UCRL-JC-129xxx Version date: March 9, 1998

Exact Results for Anisotropic Poroelasticity Arising from Oriented Isotropic Inclusions

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Abstract

Berryman and Milton [1991] have shown that exact results relating the Biot-Willis parameter to bulk moduli of isotropic constituents can be obtained when there are two isotropic constituents in welded contact throughout a composite poroelastic medium. These earlier results assumed that the constituents were randomly oriented so the macroscopic composite was isotropic. Here we show that these results may be generalized to permit the isotropic constituents to be preferentially oriented so that the macroscopic medium is arbitrarily anisotropic. One example of the results considered is a layered two-component VTI medium. We show that known exact results for such media are easily recovered using the new results. The general approach presented allows immediate use of known results on two-component elastic composites to be applied for determination of poroelastic coefficients.

1 Introduction

It is relatively rare to find exact results in the theory of composite materials (for background material on applications of the theory of composites to rocks see Berryman [1995] and for more general applications of exact results in the theory of composites see Milton [1996]). However, two of the most interesting exceptions to that general rule occur in the complex fields of poroelasticity and thermoelasticity. Curiously, the extra complication of having more than one kind of independent field that can influence the state of strain of the medium introduces an additional simplification in some cases. This fact was first noticed in thermoelasticity by Cribb [1968], Hashin [1984], and Schulgasser [1989], and later in poroelasticity by Berryman and Milton [1991] and Norris [1992].

The argument leading to these exact results is quite elementary. If we assume that we have a composite material composed of two types of constituents and those constituents are both isotropic, poroelastic or thermoelastic materials, then it is always possible to find some ratio of the two constant applied fields (stress and fluid pressure in poroelasticity/stress and temperature in thermoelasticity) such that the strain in both constituents is the same. If the strain in both constituents is everywhere the same, then the strain throughout the medium, and therefore the composite as a whole, is the same. This argument is powerful because, since the applied fields for this special case are constant, the conditions for stress equilibrium are satisfied trivially and the entire problem then reduces to simple algebra. The equal strain relations then imply a set of equations that can be solved for the Biot-Willis parameter (in poroelasticity) or the thermal expansion coefficient (in thermoelasticity) of the composite in terms of the bulk moduli of the two constituents and the overall bulk modulus of the composite system.

We wish to point out here that, although these arguments have generally been applied to macroscopically isotropic composites (usually with isotropic constituents, but sometimes with anisotropic constituents when polycrystals are considered), the same arguments apply with almost no change to macroscopic composites having arbitrary anisotropy in both poroelasticity and thermoelasticity as long as the constituents are isotropic and limited to two in number. We will restrict the discussion here to poroelasticity, but the same arguments follow the same lines of argument for thermoelasticity.

2 Porous Elastic Materials Containing Fluids

We consider elastic materials containing voids or pores, where the pores may be fluid-filled. The volume fraction of the medium containing voids is the porosity ϕ.

One simplification that arises immediately is due to the fact that the presence of pore fluids has no mechanical effect on the shear moduli, so $\mu_{dr} = \mu$. There may be other effects on the shear moduli due to the presence of pore fluids, such as softening of cementing materials or expansion of interstitial clays, which we call "chemical" effects to distinguish them from the purely mechanical effects to be considered here. We neglect all such chemical effects in the following analysis. This means that analysis for the effective shear moduli in unaltered in the presence of fluids. Thus, we can simplify our system of equations in order to focus on the parts of the analysis that do change in the presence of fluids.

