ii} \end{pmatrix}, \tag{37}$$

where $D = a_{11}a_{22}a_{33} - a_{12}^2a_{33} - a_{13}^2a_{22}$. Taking the divergence of (27), then substituting (29) and (37), and finally taking the spatial Fourier transform (having wavenumber k) gives the complex eigenvalue problem associated with the wave propagation problem:

$$\begin{pmatrix} K_u + \frac{4}{3}\mu & B^{(1)}K_u & B^{(2)}K_u \\ -a_{12}a_{33}/D & (a_{11}a_{33} - a_{13}^2)/D & a_{12}a_{13}/D \\ -a_{13}a_{22}/D & a_{12}a_{13}/D & (a_{11}a_{22} - a_{12}^2)/D \end{pmatrix} \begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix} =$$

$$v^2(\omega) \begin{pmatrix} 1 & & \\ & \frac{1}{(1-v^{(2)})\phi^{(1)}} & \\ & & \frac{1}{v^{(2)}\phi^{(2)}} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & \frac{1}{(1-v^{(2)})\phi^{(1)}} & \\ 1 & & \frac{1}{v^{(2)}\phi^{(2)}} \end{pmatrix} \begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix}, \tag{38}$$

where the eigenvalue $v^2(\omega) = \omega^2/k^2$ has the physical significance of being the square of the complex wave velocity. With obvious definitions for the matrices $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{Q}$, while $\mathbf{R}$ was previously defined in (29), we rewrite (38) as

$$\mathbf{A} \begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix} = v^2(\omega)\mathbf{PQR} \begin{pmatrix} e \\ -\zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix}, \tag{39}$$

and then, in terms of these matrices, the dispersion relation determining $v^2(\omega)$ at all angular frequencies ω is

$$\det(\mathbf{A} - v^2(\omega)\mathbf{PQR}) = 0. \tag{40}$$

This is a 3×3 determinant of complex numbers that must be solved for v^2. A method for finding the three solutions is discussed in the next subsection.

4.2 Solution of the dispersion relation

A variety of numerical methods may be used to solve (40), including for example Crout's reduction method [Crout, 1941]. However, since the system is relatively small (3×3) and since our purposes include gaining some physical insight into the processes involved, it will prove instructive to do some more analysis on the problem prior to the ultimate numerical calculations.

First, note that our analysis will be considerably simplified by an obvious rearrangement of the determinant (40) so that

$$\det(\mathbf{G} - v^2\mathbf{I}) = 0, \tag{41}$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{G} = \mathbf{A}\mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{P}^{-1}, \tag{42}$$

having matrix elements g_{ij}, for $i, j = 1, 2, 3$. The matrix $\mathbf{R}$ was given previously in (29) and is clearly invertible for finite values of the two porosities. Matrix $\mathbf{Q}$ is also always invertible for realistic choices of material parameters.

Using the properties of determinants, it is not hard to show that the 3×3 determinant (41) can be reduced to a 2×2 determinant in either of two convenient forms:

$$\det\begin{pmatrix} (g_{11}-v^2)(g_{22}-v^2)-g_{12}g_{21} & -1 \\ g_{12}g_{23}g_{31}+g_{13}g_{32}g_{21}-(g_{22}-v^2)g_{13}g_{31}-(g_{11}-v^2)g_{23}g_{32} & g_{33}-v^2 \end{pmatrix} =$$
$$\det\begin{pmatrix} (g_{11}-v^2)(g_{33}-v^2)-g_{13}g_{31} & -1 \\ g_{12}g_{23}g_{31}+g_{13}g_{32}g_{21}-(g_{33}-v^2)g_{12}g_{21}-(g_{11}-v^2)g_{23}g_{32} & g_{22}-v^2 \end{pmatrix} = 0. \tag{43}$$

It is useful to write the determinant in each of these two ways in order to make connections with single-porosity models. The first version in (43) has as its upper left-hand element an expression that corresponds precisely to the determinant for the 2×2 system when the only porosity present is the matrix (or storage) porosity. Similarly, the second version in (43) has as its element in the upper left-hand position a term that corresponds to the determinant for the 2×2 system when the only porosity present is the fracture/crack porosity. In each case the remaining terms determine the effect of coupling to the second type of porosity. Depending on other system parameters such as permeabilities and porosities, either of these limits may be useful to consider, and may also provide good starting points for the iterative (Newton-Raphson) procedure that we will use to solve the determinant equation for the complex velocity v.

If we express the determinant (41) as a polynomial $D(x)$ where $x = v^2$, then

$$\begin{aligned} D(x) = -x^3 &+ (g_{11}+g_{22}+g_{33})x^2 - (g_{22}g_{33}+g_{33}g_{11}+g_{11}g_{22}-g_{23}g_{32}-g_{13}g_{31}-g_{12}g_{21})x \\ &+ (g_{11}g_{22}g_{33}-g_{23}g_{32}g_{11}-g_{13}g_{31}g_{22}-g_{12}g_{21}g_{33}+g_{12}g_{23}g_{31}+g_{13}g_{32}g_{21}) = 0. \end{aligned} \tag{44}$$

The Newton-Raphson method [Hildebrand, 1956] for solving this equation for x is an iteration process starting from some initial choice x_0 and computing

$$x_i = x_{i-1} - D(x_{i-1})/D'(x_{i-1}), \qquad \text{for} \qquad i = 1, 2, \ldots, i_c \tag{45}$$

(where D' is the first derivative of D with respect to x) until some convergence criterion has been met at $i = i_c$. The fact that the coefficients and the solution of the problem are complex adds no special complication to this procedure. However, since the polynomial is complex, it is important that good starting values x_0 be obtained for each of the three roots of (44), as a search procedure in the complex plane would be considerably more difficult to implement than Newton-Raphson.

