Elastic and poroelastic analysis of Thomsen parameters for seismic waves in finely layered VTI media

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ABSTRACT

Layered earth models are well justified by experience, and provide a simple means of studying fairly general behavior of the elastic and poroelastic characteristics of seismic waves in the earth. Thomsen's anisotropy parameters for weak elastic and poroelastic anisotropy are now commonly used in exploration, and can be conveniently expressed in terms of the layer averages of Backus. Since our main interest is usually in the fluids underground, it would be helpful to have a set of general equations relating the Thomsen parameters as directly as possible to the fluid properties. This end can be achieved in a rather straightforward fashion for these layered earth models, and the present paper develops and then discusses these relations. It is found that, although there are five effective shear moduli for any layered VTI medium, one and only one effective shear modulus for the layered system contains all the dependence of pore fluids on the elastic or poroelastic constants that can be observed in vertically polarized shear waves in VTI media. The effects of the pore fluids on this effective shear modulus can be substantial (as much as a factor of 5 in the examples presented here) when the medium behaves in an undrained fashion, as might be expected at higher frequencies such as sonic and ultrasonic waves for well-logging or laboratory experiments, or at seismic wave frequencies for low permeability regions of reservoirs, prior to hydrofracing. The results presented are strictly for velocity analysis, not for amplitude or attenuation.

INTRODUCTION

Gassmann's fluid substitution formulas for bulk and shear moduli (Gassmann, 1951) were originally derived for the quasi-static mechanical behavior of fluid saturated rocks. It has been shown recently (Berryman and Wang, 2001) that it is possible to understand deviations from Gassmann's results at higher frequencies when the rock is heterogeneous, and in particular when the rock heterogeneity anywhere is locally anisotropic. On the other hand, a well-known way of generating anisotropy in the earth is through fine layering. Then, Backus' averaging (Backus, 1962) of the mechanical behavior of the layered isotropic media at the microscopic level produces anisotropic mechanical behavior at the macroscopic level. For our present purposes, the Backus averaging concept can also be applied to fluid-saturated porous media,

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and thereby permits us to study how deviations from Gassmann's predictions could arise in an analytical and rather elementary fashion. We consider both closed-pore and open-pore boundary conditions between layers within this model in order to study in detail how violations of Gassmann's predictions can arise in undrained versus drained conditions, or for high versus low frequency waves.

We review some standard results concerning layered VTI media in the first two sections. Then, we discuss singular value composition of the elastic (or poroelastic) stiffness matrix in order to introduce the interpretation of one shear modulus (out of the five shear moduli present) that has been shown recently (Berryman, 2003) to contain all the important behavior related to pore fluid influence on the shear deformation response. These results are then incorporated into our analysis of the Thomsen parameters for weak anisotropy. For purposes of analysis, expressions are derived for the quasi-P- and quasi-SV-wave speeds and these results are then discussed from this new point of view. Numerical examples show that the approximate analysis presented is completely consistent with the full theory for layered media. Our conclusions are summarized in the final section of the paper.

NOTATION AND SOME PRIOR RESULTS

Notation for VTI media

We begin by recalling some notation needed in the remainder of the paper. For transversely isotropic media with vertical symmetry axis, the relationship between components of stress σ_{kl} and strain $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (where u_j is the jth component of the displacement vector) is given by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \tag{1}$$

where $a = b + 2m$ (*e.g.*, Musgrave, 1970; Auld, 1973), with i, j, k, l obviously each ranging from 1 to 3 in Cartesian coordinates. The matrix describes isotropic media in the special case when $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$.

The Thomsen (1986) parameters ϵ, δ, and γ are related to these stiffnesses by

$$\epsilon \equiv \frac{a - c}{2c}, \tag{2}$$

$$\delta \equiv \frac{(f + l)^2 - (c - l)^2}{2c(c - l)}, \tag{3}$$

$$\gamma \equiv \frac{m-l}{2l}. \tag{4}$$

For P-wave propagation in the earth near the vertical, the important anisotropy parameter is δ. For SV-wave propagation near the vertical, the combination $(\epsilon - \delta)$ plays essentially the same role as δ does for P-waves. For SH-waves, the pertinent anisotropy parameter is γ. All three of the Thomsen parameters vanish for an isotropic medium.

It is also useful to note for later reference that

$$a = c(1+2\epsilon), \qquad m = l(1+2\gamma), \qquad \text{and} \qquad f \simeq c(1+\delta) - 2l. \tag{5}$$

In TI media, c and l are the velocities normal to the layering. Then, ϵ, γ, and δ measure the deviations from these normal velocities at other angles. We present the relevant details of the phase velocity analysis later in the paper.

Gassmann results for isotropic poroelastic media

To understand the significance of the results to follow, we briefly review a well-known result due to Gassmann (1951) [also see Berryman (1999b) for a tutorial]. Gassmann's equation relates the bulk modulus K^* of a saturated, undrained isotropic porous medium to the bulk modulus K_{dr} of the same medium in the drained case:

$$K^* = K_{dr}/(1-\alpha B), \tag{6}$$

where the parameters α and B [respectively, the Biot-Willis parameter (Biot and Willis, 1957) and Skempton's pore-pressure buildup coefficient (Skempton, 1954)] depend on the porous medium and fluid compliances. For the shear moduli of the drained (μ_{dr}) and saturated (μ^*) media, Gassmann's quasi-static theory gives

$$\mu^* = \mu_{dr}. \tag{7}$$

We want to emphasize once more that (7) is a result of the theory, not an assumption. It follows immediately from (6) for any isotropic poroelastic medium. Furthermore, the two equations (6) and (7) taken together show that, for isotropic microhomogeneous media, the fluid effect is all contained in the parameter $\lambda^* = K^* - \frac{2}{3}\mu^*$, where λ and μ are the well-known Lamé parameters. This result is crucial for understanding the significance of our later results to oil and gas exploration.

Backus averaging

Backus (1962) presented an elegant method of producing the effective constants for a thinly layered medium composed of either isotropic or anisotropic elastic layers. This method applies either to spatially periodic layering or to random layering, by which we mean either that the material constants change in a nonperiodic (unpredictable) manner from layer to layer or that the layer thicknesses might also be random. For simplicity, we will assume that the physical properties of the individual layers are isotropic. The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying coefficients multiply slowly varying stresses or strains.