A saturating pore fluid introduces the possibility of an additional control field and an additional type of strain variable. The pressure p_f in the fluid is the new field parameter that can be controlled. Allowing sufficient time for global pressure equilibration will permit us to consider p_f to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change ζ in the amount of fluid mass contained in the pores (see Berryman and Thigpen [1985]) is the new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume while including the effects of compression or expansion of the pore fluid itself due to changes in p_f. It is most convenient to write the resulting equations in terms of compliances rather than stiffnesses, so the basic equation to be considered takes the form:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & -\beta \\ s_{12} & s_{11} & s_{12} & -\beta \\ s_{12} & s_{12} & s_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{1}$$

The vector on the left contains the three volume strains e_{11}, e_{22}, e_{33} and the increment of fluid content ζ, which is defined by $\zeta \equiv \phi(\delta V_f/V_f - \delta V_\phi/V_\phi)$ where V_f is the initial fluid volume, V_ϕ the initial pore volume ($V_\phi = \phi V$), and δV_f and δV_ϕ are the changes in these quantities in response to the applied stresses. The vector on the right contains the stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ and the fluid pressure p_f. The constant coefficients appearing in the matrix on the right will be defined in the following two paragraphs. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances s_{ij} appearing in (1) are simply related to the drained Lamé parameters λ_{dr} and μ_{dr} in the same way they are related in normal elasticity, whereas the individual stiffnesses obtained by inverting the equation in (1) must contain coupling terms through the parameters β and γ that depend on the pore and fluid compliances. Thus, we find that

$$s_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + \mu}{\mu(3\lambda_{dr} + 2\mu)} \tag{2}$$

and

$$s_{12} = -\frac{\nu_{dr}}{E_{dr}}, \tag{3}$$

where the drained Young's modulus E_{dr} is defined by the second equality of (2) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + \mu)}. \tag{4}$$

When the external stress is hydrostatic so $\sigma = \sigma_{11} = \sigma_{22} = \sigma_{33}$, equation (1) for an isotropic constituent telescopes down to

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K_{dr} & -\alpha/K_{dr} \\ -\alpha/K_{dr} & \alpha/BK_{dr} \end{pmatrix} \begin{pmatrix} \sigma \\ -p_f \end{pmatrix}, \tag{5}$$

where $e = e_{11} + e_{22} + e_{33}$, $K_{dr} = \lambda_{dr} + \frac{2}{3}\mu$ is the drained bulk modulus, $\alpha = 1 - K_{dr}/K_m$ is the Biot-Willis parameter [Biot and Willis, 1957] with K_m being the bulk modulus of the solid minerals present, and Skempton's pore-pressure buildup parameter B [Skempton, 1954] is given by

$$B = \frac{1}{1 + K_p(1/K_f - 1/K_m)}. \tag{6}$$

New parameters appearing in (6) are the bulk modulus of the pore fluid K_f and the pore modulus $K_p = \phi K_{dr}/\alpha$ where ϕ is the porosity. We are supposing here that the constituents are pure, single solid component materials with pores, which we will sometimes refer to as Gassmann materials [Gassmann, 1951].

Comparing (1) and (5), we find that, for the ith constituent,

$$\beta^{(i)} = \frac{\alpha^{(i)}}{3K_{dr}^{(i)}} \tag{7}$$

and

$$\gamma^{(i)} = \frac{\alpha^{(i)}}{B^{(i)}K_{dr}^{(i)}}. \tag{8}$$

Thus, we conclude that for the ith isotropic constituent, we have

$$e_{11}^{(i)} = e_{22}^{(i)} = e_{33}^{(i)} = \frac{1}{3K_{dr}^{(i)}} \left(\sigma + \alpha^{(i)} p_f\right), \tag{9}$$

when a uniform stress σ and uniform fluid pressure p_f are applied.

Now, if the overall composite is anisotropic due to preferential alignment of the constituent particles (which we might think of as being general ellipsoids for simplicity, but this does not affect the argument), we consider the anisotropic generalization of (1)

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11}^* & s_{12}^* & s_{13}^* & -\beta_1^* \\ s_{12}^* & s_{22}^* & s_{23}^* & -\beta_2^* \\ s_{13}^* & s_{23}^* & s_{33}^* & -\beta_3^* \\ -\beta_1^* & -\beta_2^* & -\beta_3^* & \gamma^* \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{10}$$

The shear terms are excluded again since they do not interact mechanically with the fluid effects. This form is not completely general in that it includes orthorhombic, cubic, hexagonal, and all isotropic systems, but excludes triclinic, monoclinic, trigonal, and some tetragonal systems that would have some nonzero off-diagonal terms in the full elastic matrix. Note that the s^*_{ij}s are all independent of the fluid, and therefore would be the same if we were treating the medium as as elastic one ignoring the fluid pressure, or assuming that the fluid saturant is air. Our problem now is to relate the β^*_js to these compliances.