From (43), we see that two choices of starting values for the complex parameter x are given by (for example)

$$x_{1,2}^{(0)} = \frac{1}{2}\left(g_{11}+g_{22} \pm \sqrt{(g_{11}-g_{22})^2 + 4g_{12}g_{21}}\right). \tag{46}$$

Once the Newton-Raphson iteration has converged from both of these starting values to their final values of x_1 and x_2, then the third solution of the dispersion relation is obtained directly by recalling that

$$D(x) \equiv \sum_{n=0}^{3} D_n(-x)^n = -(x-x_1)(x-x_2)(x-x_3), \tag{47}$$

which shows that

$$x_3 = D_2 - x_1 - x_2 = \frac{D_1 - x_1x_2}{x_1+x_2} = \frac{D_0}{x_1x_2}. \tag{48}$$

Any one of these three formulas may be used to calculate x_3 directly, or combinations of them (such as the geometric mean of any two) may be used to reduce the errors that might be introduced by any lack of convergence of the values used for x_1 and x_2.

For each value of x that solves the dispersion equation, we can then compute the wave velocity and inverse of the quality factor Q using the definition

$$\frac{1}{\sqrt{x}} \equiv \frac{1}{v(\omega)} \equiv \frac{1}{v_r}\left(1 + \frac{i}{2Q}\right). \tag{49}$$

This definition of $1/Q$ is accurate when attenuation losses are low, but should be carefully interpreted for high loss situations (*e.g.,* low frequency diffusive modes). In particular, it is often stated that, when losses are high, (49) needs to be modified so that if

$$\frac{1}{v(\omega)} \equiv \frac{1}{v_r}\left(1 + \frac{i\alpha v_r}{\omega}\right), \tag{50}$$

where α is the attenuation coefficient (having units of inverse length), then (as Hamilton [1972] and Bourbié *et al.* [1987] show)

$$\frac{1}{Q} \equiv \frac{2\alpha v_r}{\omega - \alpha^2 v_r^2/\omega}. \tag{51}$$

However, for diffusive modes the imaginary and real parts of the velocity are of comparable size, and therefore it is possible that the formula (51) will be singular at low frequencies for such modes. To avoid this complication, we use the definition (49) in all cases, and then interpret those situations where in which $1/Q \to 2$ as an indication that the mode under consideration is actually diffusive rather than propagatory.

4.3 Higher frequency behavior and analyticity

For the purposes of this paper, we will take (25) and (26) to be the low frequency limits of the drag coefficients and also assume that $b_{23} \equiv 0$. Then, the coefficients b_{12} and b_{13} must be modified at higher frequencies in order to assure that the analytic structure of the theory as a whole preserves obvious physical requirements such as nonnegative dissipation for all modes at all times. This issue arises naturally when we have obtained the solution $x = v^2$ to (45) for any one of the three compressional modes. Then, taking the complex square root, we get two roots that differ only by $+$ and $-$ signs. We want the solution that has both positive real velocity and a negative imaginary part. This is so because $k = \omega/v = k_0 + i\alpha$, where α should be a physical attenuation coefficient such that

$$\exp i(kz - \omega t) = \exp(-\alpha z)\exp ik_0(z - v_0 t) \tag{52}$$

leads to a decrease in the overall amplitude of the compressional mode. If, for any of the three compressional modes, no root exists with both positive real velocity and negative imaginary part, then the dispersion relation is unphysical and the approximations we have made in deriving it are suspect.

In our examples to follow, we will tentatively take the results from Appendix B as the proper way to modify the drag coefficients at higher frequencies, but must always be careful to check that this choice does not lead to unphysical behavior.

5 EXAMPLE

The example we present here is for Berea sandstone saturated with water. The parameters used in the calculations are taken from Tables 1 and 2. Most of the mechanical properties were obtained from measurements made by Coyner [1984]. The permeability values for the matrix $k^{(11)}$ and the fractures $k^{(22)}$ are the same as those used by Lewallen and Wang [1998].

The approach described in the preceding text, together with the results obtained in Appendices A and B, has been implemented by writing a Fortran code and computing the eigenvalues for the three compressional modes and their corresponding eigenvectors in the frequency range 10.0 Hz to 1 MHz. The results for the computed velocities and inverse quality factors are then displayed in Figures 1–6. The results for the eigenvectors will be described but not displayed here. We will not present results for the shear modes, but expect them to differ little from results for single-porosity calculations for the same material, since the pore fluid does not significantly affect the shear moduli in these models.

Figures 1 and 2 show that the first compressional wave is dispersive and has its main contributions to attenuation ($1/Q \simeq .001$) centered at about 3 kHz, with significant decrease in the attenuation envelope (by about an order of magnitude) at 100 Hz and 100 kHz. Wave velocity dispersion is localized approximately to the frequency range 1 kHz to 10 kHz, and the total dispersion is less than 1 %. The eigenvector for this mode shows that the storage pore fluid is essentially moving in concert with the solid frame throughout the frequency range considered, with some small but largely negligible deviations above 1 kHz. On the other hand, the fracture pore fluid oscillates out of phase with respect to the solid frame with an amplitude as much as about one half that of the frame amplitude above about 10 kHz. The observed dissipation for this mode is clearly tied to the out of phase motion of the fracture fluid.