The derivation has been given many places including Schoenberg and Muir (1989) and Berryman (1999a). One illuminating derivation given recently by Milton (2002) will be followed here, with the main difference being that we assume the layering direction is z or 3. We break the equation down into 3×3 pieces so that

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = A_{11} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \end{pmatrix} + A_{13} \begin{pmatrix} e_{33} \\ e_{23} \\ e_{31} \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = A_{31} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \end{pmatrix} + A_{33} \begin{pmatrix} e_{33} \\ e_{23} \\ e_{31} \end{pmatrix}, \tag{9}$$

where the 3×3 matrices are

$$A_{11} = \begin{pmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}, \quad A_{13} = A_{31}^T = \begin{pmatrix} \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{33} = \begin{pmatrix} \lambda+2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}. \tag{10}$$

Noting that the variables σ_{11}, σ_{22}, σ_{12}, e_{33}, e_{23}, and e_{31} are fast variables in the layers, and all the remaining variables are slow (actually constant), it is advantageous to rearrange these equations so the slow variables multiply the elastic parameter matrices and are all on one side of the equations, while the fast variables are all alone on the other side of the equations. Then, it is trivial to perform the layer averages, since they depend only on the (assumed known) values of the elastic parameters in the layers and are multiplied by constants. Having done this, we can then transform back into the standard forms of (8) and (9) with the stresses and strains now reinterpreted as the overall values, and find the following relationships (where the star indicates the effective property of the layered system):

$$A_{33}^* = \left\langle A_{33}^{-1} \right\rangle^{-1}, \tag{11}$$

$$A_{13}^* = \left(A_{31}^*\right)^T = \left\langle A_{13} A_{33}^{-1} \right\rangle A_{33}^*, \tag{12}$$

and

$$A_{11}^* = \langle A_{11} \rangle + A_{13}^* \left(A_{33}^*\right)^{-1} A_{31}^* - \left\langle A_{13} A_{33}^{-1} A_{31} \right\rangle. \tag{13}$$

The brackets $\langle x \rangle$ indicate the volume (or equivalaently the one-dimensional layer) average of the quantity x in the simple layered medium under consideration. It follows that the anisotropy coefficients in equation (1) are then related to the layer parameters by the following expressions:

$$c = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}, \tag{14}$$

$$f = c \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle, \tag{15}$$

$$l = \left\langle \frac{1}{\mu} \right\rangle^{-1}, \tag{16}$$

$$m = \langle \mu \rangle, \tag{17}$$

$$a = \frac{f^2}{c} + 4m - 4 \left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle, \tag{18}$$

and

$$b = a - 2m. \tag{19}$$

When the layering is fully periodic, these results may be attributed to Bruggeman (1937) and Postma (1955), while for more general layered media including random media they should be attributed to Backus (1962). The constraints on the Lamé parameters λ and μ for each individual layer are $0 \le \mu \le \infty$ and $-\frac{2}{3}\mu \le \lambda \le \infty$. Although, for physically stable materials, μ and the bulk modulus $K = \lambda + \frac{2}{3}\mu$ must both be nonnegative, λ (and also Poisson's ratio) may be negative. Large fluctuations in λ for different layers are therefore entirely possible, in principle, but may or may not be an issue for any given region of the earth.

One very important fact that is known about these equations (Backus, 1962) is that they reduce to isotropic results with $a = c$, $b = f$, and $l = m$, if the shear modulus is a constant ($= \mu$), regardless of the behavior of λ. This fact is also very important for applications involving partial and/or patchy saturation (Mavko *et al.*, 1998; Johnson, 2001). Furthermore, it is closely related to the well-known bulk modulus formula of Hill (1963) for isotropic composites having uniform shear modulus, and also to the Hashin-Shtrikman bounds (Hashin and Shtrikman, 1961).

THOMSEN PARAMETERS ϵ AND δ

Thomsen's ϵ

An important anisotropy parameter for quasi-SV-waves is Thomsen's parameter ϵ, defined in equation (2). Formula (18) for a may be rewritten as

$$a = \left\langle \frac{(\lambda + 2\mu)^2 - \lambda^2}{\lambda + 2\mu} \right\rangle + c \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2, \tag{20}$$

which can be rearranged into the convenient and illuminating form

$$a = \langle\lambda + 2\mu\rangle - c\left[\left\langle\frac{\lambda^2}{\lambda+2\mu}\right\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^2\right]. \tag{21}$$

This formula is very instructive because the term in square brackets is in Cauchy-Schwartz form $[\langle\alpha^2\rangle\langle\beta^2\rangle \geq \langle\alpha\beta\rangle^2]$, so this factor is nonnegative. Furthermore, the magnitude of this term depends mainly on the fluctuations in the λ Lamé parameter, largely independent of μ, since μ appears only in the weighting factor $1/(\lambda+2\mu)$. Clearly, if $\lambda = constant$, then this bracketed factor vanishes identically, regardless of the behavior of μ. Large fluctuations in λ will tend to make this term large. If in addition we consider Thomsen's parameter ϵ written in a similar fashion as

$$2\epsilon = \left[\langle\lambda+2\mu\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - 1\right] - \left[\left\langle\frac{\lambda^2}{\lambda+2\mu}\right\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^2\right], \tag{22}$$

we find that the term enclosed in the first bracket on the right hand side is again in Cauchy-Schwartz form showing that it always makes a positive contribution unless $\lambda+2\mu = constant$, in which case it vanishes. Similarly, the term enclosed in the second set of brackets is always non-negative, but the minus preceding the second bracket causes this contribution to make a negative contribution to 2ϵ unless $\lambda = constant$, in which case it vanishes. So, the sign of ϵ is indeterminate. The Thomsen parameter ϵ may have either a positive or a negative sign for a TI medium composed of arbitrary thin isotropic layers.

Helbig and Schoenberg (1987) discuss an interesting case where large fluctuations in μ combined with large fluctuations in λ, including $\lambda < 0$ for one component, lead to wavefronts with anomalous polarizations in layered TI media. Schoenberg (1994) also discusses several shale examples having large fluctuations in both λ and μ.

