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Depth migration by an unconditionally stable explicit finite-difference method

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ABSTRACT

We develop an unconditionally stable, explicit depth migration method. The downward continuation operator derived by a finite-difference approximation of the one-way equation is given by the exponential of a banded matrix. We approximate this exponential by decomposing the banded matrix into block diagonal matrices, of which the exponential can be computed analytically. The derived downward continuation operator is explicit and unconditionally stable, and thus it may be efficiently implemented on either vector or parallel computers.
First, we apply this algorithm to a 15-degree migration. To increase the accuracy at steep dip, we add more terms to the Taylor-series expansion of the square-root operator. However, the improvement in accuracy gained adding more terms to the Taylor series, it is at the expense of higher computational cost.

INTRODUCTION

There has been a great deal of debate over explicit versus implicit methods for seismic wavefield extrapolation. Stability has traditionally been one of the most compelling advantages of implicit methods. Conventional 45-degree finite-difference migration, for example, is based on an implicit wavefield extrapolation that is guaranteed to be stable (Claerbout, 1985). In contrast, the derivation of stable explicit downward continuation operators is more difficult; their derivation, and proof of stability, assumes the velocity to be locally constant (Blanquiere, 1989; Hale, 1990)

Despite the stability problem, explicit methods are attractive because they are efficient on either vector or parallel computers. In addition to efficiency, another advantage shared by explicit methods for depth migration of seismic wavefields is the ease with which they can be extended for use in 3-D depth migration. On the contrary, the solution of the linear system of equations required by implicit methods is particularly expensive in 3-D.

Recently, a new method for deriving unconditionally stable explicit time extrap-

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olators was presented by Richardson et al. (1991). Their method is based on the decomposition of the banded matrix derived by a finite-difference approximation of the wave equation. This banded matrix is decomposed in block matrices, that can be analytically exponentiated. We apply this new method to derive a stable downward continuation operator from the one-way wave equation. The banded matrix in our case comes from a Taylor approximation of the square root of a matrix. When the Taylor series is truncated after the second term, we get the conventional 15-degree approximation. Adding more terms to the Taylor series improves the accuracy for steep dips. However, the downward continuation operator becomes longer, and its derivation becomes fairly complicated.

ONE-WAY WAVE EQUATION AND ITS APPROXIMATION

Most extrapolation methods used in migration and modeling are based on the scalar wave equation. For a 2-D acoustic earth model, the two-way scalar wave equation in the (ω, x, z) domain is of the following form:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = -\frac{\omega^2}{v^2} P, \tag{1}$$

where P is the pressure field, ω is the frequency, z is depth, and v is velocity. Solving the scalar wave equation requires the first depth derivative of the wavefield. To obtain a one-way solution to the two-way equation, without having to know the vertical derivatives, we can use the one-way scalar wave equation

$$\frac{\partial P}{\partial z} = \pm i \frac{\omega}{v} \sqrt{1 + \frac{v^2}{w^2} \frac{\partial^2}{\partial x^2}} P, \tag{2}$$

where the positive sign is used in modeling, the negative sign in migration, and the velocity is half of the true velocity. While the one-way equation eliminates the need to know the vertical derivatives of the wavefields, it contains a square-root of a differential operator that cannot easily be implemented numerically.

There are two methods of rationalizing the square-root equation: by Taylor series and by continued-fraction (Claerbout, 1985). The Taylor-series method leads to an explicit scheme whereas the continued-fraction method leads an implicit scheme. The continued-fraction expansion was first used by Muir (Claerbout, 1985) to approximate the square-root operator,

$$R = \sqrt{1 - X^2}, \tag{3}$$

where

$$X^2 = -\frac{v^2}{w^2} \frac{\partial^2}{\partial x^2}.$$

The expansion is

$$R_n = 1 - \frac{X^2}{1 + R_{n-1}}, \tag{4}$$

where n is the order of approximation, and usually $R_0 = 1$ (Claerbout, 1985). In implicit algorithms, the fraction expansion method for approximation is implemented recursively. In the explicit scheme this method makes implementation difficult because it produces another function of a matrix and as a result, a Taylor-series expansion is generally used. By Taylor-series expansion, the square root operator given in equation (3) can be written as

$$\sqrt{1-X^2} = 1 - \frac{X^2}{2} - \frac{X^4}{8} - \frac{X^6}{16} - \cdots . \tag{5}$$