Figures 3 and 4 show that the second compressional mode is diffusive at low frequencies, but becomes propagatory with a $Q \simeq 5$ or greater at about 10 kHz. The wave speed is quite small at these higher frequencies (about 550 m/s), indicating that the wave is probably propagating mostly through some pore fluid along a tortuous path. The eigenvector analysis shows that the storage fluid excitation is again quite small compared to that of the fracture fluid, although it is about two orders of magnitude larger than that observed for the first compressional wave. The main effect observed for the second compressoinal wave is a large oscillation of the fracture fluid relative to the solid frame, so that this mode can be properly characterized in this example as a slow wave (in the sense of single-porosity poroelasticity) through the fracture fluid.

Figures 5 and 6 show that the third compressional mode is diffusive at all frequencies in the range considered. The apparent velocity is much lower (less than 100 m/s) — even than that of the second compressional mode. The eigenvector analysis for this mode shows that both the storage fluid and the fracture fluid are oscillating with significant and comparable amplitudes, larger than that of the solid frame. The two fluids are also oscillating out of phase with each other. The amplitude of the storage fluid oscillation slightly dominates that of the fracture fluid, which partly explains the increased attenuation for this mode. For this example, we might characterize this mode as a slow wave through the storage fluid, but note that this interpretation is slightly over simplified.

Finally, we note that other examples have been computed in an attempt to verify the nature of the dependence of the mode parameters on the input parameters. It has been observed for example that the second and third compressional wave "velocities" at low frequencies are proportional to the square root of the fluid bulk modulus as would be expected for a single-porosity slow wave.

6 DISCUSSION AND CONCLUSIONS

The analyticity issue is an important one, but not one we have not dwelt upon in this paper. Because of the complex nature of the dispersion equations and the existence of three compressional modes for this theory, it will require some substantial amount of effort to clarify the proper analytic structure for the theory so that unphysical results are not generated. The main issues here are whether it is always appropriate or not to use the results presented in Appendix B. It is known that these types of results are valid for single-porosity analysis, but when used together with some of the approximations we have made here — such as (for example) the neglect of cross-terms in the quasistatic permeability equations — may introduce some unphysical behavior at high frequencies for some combinations of parameters. We leave this analysis to future work.

Another important practical issue involves the careful comparison of these results with the extensive literature on wave propagation and attenuation in fractured earth materials. These comparisons will also be left to the future.

We conclude that the double-porosity dual-permeability analysis that has been presented here has the capability to explain both wave propagation and attenuation in earth materials when the attenuation is due to out-of-phase motion of pore fluids in storage and fracture porosity. However, there remains quite a lot of work to do yet both on the theory and on its applications to real data before we can consider the story to be complete.

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Appendix A

Tortuosity for Double-Porosity Media

Theoretical estimates of tortuosity for the matrix and fracture components of the double-porosity medium may be obtained by noting that equation (7) implies

$$\tau^{(1)} = \frac{1}{2}\left(1 + \frac{1}{\phi^{(1)}}\right) \tag{53}$$

for storage porosity that is spherical in shape, while

$$\tau^{(2)} = 1, \tag{54}$$

for the fracture porosity, because $\phi^{(2)} = 1$ by assumption.

It is more difficult to estimate the overall tortuosity τ, but a reasonable physical value can be obtained by considering the Hashin-Shtrikman bounds on electrical conductivity of a composite medium [Hashin and Shtrikman, 1962]. These bounds show that for a two-component medium the effective conductivity will lie between the values $\sigma_{HS}^{\pm}$ given by the formula [Berryman, 1995]

$$\frac{1}{\sigma_{HS}^{\pm} + 2\sigma_m} = \frac{1 - v^{(2)}}{\sigma_1 + 2\sigma_m} + \frac{v^{(2)}}{\sigma_2 + 2\sigma_m}, \tag{55}$$

where

$$\sigma_m = \max\left(\sigma_1, \sigma_2\right) \quad \text{for} \quad \sigma_{HS}^{+} \tag{56}$$

and

$$\sigma_m = \min\left(\sigma_1, \sigma_2\right) \quad \text{for} \quad \sigma_{HS}^{-}. \tag{57}$$

The notation means that σ_{HS}^{+} is the upper bound, while σ_{HS}^{-} is the lower bound.

Recalling that electrical tortuosity is related to formation factor F by $\tau = \phi F$, where $F = \sigma_f/\sigma$, we find that the tortuosity bounds for the double-porosity medium are:

$$\frac{1}{\phi/\tau^{\pm} + 2/F_m} = \frac{1 - v^{(2)}}{\phi^{(1)}/\tau^{(1)} + 2/F_m} + \frac{v^{(2)}}{1 + 2/F_m}. \tag{58}$$

We will assume that the overall tortuosity of the fractured double-porosity medium is in fact dominated by the fractures, in which case it is appropriate to assume that the actual electrical conductivity will be close to the upper bound σ_{HS}^{+}. In this case we choose $F_m = 1$ and, after rearranging the formula, we find

$$\tau \simeq \phi \frac{v^{(2)}\phi^{(1)} + (3 - v^{(2)})\tau^{(1)}}{(3 - 2v^{(2)})\phi^{(1)} + 2v^{(2)}\tau^{(1)}}. \tag{59}$$

Also, recall that the overall porosity is given by $\phi = (1 - v^{(2)})\phi^{(1)} + v^{(2)}$. The formula (59) is expected to be valid for situations in which $v^{(2)} \ll 1$. For applications to media in which such an assumption is not valid, the formula (55) should generally be used instead of (59).