Fluctuations of λ in the earth have important implications for oil and gas exploration. As we recalled in our earlier discussion, Gassmann's well-known results (Gassmann, 1951) show that, when isotropic porous elastic media are saturated with any fluid, the fluid has no mechanical effect on the shear modulus μ, but — when these results apply — it can have a significant effect on the bulk modulus $K = \lambda + \frac{2}{3}\mu$, and therefore on λ. Thus, observed variations in layer μ should have no direct information about fluid content, while observed variations in layer λ, especially if they are large variations, may contain important clues about variations in fluid content. So the observed structure of ϵ in (22) strongly suggests that small positive and all negative values of ϵ may be important indicators of significant fluctuations in fluid content.

From (21), we can infer in general that

$$a \leq \langle\lambda+2\mu\rangle\,, \tag{23}$$

so we have an upper bound on ϵ in finely layered media stating that

$$2\epsilon \leq \left(\langle\lambda+2\mu\rangle\left\langle\frac{1}{\lambda+2\mu}\right\rangle - 1\right) = \left(\langle\rho v_p^2\rangle\langle\rho^{-1}v_p^{-2}\rangle - 1\right), \tag{24}$$

where ρ is the density.

Thomsen's δ

Thomsen's parameter δ defined by Eq. (3) is pertinent for near vertical quasi-P-waves and can also be rewritten as

$$\delta = -\frac{(c+f)(c-f-2l)}{2c(c-l)}. \tag{25}$$

This parameter is considerably more difficult to analyze than either γ or ϵ for various reasons, some of which we will enumerate shortly.

Because of a controversy surrounding the sign of δ for finely layered media (*e.g.*, Levin, 1988; Thomsen, 1988; Anno, 1997), Berryman *et al.* (1999) performed a series of Monte Carlo simulations with the purpose of establishing the existence or nonexistence of layered models having positive δ. Those simulation results should be interpreted neither as modeling of natural sedimentation processes nor as an attempt to reconstruct any petrophysical relationships. The main goal was to develop a general picture of the distribution of the sign of δ using many choices of constituent material properties.

The analysis of Berryman *et al.* (1999) established a similarity in the circumstances between the occurrence of positive δ and the occurrence of small positive ϵ (*i.e.*, both occur when Lamé λ is fluctuating greatly from layer to layer). The positive values of δ are in fact most highly correlated with the smaller positive values of ϵ. We should also keep in mind the fact that $\epsilon - \delta \geq 0$ is always true for layered models and this fact also plays a role in these comparisons, determining the unoccupied upper left hand corner of a δ vs. ϵ plot.

SINGULAR VALUE DECOMPOSITION

The singular value decomposition (SVD), or equivalently the eigenvalue decomposition in the case of a real symmetric matrix, for (1) is relatively easy to perform. We can immediately write down four eigenvectors:

$$\begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}, \quad \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix}, \quad \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}, \quad \begin{pmatrix} 1\\-1\\0\\0\\0\\0 \end{pmatrix}, \tag{26}$$

and their corresponding eigenvalues, respectively $2l$, $2l$, $2m$, and $a-b=2m$. All four correspond to shear modes of the system. The two remaining eigenvectors must be orthogonal to all four of these and therefore both must have the general form

$$\begin{pmatrix} 1 \\ 1 \\ X \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{27}$$

with the corresponding eigenvalue

$$\chi = a + b + fX. \tag{28}$$

The remaining condition that determines both X and χ is

$$\chi X = 2f + cX, \tag{29}$$

which, after substitution for χ, leads to a quadratic equation having the solutions

$$X_\pm = \frac{1}{2}\left(-\left[\frac{a+b-c}{f}\right] \pm \sqrt{8 + \left[\frac{a+b-c}{f}\right]^2} \right). \tag{30}$$

The ranges of values for $X_\pm$ are $0 \le X_+ \le \infty$ and, since $X_- = -2/X_+$, $-\infty \le X_- \le 0$. The interpretation of the solutions $X_\pm$ is simple for the isotropic limit where $X_+ = 1$ and $X_- = -2$, corresponding respectively to pure compression and pure shear modes. For all other cases, these two modes have mixed character, indicating that pure compression cannot be excited in the system, and must always be coupled to shear. Some types of pure shear modes can still be excited even in the nonisotropic cases, because the other four eigenvectors in (26) are unaffected by this coupling, and they are all pure shear modes. Pure compressional and shear modes are obtained as linear combinations of these two mixed modes according to

$$\alpha \begin{pmatrix} 1 \\ 1 \\ X_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ X_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1+\alpha) \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{31}$$

with $\alpha = -2(X_+ - 1)/[X_+(X_+ + 2)]$ for pure shear, and

$$\begin{pmatrix} 1 \\ 1 \\ X_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ X_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1+\beta) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{32}$$

and with $\beta = X_+(X_+ - 1)/(X_+ + 2)$ for pure compression.

To understand the behavior of X_+ in terms of the layer property fluctuations or, alternatively, in terms of the Thomsen parameters, it is first helpful to note that the pertinent functional $F(x) = \frac{1}{2}\left[-x + \sqrt{8+x^2}\right]$ is easily shown to be a monotonic function of its argument x. So it is sufficient to study the behavior of the argument $x = (a+b-c)/f$.

Exact results in terms of layer elasticity parameters

Combining results from Eqs. (18)–(15), we find after some work on rearranging the terms that

$$\frac{a+b-c}{f} = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^{-1} \left[\left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle + 6 \left\langle \frac{m-\mu}{\lambda+2\mu} \right\rangle \right. \\ \left. -8 \left\{ \left\langle \frac{\mu^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda+2\mu} \right\rangle^2 \right\} \right], \tag{33}$$

where the correction involving $m-\mu$ in the numerator is the difference of the shear modulus from the layer-averaged shear modulus m, and will be the dominant correction when fluctuations in μ are small. The fact that $\langle (m-\mu)/\mu \rangle = \langle \mu \rangle \langle 1/\mu \rangle - 1 \geq 0$, suggests that this dominant correction to unity (since the leading term is exactly unity) for this expression will be positive if λ and μ are positively correlated throughout all the layers, but the correction could be negative in cases where there is a strong negative correlation between λ and μ. On the other hand, the term in curly brackets in (33) is again in Cauchy-Schwartz form (*i.e.*, $\langle \alpha^2 \rangle \langle \beta^2 \rangle - \langle \alpha\beta \rangle^2 \geq 0$), and therefore is always non-negative. But, since it is multiplied by -1, the contribution to this expression is non-positive. This term is also quadratic in the deviations of μ from its layer average, and thus is of higher order than the term explicitly involving $m-\mu$. So, if the fluctuations in shear modulus are very large throughout the layered medium, the quadratic terms can dominate — in which case the overall result could be less than unity. Numerical examples developed by applying a code of V. Grechka [used previously in a similar context by Berryman *et al.* (1999)] confirm these analytical results.