If the rock is composed of isotropic constituents A and B, then there exists a set of choices of σ and p_f such that (for example)

$$e^{(A)}_{11} = e^{(B)}_{11} = \frac{1}{3K^{(A)}_{dr}}\left(\sigma + \alpha^{(A)} p_f\right) = \frac{1}{3K^{(B)}_{dr}}\left(\sigma + \alpha^{(B)} p_f\right). \tag{11}$$

The appropriate set of choices is determined by the ratio

$$\frac{\sigma}{p_f} = \frac{\alpha^{(B)} K^{(A)} - \alpha^{(A)} K^{(B)}}{K^{(B)} - K^{(A)}}, \tag{12}$$

which easily follows from (11). Because the constituents are isotropic the same ratio (12) also produces the same strains in the other two directions in both constituents. Furthermore, we must therefore have that

$$e_{11} = e_{22} = e_{33} = e^{(A)}_{11} = e^{(B)}_{11} = \ldots, \tag{13}$$

for the overall strain of the composite when (12) is valid. Thus, we find that

$$\begin{aligned}\beta^*_1 + \left(s^*_{11} + s^*_{12} + s^*_{13}\right)\frac{\sigma}{p_f} &= \beta^*_2 + \left(s^*_{12} + s^*_{22} + s^*_{23}\right)\frac{\sigma}{p_f} = \beta^*_3 + \left(s^*_{13} + s^*_{23} + s^*_{33}\right)\frac{\sigma}{p_f} \\ &= \frac{1}{3K^{(A)}}\left(\alpha^{(A)} + \frac{\sigma}{p_f}\right) = \frac{1}{3}\left(\frac{\alpha_{(B)} - \alpha^{(A)}}{K^{(B)} - K^{(A)}}\right),\end{aligned} \tag{14}$$

where the ratio σ/p_f can be replaced everywhere by (12), as has been done to produce the final equality on the right. Equation (14) is the main result of the paper. It provides a general relationship between the β^*s and the compressibilities for a two-component composite poroelastic medium as desired.

To demonstrate the validity of these formulas, we work an example in the next section.

3 Layered Example

Because of the generality of the arguments leading to (14), there are no assumptions about geometry imbedded in this result and it is therefore acceptable to consider arbitrary geometries. One geometry that can be chosen for which exact results are known is the case of layers of isotropic components. This problem can be solved exactly for any number of components using the methods originally introduced by Backus [1962].

The trick in using the Backus method is to rearrange the equations so all the fast variables are in the left-hand vector and all the slow (actually constant for quasistatics) variables

are in the right-hand vector which multiplies the matrix of variable coefficients. Making this transformation from (1) using well-known identities (2) and (3) for the isotropic case, we obtain

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ -e_{33} \\ \zeta \end{pmatrix} = \begin{pmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & \nu/(1-\nu) & \beta E/(1-\nu) \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & \nu/(1-\nu) & \beta E/(1-\nu) \\ \nu/(1-\nu) & \nu/(1-\nu) & -(1-\nu-2\nu^2)/(1-\nu)E & \beta(1+\nu)/(1-\nu) \\ \beta E/(1-\nu) & \beta E/(1-\nu) & \beta(1+\nu)/(1-\nu) & -\left[\gamma - 2\beta^2 E/(1-\nu)\right] \end{pmatrix} \times \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}, \quad (15)$$

with the fast variables on the left and the slow variables (actually constant) in the vector on the right. Signs have been chosen so the matrix is symmetric. We have also dropped the subscript dr from the drained constants ν and E in (15) as there should be no confusion. Note that the 3×3 submatrix in the upper left is exactly the same as what would be found in the purely elastic case.