Appendix B

Frequency dependent permeability, tortuosity, and viscosity

Johnson *et al.* [1987] have shown that for a single-porosity medium the frequency dependence of the dynamic permeability and tortuosity can be well-approximated by

$$k(\omega) = \frac{k_0}{\left[1 - 4ik_0^2\tau_\infty^2\rho_f\omega/\eta\Lambda^2\phi^2\right]^{\frac{1}{2}} - ik_0\tau_\infty\rho_f\omega/\eta\phi} \tag{60}$$

and

$$\tau(\omega) = \tau_\infty + \frac{i\eta\phi}{k_0\rho_f\omega}\left[1 - 4ik_0^2\tau_\infty^2\rho_f\omega/\eta\Lambda^2\phi^2\right]^{\frac{1}{2}}. \tag{61}$$

The symbol η stands for the fluid viscosity (in units of $kgm/m \cdot s$), while η/ρ_f is the kinematic viscosity (in units of m^2/s). The new symbols appearing in these formulas are the d.c. permeability k_0, the high frequency tortuosity τ_∞, and the lambda parameter Λ introduced by Johnson *et al.* [1986]. The values of the high frequency tortuosity τ_∞ for double-porosity media were considered here in Appendix A. The d.c. permeabilities for double-porosity media have been considered in Berryman and Wang [1995] and in Lewallen and Wang [1998].

For single porosity media, Johnson *et al.* [1987] show that the lambda parameter approximately satisfies

$$\Lambda^2 = 8k_0F = 8k_0\tau_\infty/\phi. \tag{62}$$

For the present purposes, we will assume that this relation holds independently for the storage porosity and the fracture porosity. Then we will have

$$\Lambda^{(1)} = \left[8k^{(11)}\tau^{(1)}/\phi^{(1)}\right]^{\frac{1}{2}} \tag{63}$$

and

$$\Lambda^{(2)} = \left[8k^{(22)}\tau^{(2)}/\phi^{(2)}\right]^{\frac{1}{2}} = \left[8k^{(22)}\right]^{\frac{1}{2}}. \tag{64}$$

Finally, we see that for the double-porosity medium, the corrections due to frequency dependence can be viewed alternatively as a frequency dependent viscosity, since euqations (60) and (61) follow by assuming that

$$\eta(\omega) \equiv \eta\left(1 - i\Lambda^2\rho_f\omega/16\eta\right)^{\frac{1}{2}} \tag{65}$$

These corrections need to be made separately for the two types of pores. This interpretation of the frequency dependence as being associated specifically with the viscosity is the one advocated by Biot [1956b], and has some advantages in the present context as it makes it quite straightforward to determine what the corrections should be for the multiple porosity problem. Note that we used (62) to simplify the factors inside the square root in (65).