Our main conclusion is that the shear modulus fluctuations giving rise to the anisotropy due to layering are (as expected) the main source of deviations of (33) from unity. But now we can say more, since positive deviations of this parameter from unity are generally associated with smaller magnitude fluctuations of the layer shear modulus, whereas negative deviations from unity must be due to large magnitude fluctuations in these shear moduli.

Approximate results for small values of Thomsen parameters

Using the definitions of the Thomsen parameters, we can also rewrite the terms appearing in (33) in order to make connection with this related point of view. Recalling (5) and the fact that $b = a - 2m$, we have

$$\frac{a+b-c}{f} \simeq 1 + \frac{3}{c-2l}(c\delta + 4l\gamma) + \frac{4}{c-2l}\left[c(\epsilon-\delta) - 4l\gamma\right], \tag{34}$$

with some higher order corrections involving powers of δ and products of δ with ϵ and γ that we have neglected here. We have added and subtracted equally some terms proportional to δ, and others proportional to γ, in order to emphasize the similarities between the form (34) and that found previously in (33). In particular, the difference $\epsilon - \delta$ is known (Postma, 1955; Berryman, 1979) to be non-negative and its deviations from zero depend on fluctuations in μ from layer to layer, behavior similar to that of the final term in (33). Since the formula (34) is only approximate and its interpretation requires the use of various other results we derive later for other purposes, we will for now delay further discussion of this point to the end of the paper.

DISPERSION RELATIONS FOR SEISMIC WAVES

The general behavior of seismic waves in anisotropic media is well known, and the equations are derived in many places including Berryman (1979) and Thomsen (1986). The results are

$$\rho\omega_\pm^2 = \frac{1}{2}\left\{(a+l)k_1^2 + (c+l)k_3^2 \pm \sqrt{[(a-l)k_1^2 - (c-l)k_3^2]^2 + 4(f+l)^2k_1^2k_3^2}\right\}, \quad (35)$$

for compressional (+) and vertically polarized shear (−) waves and

$$\rho\omega_s^2 = mk_1^2 + lk_3^2, \quad (36)$$

for horizontally polarized shear waves, where ρ is the overall density, ω is the angular frequency, k_1 and k_3 are the horizontal and vertical wavenumbers (respectively), and the velocities are given simply by $v = \omega/k$ with $k = \sqrt{k_1^2 + k_3^2}$. The SH wave depends only on elastic parameters l and m, which are not dependent in any way on layer λ and therefore will play no role in the poroelastic analysis. Thus, we can safely ignore SH except when we want to check for shear wave splitting (bi-refringence) – in which case the SH results will be useful for the comparisons.

The dispersion relations for quasi-P- and quasi-SV-waves can be rewritten in a number of instructive ways. One of these that we will choose for reasons that will become apparent shortly is

$$\rho\omega_\pm^2 = \frac{1}{2}\Big[(a+l)k_1^2 + (c+l)k_3^2 \pm \sqrt{[(a+l)k_1^2 + (c+l)k_3^2]^2 - 4[(ak_1^2 + ck_3^2)lk^2 + \{(a-l)(c-l) - (f+l)^2\}k_1^2k_3^2]}\Big]. \quad (37)$$

Written this way, it is then obvious that the following two relations hold:

$$\rho\omega_+^2 + \rho\omega_-^2 = (a+l)k_1^2 + (c+l)k_3^2, \quad (38)$$

and

$$\rho\omega_+^2 \cdot \rho\omega_-^2 = (ak_1^2 + ck_3^2)lk^2 + [(a-l)(c-l) - (f+l)^2]k_1^2k_3^2, \quad (39)$$

either of which could have been obtained directly from (35) without the intermediate step of (37).

We are motivated to write the equations in this way in order to try to avoid evaluating the square root in (35) directly. Rather, we would like to arrive at a natural approximation that is quite accurate, but does not involve the square root operation. From a general understanding of the problem, it is clear that a reasonable way of making use of (38) is to make the identifications

$$\rho\omega_+^2 \equiv ak_1^2 + ck_3^2 - \Delta, \tag{40}$$

and

$$\rho\omega_-^2 \equiv lk^2 + \Delta, \tag{41}$$

with Δ still to be determined. Then, substituting these expressions into (39), we find that

$$(ak_1^2 + ck_3^2 - lk^2 - \Delta)\Delta = [(a-l)(c-l) - (f+l)^2]k_1^2k_3^2 \tag{42}$$

Solving (42) for Δ would just give the original results back again. So the point of (42) is not to solve it exactly, but rather to use it as the basis of an approximation scheme. If Δ is small, then we can presumably neglect it inside the parenthesis on the left hand side of (42), or we could just keep a small number of terms in an expansion.

The leading term, and the only one we will consider here, is

$$\Delta = \frac{[(a-l)(c-l) - (f+l)^2]k_1^2k_3^2}{(a-l)k_1^2 + (c-l)k_3^2 - \Delta} \simeq \frac{[(a-l)(c-l) - (f+l)^2]}{(a-l)/k_3^2 + (c-l)/k_1^2}. \tag{43}$$

The numerator of this expression is known to be a positive quantity for layered materials (Postma, 1955; Berryman, 1979). Furthermore, it can be rewritten in terms of Thomsen's parameters as

$$[(a-l)(c-l) - (f+l)^2] = 2c(c-l)(\epsilon - \delta). \tag{44}$$

Using the first of the identities noted earlier in (5), we can also rewrite the first elasticity factor in the denominator as $a - l = (c-l)[1 + 2c\epsilon/(c-l)]$. Combining these results in the limit of $k_1^2 \to 0$ (for relatively small horizontal offset), we find that

$$\rho\omega_+^2 \simeq ck^2 + 2c\delta k_1^2, \tag{45}$$

and

$$\rho\omega_-^2 \simeq lk^2 + 2c(\epsilon - \delta)k_1^2, \tag{46}$$

with $\Delta \simeq 2c(\epsilon - \delta)k_1^2$. Improved approximations to any desired order can be obtained with only a little more effort by using (42) or (43) instead of the first approximation used here. But (45) and (46) are satisfactory for our present purposes.