Once we have (15), the averaging is trivial being just the volume average of each matrix element appearing in the equation. Having performed thess averages, we must then backsolve to determine the relationship between these average matrix elements and the quantities of interest. Using (2) and (3) to return to expressions involving λ and μ, the results of present interest for a general multicomponent layered poroelastic medium are:

$$\beta_1 = \beta_2 = \left\langle \frac{\alpha\mu}{\lambda+2\mu} \right\rangle \left\langle \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} \right\rangle^{-1}, \quad (16)$$

and

$$\beta_3 = \left[\left\langle \frac{\alpha}{\lambda+2\mu} \right\rangle \left\langle \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} \right\rangle - 2 \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{\alpha\mu}{\lambda+2\mu} \right\rangle \right] \left\langle \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} \right\rangle^{-1}. \quad (17)$$

One means of checking the validity of these results is to consider the limit in which the shear modulus μ is a constant. It is known that the layered elastic medium becomes isotropic in this case, and we expect that $\beta_1 = \beta_2 = \beta_3$ should also hold. In this limit, we easily find that

$$\beta_1 = \beta_3 = \left\langle \frac{\alpha}{\lambda+2\mu} \right\rangle \left\langle \frac{3\lambda+2\mu}{\lambda+2\mu} \right\rangle^{-1}. \quad (18)$$

For two-component layered media, our general results (14) imply that

$$\beta_1 = \frac{1}{3}\left(\frac{\alpha^{(B)} - \alpha^{(A)}}{K^{(B)} - K^{(A)}} \right) - \left(\frac{\alpha^{(B)} K^{(A)} - \alpha^{(A)} K^{(B)}}{K^{(B)} - K^{(A)}} \right) \left\langle \frac{\mu}{\lambda+2\mu} \right\rangle \left\langle \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} \right\rangle^{-1}, \quad (19)$$

and

$$\beta_3 = \frac{1}{3}\left(\frac{\alpha^{(B)}-\alpha^{(A)}}{K^{(B)}-K^{(A)}}\right) - \left(\frac{\alpha^{(B)}K^{(A)}-\alpha^{(A)}K^{(B)}}{K^{(B)}-K^{(A)}}\right) \times \left[\left\langle\frac{1}{\lambda+2\mu}\right\rangle - 2\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle\left\langle\frac{\mu}{\lambda+2\mu}\right\rangle\left\langle\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\right\rangle^{-1}\right]. \quad (20)$$

It is nontrivial to show that these equations are consistent with (16) and (17), but the algebra becomes relatively straightforward if we first notice the two-component identities:

$$\left\langle\frac{\alpha}{\lambda+2\mu}\right\rangle = \frac{1}{3}\left(\frac{\alpha^{(B)}-\alpha^{(A)}}{K^{(B)}-K^{(A)}}\right)\left\langle\frac{3\lambda+2\mu}{\lambda+2\mu}\right\rangle - \left(\frac{\alpha^{(B)}K^{(A)}-\alpha^{(A)}K^{(B)}}{K^{(B)}-K^{(A)}}\right)\left\langle\frac{1}{\lambda+2\mu}\right\rangle, \quad (21)$$

and

$$\left\langle\frac{\alpha\mu}{\lambda+2\mu}\right\rangle = \frac{1}{3}\left(\frac{\alpha^{(B)}-\alpha^{(A)}}{K^{(B)}-K^{(A)}}\right)\left\langle\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\right\rangle - \left(\frac{\alpha^{(B)}K^{(A)}-\alpha^{(A)}K^{(B)}}{K^{(B)}-K^{(A)}}\right)\left\langle\frac{\mu}{\lambda+2\mu}\right\rangle. \quad (22)$$

Substituting (22) into (16) gives (19), while substituting both (21) and (22) into (17) gives (20).

This completes the demonstration that the general results (14) are valid for the two-component layered model in poroelasticity. Next we show how to make use of the results when elastic measurements are available for rocks.

Acknowledgments

Work of both authors was performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48 and supported specifically by the Geosciences Research Program of the DOE Office of Energy Research within the Office of Basic Energy Sciences, Division of Engineering and Geosciences, as part of the Advanced Computational Technology Initiative.

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