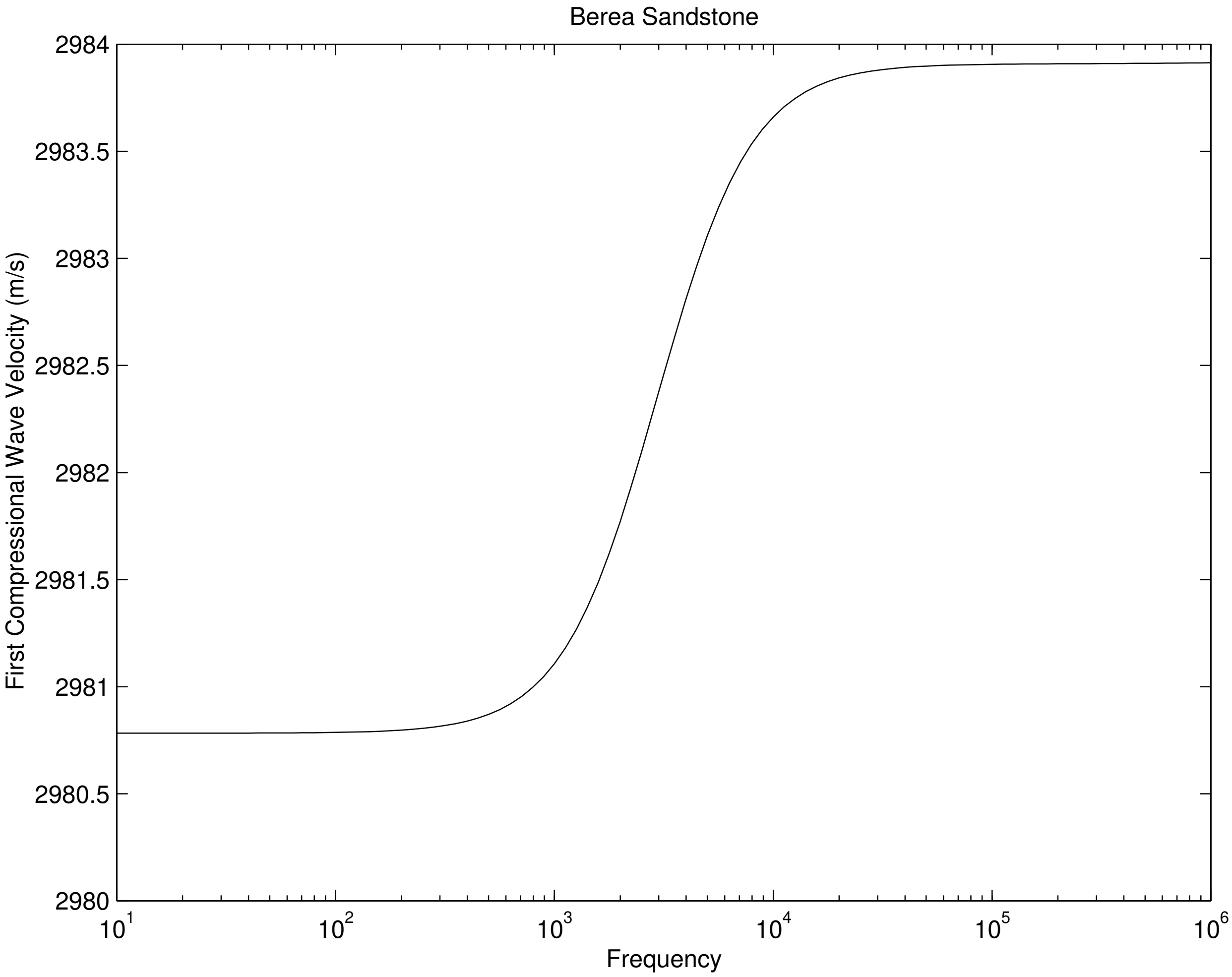

Figure 1: First compressional wave velocity as a function of frequency for fractured Berea sandstone.

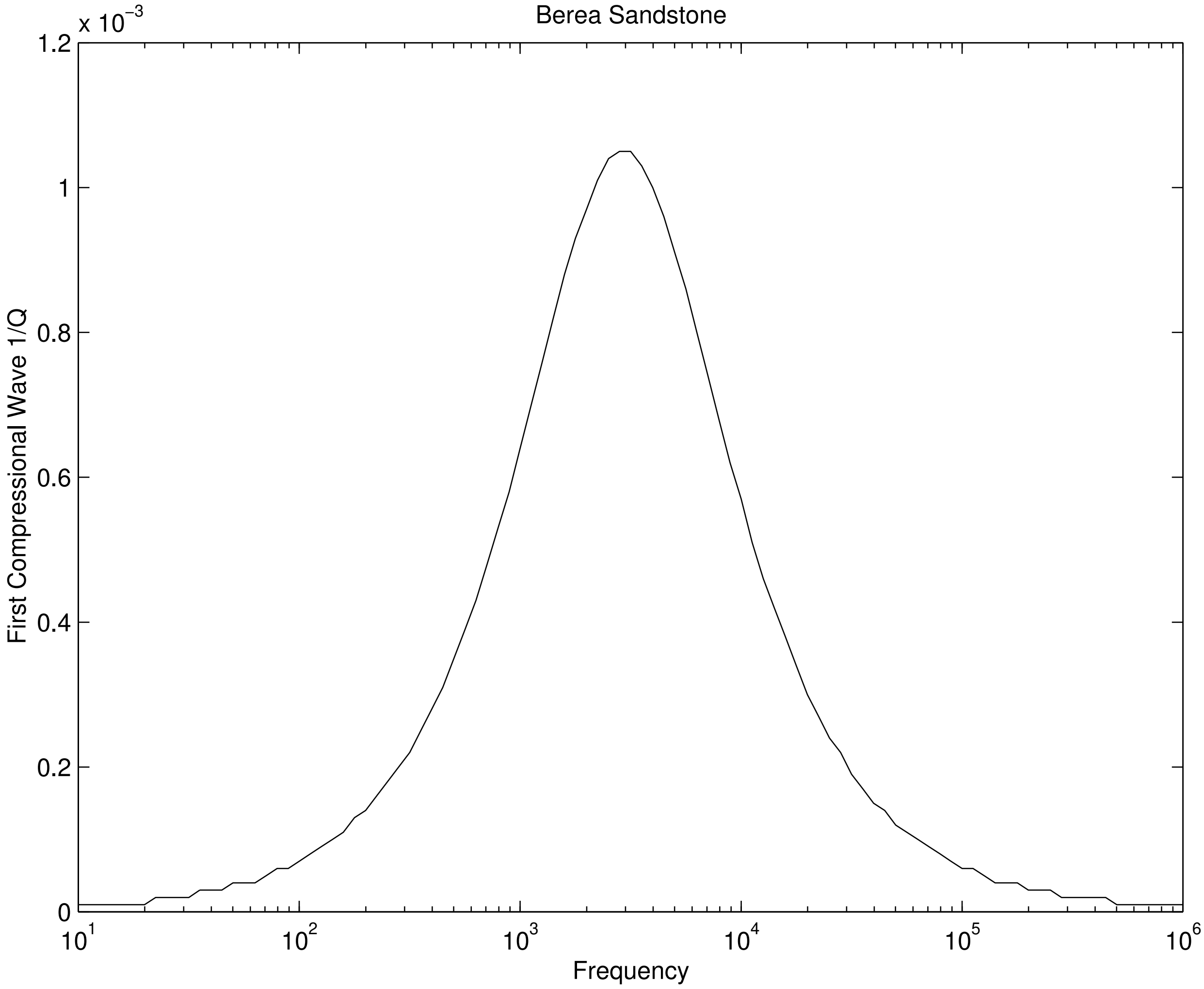

Figure 2: First compressional wave inverse quality factor $1/Q$ as a function of frequency for fractured Berea sandstone.