INTERPRETATION OF P AND SV COEFFICIENTS FOR LAYERED MEDIA

General analysis for VTI media

The correction terms for SV waves in weakly anisotropic media are proportional to the factor

$$\mathcal{A} \equiv (a-l)(c-l) - (f+l)^2 = 2c(c-l)(\epsilon-\delta), \tag{47}$$

which is sometimes called the *anellipticity parameter*. For the case of weak anisotropy that we are considering here, the presence of this term in (46) just introduces ellipticity into the move out, but the higher order corrections that we neglected can introduce deviations from ellipticity, hence anellipticity.

Clearly, from (46) for quasi-SV-waves [and in layered media at this order of approximation], the anellipticity parameter holds all the imformation about presence or absence of fluids that is not already contained in the density factor ρ. So it will be worth our time to study this factor in more detail. First note that, after rearrangement, we have the general identity

$$\mathcal{A} = (f+l)(a+c-2f-4l) + (a-f-2l)(c-f-2l), \tag{48}$$

which is true for all transversely isotropic media.

In some earlier work (Berryman, 2003), the author has shown that it is convenient to introduce two special effective shear moduli μ_1^* and μ_3^* associated with a and c, namely,

$$\mu_1^* \equiv a-m-f \qquad \text{and} \qquad 2\mu_3^* \equiv c-f. \tag{49}$$

Furthermore, it was shown that the combination defined by

$$G_{eff} = (\mu_1^* + 2\mu_3^*)/3 \tag{50}$$

plays a special role in the theory, as it is the only effective shear modulus for the anisotropic system that may also contain information about fluid content. It turns out that (48) can be rewritten in terms of this effective shear modulus if we first introduce two more parameters:

$$\mathcal{K} = f+l+\left[\frac{1}{a-f-2l} + \frac{1}{c-f-2l}\right]^{-1} \tag{51}$$

and

$$\mathcal{G} = \left[3G_{eff} + m - 4l\right]/3. \tag{52}$$

Then, (48) can be simply rewritten as

$$\mathcal{A} = 3\mathcal{K}\mathcal{G}. \tag{53}$$

This result is analogous to, but distinct from, a product formula relating the effective shear modulus G_{eff} and the bulk modulus

$$K = f+\left[\frac{1}{a-m-f} + \frac{1}{c-f}\right]^{-1} \tag{54}$$

to the eigenvalues of the elastic matrix according to

$$\chi_+\chi_- = 6KG_{eff}. \tag{55}$$

In the isotropic limit for layered materials, when $\mu \to constant$, we have $K \to f + 2\mu/3$, while $\mathcal{K} \to f + \mu$. So these two parameters are not the same, but they do have strong similarities in their behavior. In contrast, $G_{eff} \to \mu$, while $\mathcal{G} \to 0$ in the same limit. It is also possible to show for layered materials that in general $l \leq \mathcal{K} - f \leq m$, with the lower limit being optimum.

Also, since Thomsen's δ plays an important role in (45), it is helpful to note that (25) can also be rewritten as

$$c\delta = -(c - f - 2l)\left[1 - \frac{c - f - 2l}{2(c-l)}\right], \tag{56}$$

which shows that, at least for weakly anisotropic media, $c\delta$ is very nearly a direct measure of the quantity $c - f - 2l$.

Analysis for layered media

The analysis presented so far is general for all VTI elastic media. But we can say more by assuming now that the anisotropy arises due to layering. Then, for example, we have the following relations

$$f + 2l = c\left\langle \frac{\lambda + 2l}{\lambda + 2\mu} \right\rangle, \tag{57}$$

$$c - f - 2l = 2c\left\langle \frac{\mu - l}{\lambda + 2\mu} \right\rangle, \tag{58}$$

and

$$a - f - 2l = 2c\left\{\left\langle \frac{2m - \mu - l}{\lambda + 2\mu} \right\rangle - 2\left[\left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle\left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle^2\right]\right\}. \tag{59}$$

Eq. (57) is an easy consequence of the Backus averaging formulas. Then, (58) shows that c differs from $f + 2l$ only by a term that measures the difference in the weighted average of μ and l. Eq. (59) shows that a differs from $f + 2l$ in a more complicated fashion that depends on the difference in the weighted average of $(2m - l)$ and μ, as well as a term that is higher order in the fluctuations of the layer μ values. Combining these results, we have

$$G_{eff} = m - \frac{4c}{3}\left[\left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle\left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle^2\right], \tag{60}$$

showing that all the interesting behavior (including strong μ fluctuations in the layers together with λ dependence) is collected in G_{eff}. Since the product of (58) and (59) is clearly of higher order in the fluctuations of the layer shear moduli, it is not hard to see that, to leading order when these fluctuation effects are small,

$$\mathcal{A} \simeq (c-l)(3G_{eff}+m-4l) \tag{61}$$

from which we can conclude that the important coefficient in (46) is given to a good approximation by

$$2c(\epsilon-\delta) \simeq 3G_{eff}+m-4l \sim 4(m-l) = 8l\gamma, \tag{62}$$

where the final expression is a statement about the limiting behavior when either the μ fluctuations are very small, or when strong undrained behavior is present together with large μ fluctuations.

To study the fluid effects, the drained Lamé parameter λ in each layer should be replaced under undrained conditions by

$$\lambda^* = K^* - 2\mu/3, \tag{63}$$

where K^* was defined by (6). Then, for small fluctuations in μ, Eq. (62) shows that the leading order terms due to these shear modulus variations contributing to $\epsilon-\delta$ actually do not depend on the fluids at all (since $m-l$ does not depend on them). There is an enhancement in the shear wave speed for SV in layered media, just due to the changes in the shear moduli, and independent of any fluids that might be present in that case, but the magnitude of this enhancement is small because the difference $m-l$ is also small. When $m-l$ is large, then the magnitude of the enhancement due to liquids in the pores can be very substantial as we will see in the following examples. So the effects of liquid on G_{eff} will generally be weak when the fluctuations in μ are weak, and strong when they are strong.