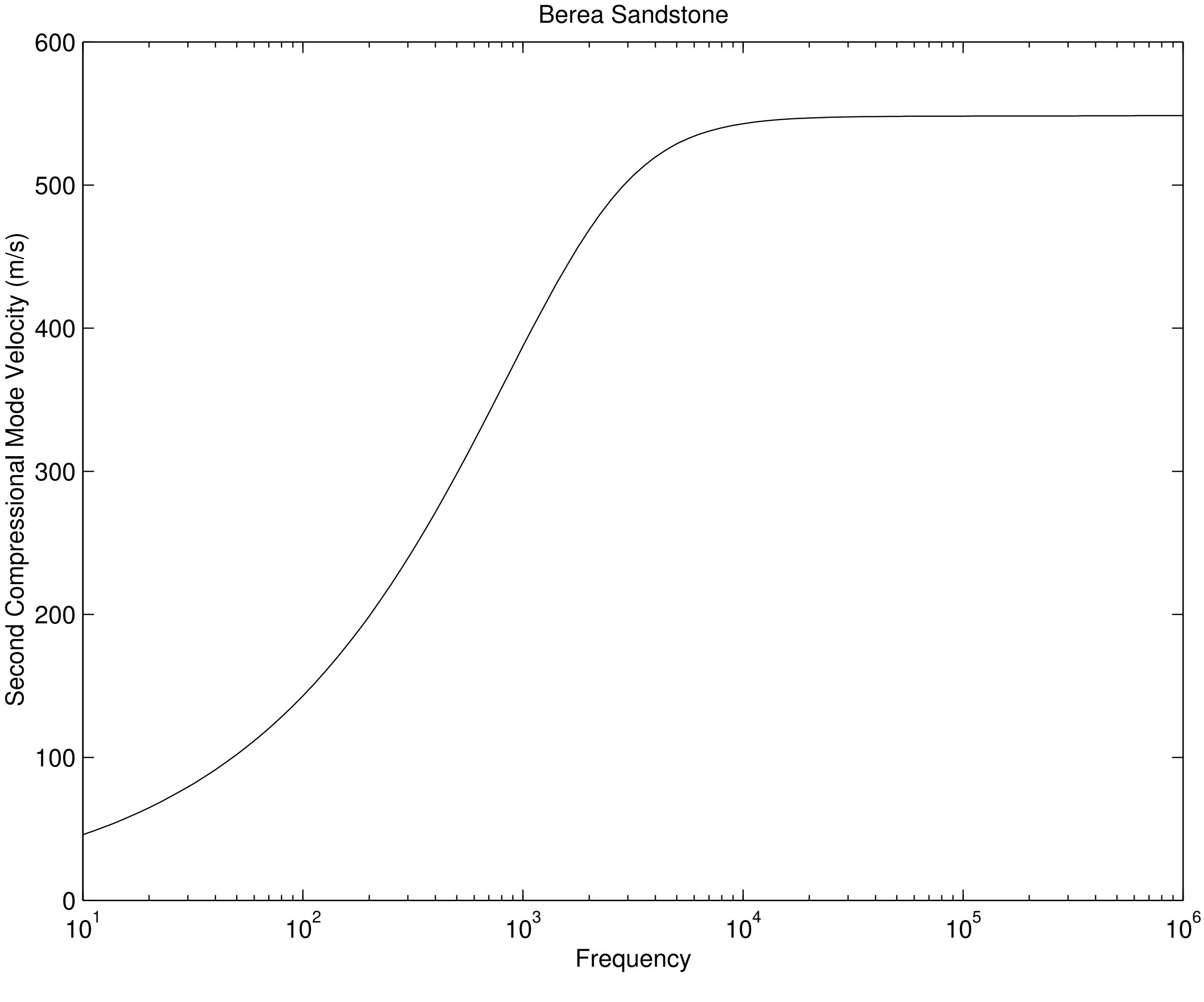

Figure 3: Second compressional wave velocity as a function of frequency for fractured Berea sandstone.

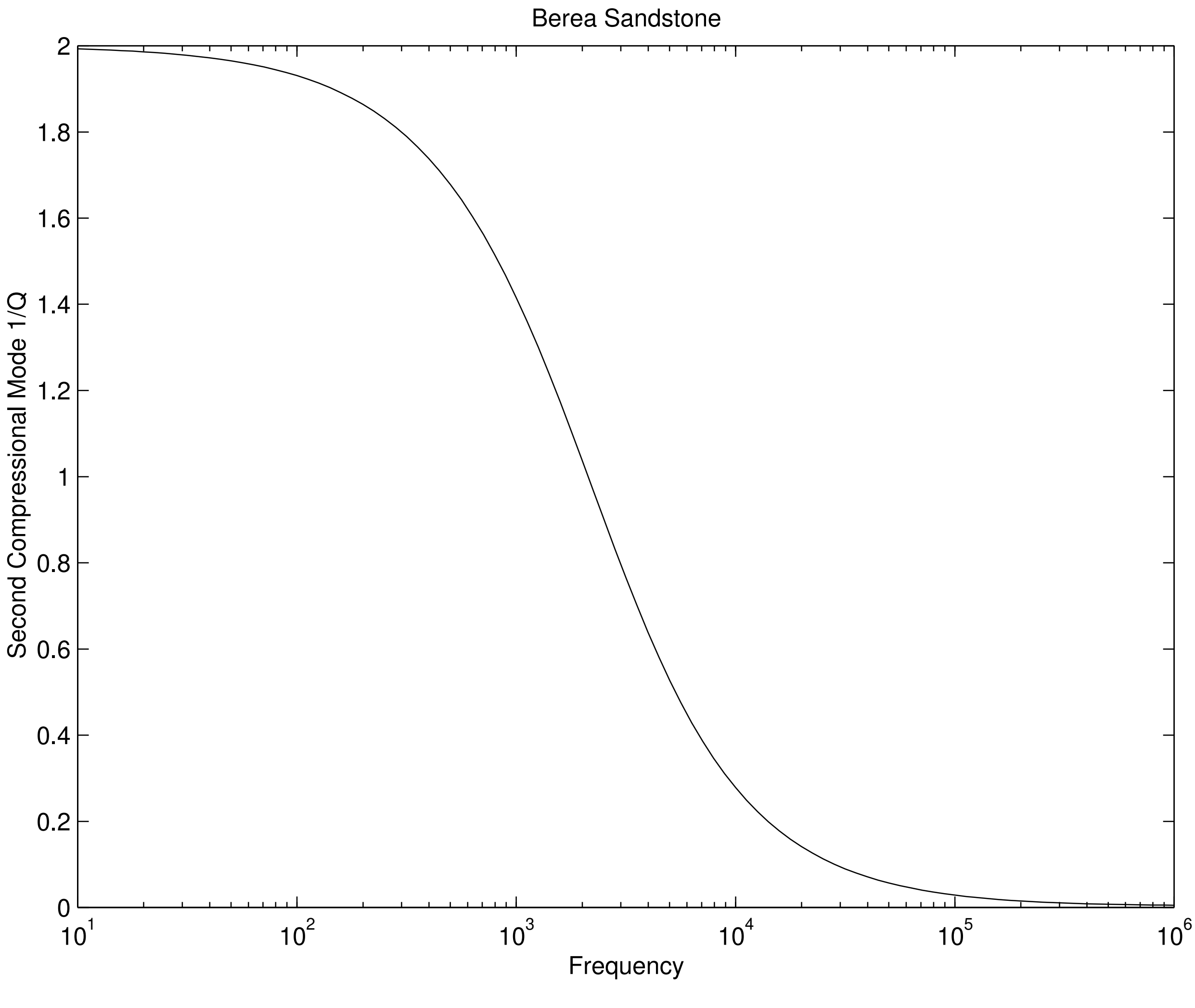

Figure 4: Second compressional wave inverse quality factor $1/Q$ as a function of frequency for fractured Berea sandstone.