To check the corresponding result for P-waves, we need to estimate δ. Making use of (56), we also have

$$c\delta = -2c\left\langle\frac{\mu-l}{\lambda+2\mu}\right\rangle\left[1-l^{-1}\left\langle\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\right\rangle^{-1}\left\langle\frac{\mu-l}{\lambda+2\mu}\right\rangle\right]. \tag{64}$$

Working to the same order as we did for the final expression in (62), we can neglect the second term in the square brackets of (64). What remains shows that pore fluids would have an effect on this result. The result is

$$c^*\delta^* \simeq -2c^*\left\langle\frac{\mu-l}{\lambda^*+2\mu}\right\rangle, \tag{65}$$

and a similar replacement should also be made for G_{eff} in (60). Eq. (65) shows that, since c^* and δ^* both depend on the λ^*'s (although in opposite ways, since one increases while the other decreases as λ^* increases), the product of these factors will have some dependence on fluids. The degree to which fluctuations in λ^* and μ are correlated or anticorrelated as they vary from layer to layer will also affect these results in predictable ways.

Now we have derived all the results needed to interpret (34) and show how it is related to (33). First, we note the some of the main terms missing from (34) are those due to approximations made to δ and the denominators of (33), which have been approximated as $f \simeq c - 2l$ instead of $f \simeq c(1+\delta) - 2l$. Then, from (62), it is easy to see that the final term in (34) vanishes to lowest order, and that the remainder is given exactly by the shear modulus fluctuation terms in brackets in (59) — in complete agreement with the final terms of (33). Then, from (64), it follows that the leading contribution to the factor $c\delta + 4l\gamma$ is

$$c\delta + 4l\gamma \simeq 2c\left\langle \frac{m-\mu}{\lambda+2\mu} \right\rangle, \tag{66}$$

in complete agreement with the second term in (33).

In the case of very strong fluctuations in the layer shear moduli, then (59) and (64) both show that effects of the pore fluids can be more strongly felt in the anisotropy correction factors $2c^*(\epsilon^* - \delta^*)$ and $2c^*\delta^*$ for undrained porous media, and therefore more easily observed in seismic, sonic, or ultrasonic data. When these effects are present, the vertically polarized quasi-shear mode will show the highest magnitude effect, the horizontally polarized shear mode will show no effect, and the quasi-compressional mode will show an effect of intermediate magnitude. It is known that these effects, when present, are always strongest at 45°, and are diminished when the angle of propagation is either 0° or 90° relative to the layering direction. We will test these analytical predictions with numerical examples in the next section.

To summarize our main result here: All the liquid dependence in the shear moduli comes into the wave dispersion formulas through coefficient a (or equivalently ϵ). Equations (59) and (60) show that

$$a = 2f - c + m + 3G_{eff}. \tag{67}$$

For small fluctuations in μ, coefficients a and c have comparable magnitude dependence on the fluid effects, but of opposite sign. For large fluctuations, the effects on a are much larger (quadratic) than those on c (linear). Propagation at normal incidence will never show much effect due to the liquids, while propagation at angles closer to 45° can show large enhancements in both quasi-P and quasi-SV waves (when shear fluctuations are large), but still no effect on SH waves.

COMPUTED EXAMPLES

From previous work (Berryman, 2003), we know that large fluctuations in the layer shear moduli are required before significant deviations from Gassmann's quasi-static constant result, thereby showing that the shear modulus dependence on fluid properties can become noticeable. To generate a model that demonstrates these results, I made use of a code of V. Grechka [used previously in a joint publication (Berryman *et al.*, 1999)] and then I arbitrarily picked one of the models that seemed to be most interesting for the present purposes. The parameters of this model are displayed in TABLE 1. The results for the various elastic coefficients and Thomsen parameters are displayed in TABLE 2. The results of the calculations for V_p and V_{sv} are shown in Figures 1 and 2.

The model calculations were simplified in one way: the value of the Biot-Willis parameter was chosen to be a uniform value of $\alpha = 0.8$ in all layers. We could have actually computed a value of α from the other layer parameters, but to do so would require another assumption about the porosity values in each layer. Doing this seemed an exercise of little value because we are just trying to show in a simple way that the formulas given here really do produce the types of results predicted analytically, and also to get a feeling for the magnitude of the effects. Furthermore, if α is a constant, then it is only the product αB that matters. Whatever choice of constant $\alpha \leq 1$ is made, it mainly determines the maximum value of the product αB for B in the range $[0,1]$. So, for a parameter study, it is only important not to choose too a small value of α, which is why the choice $\alpha = 0.8$ was made. This means that the maximum amplification of the bulk modulus due to fluid effects can be as high as a factor of 5 $[= 1/(1-\alpha)]$ for the present examples.

TABLE 1. Layer parameters for the three materials in a simple layered medium used to produce the examples in Figures 1 and 2.

Constituent	K (GPa)	μ (GPa)	z (m/m)
1	9.4541	0.0965	0.477
2	14.7926	4.0290	0.276
3	43.5854	8.7785	0.247

We took the porosity to be $\phi = 0.2$, and the overall density to be $\rho = (1-\phi)\rho_s + \phi S \rho_l$, where $\rho_s = 2650.0$ kg/m^3, S is liquid saturation ($0 \leq S \leq 1$), and $\rho_l = 1000.0$ kg/m^3. Then, three cases were considered: (1) Gas saturation $S = 0$ and $B = 0$, which is also the drained case, assuming that the effect of the saturating gas on the moduli is negligible. (2) Partial liquid saturation $S = 0.95$ and $B = \frac{1}{2}$ [which is intended to model a case of partial liquid saturation], intermediate between the other two cases. For smaller values of liquid saturation, the effect of the liquid might not be noticeable, since the gas-liquid mixture when homogeneously mixed will act much like the pure gas in compression, although the density effect is still present. When the liquid fills most of the pore-space, and the gas occupies less than about 3% of the entire volume of the rock, the gas starts to become disconnected, and we expect the effect the liquid to start becoming more noticeable, and therefore we choose $B = \frac{1}{2}$ to be representative of this case. And, finally, (3) full liquid saturation $S = 1$ and $B = 1$, which is also the fully undrained case. We assume for the purposes of this example that a fully saturating liquid has the maximum possible stiffening effect on the locally microhomogeneous, poroelastic medium.

TABLE 2. The VTI elastic coefficients and Thomsen parameters for the materials (see Table 1) used in the computed examples of Figures 1 and 2.

Elastic Parameters and Density	*Case* $B=0$	*Case* $B=\frac{1}{2}$	*Case* $B=1$
a (GPa)	33.8345	50.3523	132.7003
c (GPa)	33.1948	50.4715	134.2036
f (GPa)	22.2062	38.5857	120.7006
l (GPa)	4.0138	4.0138	4.0138
m (GPa)	6.7777	6.7777	6.7777
G_{eff} (GPa)	5.2797	5.8841	6.2417
δ	-0.0847	-0.0733	-0.0399
$\epsilon-\delta$	0.0943	0.0745	0.0343
γ	0.3443	0.3443	0.3443
ρ (kg/m^3)	2120.0	2310.0	2320.0

The results shown in the Figures are in complete qualitative and quantitative agreement with the analytical predictions described, as expected.

CONCLUSIONS

Although there are five effective shear moduli for any layered VTI medium, the main result of the paper is that there is just one effective shear modulus for the layered system that contains *all* the dependence of pore fluids on the elastic or poroelastic constants — all that can be observed in vertically polarized shear waves in VTI media. The relevant modulus G_{eff} is related to uniaxial shear strain and the relevant axis of symmetry is the vertical one, normal to the bedding planes. The pore-fluid effects on this effective shear modulus can be substantial when the medium behaves in an undrained fashion, as might be expected at higher frequencies such as sonic and ultrasonic for well-logging or laboratory experiments, or at seismic frequencies for lower permeability regions of reservoirs. These predictions are clearly illustrated by the example in Figure 2.

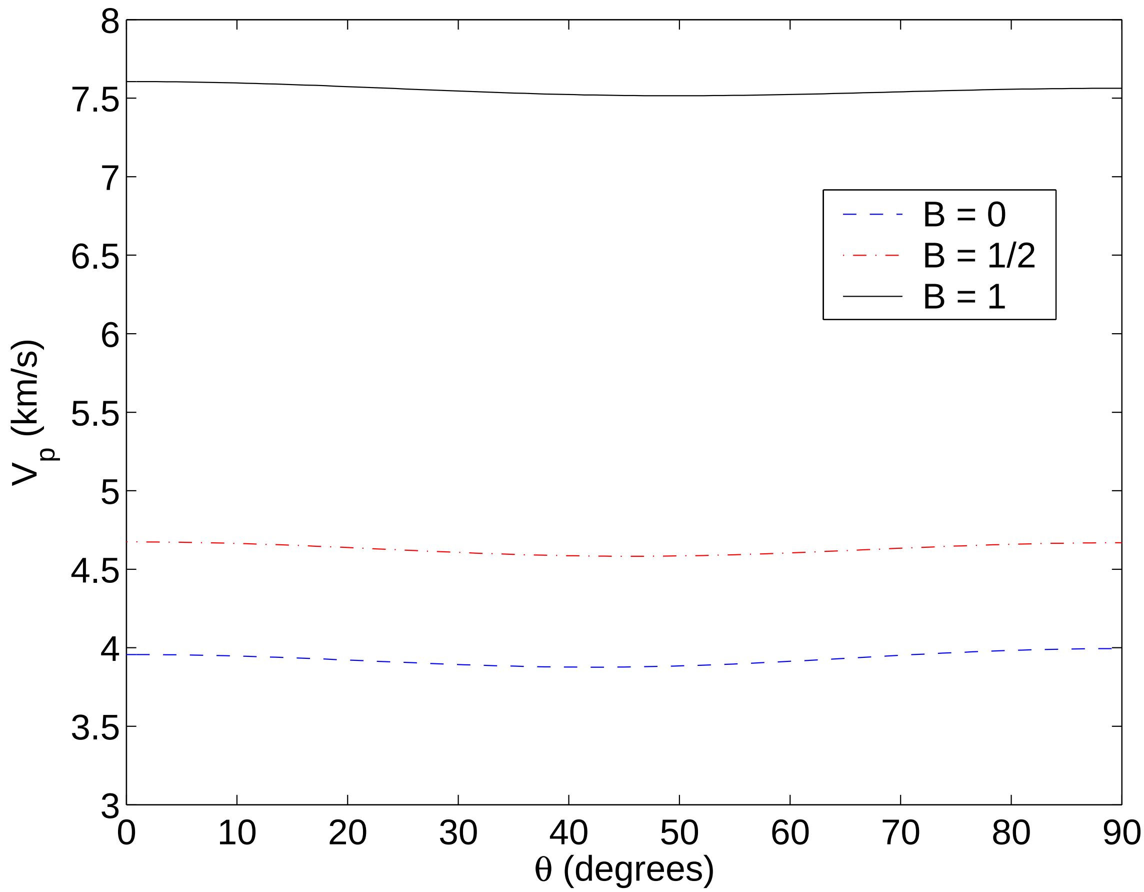

Figure 1: Compressional wave speed V_p as a function of angle θ from the vertical. Two curves shown correspond to choices of Skempton's coefficient $B = 0$ for the drained case (dashed line) and $B = 1$ for the undrained case (solid line). The case $B = \frac{1}{2}$ (dot-dash line) is used to model partial saturation conditions as described in the text. The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers. $\boxed{\text{vp}}$ [NR]