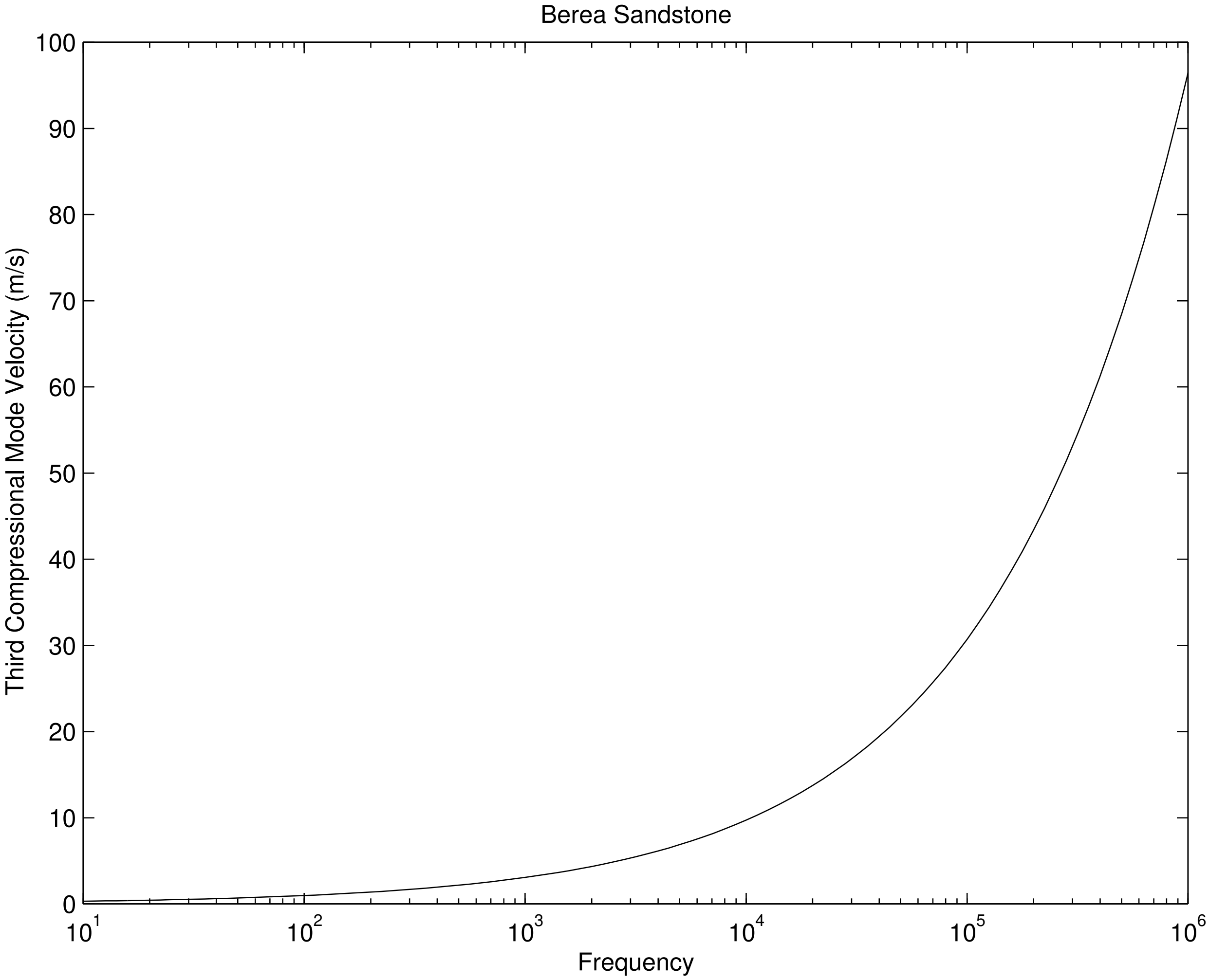

Figure 5: Third compressional wave velocity as a function of frequency for fractured Berea sandstone.

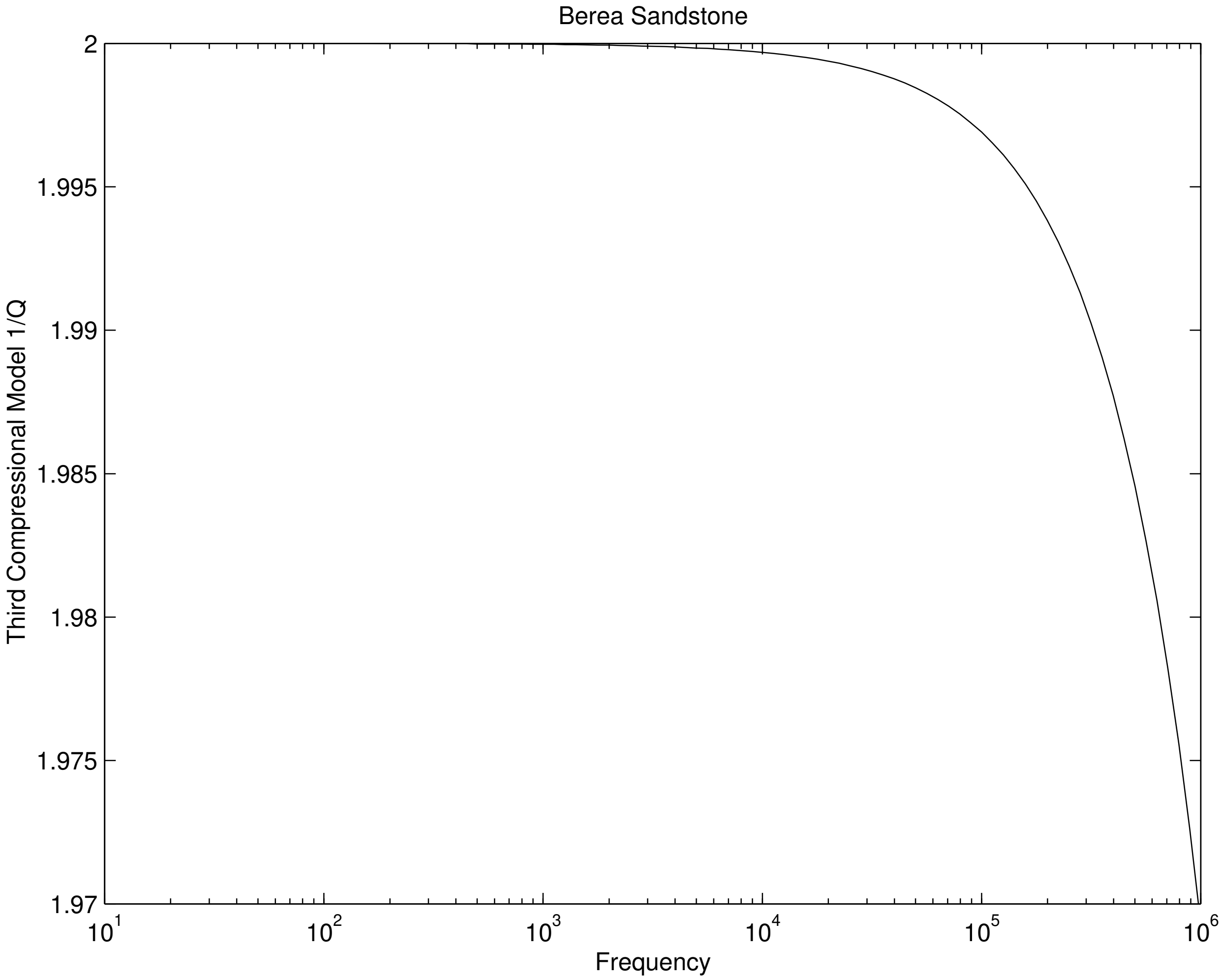

Figure 6: Third compressional wave inverse quality factor $1/Q$ as a function of frequency for fractured Berea sandstone.