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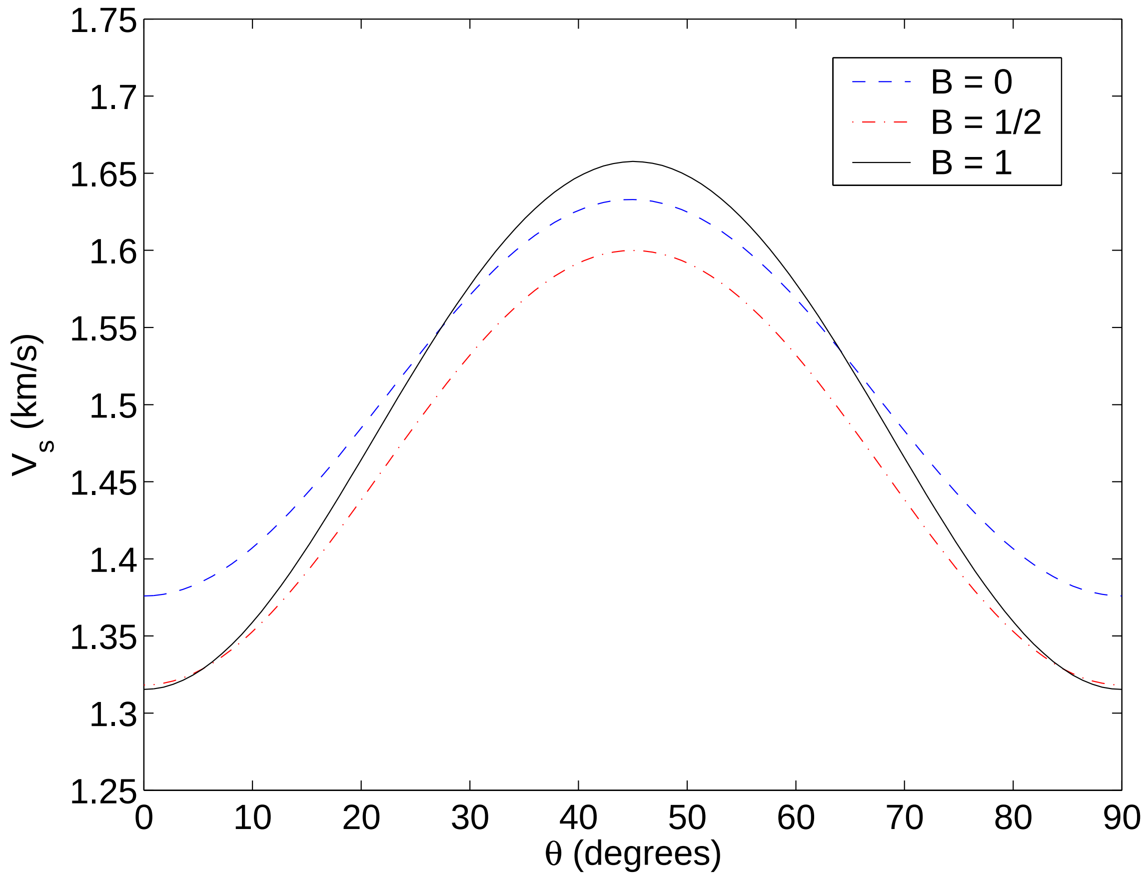

Figure 2: Vertically polarized shear wave speed V_{sv} as a function of angle θ from the vertical. Two curves shown correspond to choices of Skempton's coefficient $B = 0$ for the drained case (dashed line) and $B = 1$ for the undrained case (solid). The case $B = \frac{1}{2}$ (dot-dash line) is used to model partial saturation conditions as described in the text. The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers. $\boxed{\text{vs}}$ [NR]

REFERENCES

Anno, P. 1997, Equivalent anisotropy for finely-layered media: Colorado School of Mines, Center for Wave Phenomena, Project Review, pp. 307–312.

Auld, B. A., 1973, *Acoustic Fields and Waves in Solids*, Vols. 1 and 2, John Wiley & Sons, New York.

Backus, G. E. 1962, Long-wave elastic anisotropy produced by horizontal layering: *J. Geophys. Res.* **67**, 4427–4440.

Berryman, J. G. 1979, Long-wave elastic anisotropy in transversely isotropic media: *Geophysics* **44**, 896–917.

Berryman, J. G., 1999a, Transversely isotropic elasticity and poroelasticity arising from thin isotropic layers: in *Theoretical and Computational Acoustics '97, Proceedings of the Third International Conference on Theoretical and Computational Acoustics, July 14–18, 1997, Newark, New Jersey*, edited by Y.-C. Teng, E.-C. Shang, Y.-H. Pao, M. H. Schultz, and A. D. Pierce (World Scientific, Singapore), pp. 457–474.

Berryman, J. G., 1999b, Origin of Gassmann's equations: *Geophysics* **64**, 1627–1629.

Berryman, J. G., 2003, Poroelastic shear modulus dependence on pore-fluid properties arising in a model of thin isotropic layers: LLNL UCRL-JC-152285, preprint, March, 2003.

Berryman, J. G., Grechka, V. Y., and Berge, P. A, 1999, Analysis of Thomsen parameters for finely layered VTI media: *Geophysical Prospecting* **47**, 959–978.

Berryman, J. G., and Wang, H. F., 2001, Dispersion in poroelastic systems: *Phys. Rev. E* **64**, 011303-1–011303-16.

Biot, M. A., and Willis, D. G., 1957, The elastic coefficients of the theory of consolidation: *J. Appl. Mech.* **24**, 594–601.

Bruggeman, D. A. G., 1937, Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen III. Die elastischen Konstanten der quasiisotropen Mischkörper aus isotropen Substanzen: *Annalen der Physik* **29**, 160–178.

Gassmann, F. 1951, Über die Elastizität poröser Medien: *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich* **96**, 1–23.

Hashin, Z., and Shtrikman, S., 1961, Note on a variational approach to the theory of composite elastic materials: *J. Franklin Inst.* **271**, 336–341.

Helbig, K., 1983, Elliptical anisotropy – Its significance and meaning: *Geophysics* **48**, 825–832.

Hill, R., 1963, Elastic properties of reinforced solids: Some theoretical results: *J. Mech. Phys. Solids* **11**, 357–372.

Johnson, D. L., 2001, Theory of frequency dependent acoustics in patchy-saturated porous media: *J. Acoust. Soc. Am.* **110**, 682–694.

Levin, F. K. 1988, Discussion on "Weak elastic anisotropy," by L. Thomsen: *Geophysics* **53**, 558-559.

Mavko, G., Mukerji, T., and Dvorkin, J., 1998, *The Rock Physics Handbook*, Cambridge, new York, 1998, pp. 206–207.

Milton, G. W., 2002, *The Theory of Composites* (Cambridge University Press, Cambridge, UK), pp. 163–165.

Musgrave, M. J. P., 1970, *Crystal Acoustics: Introduction to the Study of Elastic Waves and Vibrations in Crystals*, Holden-Day, San Francisco, California.

Postma, G. W. 1955, Wave propagation in a stratified medium: *Geophysics* **20**, 780–806.

Schoenberg, M. 1994, Transversely isotropic media equivalent to thin isotropic layers: *Geophys. Prospecting* **42**, 885–915.

Schoenberg, M., and Muir, F., 1989, A calculus for finely layered anisotropic media: *Geophysics* **54**, 581–589.

Skempton, A. W. 1954, The pore-pressure coefficients A and B: *Geotechnique* **4**, 143–147.

Thomsen, L. 1986, Weak elastic anisotropy: *Geophysics* **51**, 1954–1966.