15-DEGREE DEPTH MIGRATION

The first-order approximations of the one-way wave equation by both continued-fraction expansion and Taylor-series expansion result in the same equation: the conventional 15-degree equation. The 15-degree migration equation in the (ω, x, z) domain (Claerbout, 1985) is given by

$$\frac{\partial P}{\partial z} = -i\left(\frac{\omega}{v} + \frac{v}{2\omega}\frac{\partial^2}{\partial x^2}\right)P. \tag{6}$$

After a second difference approximation to the second partial derivative and subsequent simplification, the equation (6) becomes

$$\frac{\partial P}{\partial z} = MP, \tag{7}$$

where the matrix M takes the form

$$M = -i\frac{\omega}{v}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} - i\frac{v}{2\omega\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

Rearranging,

$$M = \begin{pmatrix} 2a & b & 0 & 0 & \cdots & 0 \\ b & 2a & b & 0 & \cdots & 0 \\ 0 & b & 2a & b & \cdots & 0 \\ \cdots & & & & & \end{pmatrix}$$

with

$$a = -\frac{i}{2}\left(\frac{\omega}{v} - \frac{v}{\omega\Delta x^2}\right)$$

and

$$b = -i\frac{v}{2\omega\Delta x^2}.$$

The solution to equation (7) is

$$P(z+\Delta z) = \exp(\Delta z M)P(z). \tag{8}$$

The calculation of the exponentiated matrix $\exp(\Delta z M)$ requires matrix diagonalization, which is numerically expensive. However, we can approximate the exponential of the matrix by writing (Richardson, et al., 1991)

$$\exp(\Delta z M) = \exp(\Delta z M_e) \exp(\Delta z M_o) + \epsilon(\Delta z^2), \tag{9}$$

where the matrix M is split into two matrices $M = M_e + M_o$, with

$$M_e = \begin{pmatrix} a & b & . & . & \dots & . \\ b & a & . & . & \dots & . \\ . & . & a & b & \dots & . \\ . & . & b & a & \dots & . \\ \dots & & & & & \end{pmatrix}$$

and

$$M_o = \begin{pmatrix} a & . & . & . & \dots & . \\ . & a & b & . & \dots & . \\ . & b & a & . & \dots & . \\ . & . & . & a & \dots & . \\ \dots & & & & & \end{pmatrix}.$$

The approximate depth-stepping operator,

$$B = \exp(\Delta z M_e) \exp(\Delta z M_o) \tag{10}$$

forms the basis for our unconditionally stable, explicit algorithm for migration. To compute the matrix exponentials, notice that both M_e and M_o are block diagonal and we need only consider the exponential of the 2 by 2 matrix

$$E = \Delta z \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

By using the eigenvalue decomposition of the matrix, we see that

$$\exp(E) = Q \exp(\Lambda) Q^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} \exp(\Delta z(a+b)) + \exp(\Delta z(a-b)) & \exp(\Delta z(a+b)) - \exp(\Delta z(a-b)) \\ \exp(\Delta z(a+b)) - \exp(\Delta z(a-b)) & \exp(\Delta z(a+b)) + \exp(\Delta z(a-b)) \end{pmatrix},$$

where Q represents the matrix whose columns are eigenvectors of the matrix E, and Λ is the diagonal eigenvalue matrix.

Since both M_e and M_o are block diagonal, exponentiating them amounts to exponentiating E. Both $\exp(\Delta z M_e)$ and $\exp(\Delta z M_o)$ are also block diagonal. Since the eigenvalues of $\exp(E)$ are $\exp(a+b)$ and $\exp(a-b)$ with a and b imaginary, it follows that $\|\exp(\Delta z M_e)\| = 1$ and $\|\exp(\Delta z M_o)\| = 1$. To prove the unconditional stability of the algorithm we need only show that $\|B\| \leq 1$. This follows immediately since $\|B\| = \|\exp(\Delta z M_e) \exp(\Delta z M_o)\| \leq \|\exp(\Delta z M_e)\| \|\exp(\Delta z M_o)\|$, each of which 1 according to the preceding discussion.

In Figure ??, the impulse responses of both implicit and explicit operators of 15-degree migration are compared for the same extrapolation step $\Delta z/\Delta x = 1$. In the case of the implicit scheme, we used a Crank-Nicholson implementation which has an accuracy on the order of Δz^2. For the explicit scheme, we performed a second-order approximation using a split $M = M_e + M_o$ whose accuracy is on the order of Δz. We see that the impulse response of the explicit method has the shape of an ellipse, which is characteristic of the 15-degree extrapolation operator. The impulse response of the explicit method shows some dispersion than that of the implicit scheme due to a poor approximation of the matrix exponent. The way to get a more accurate approximation is discussed in the following section.

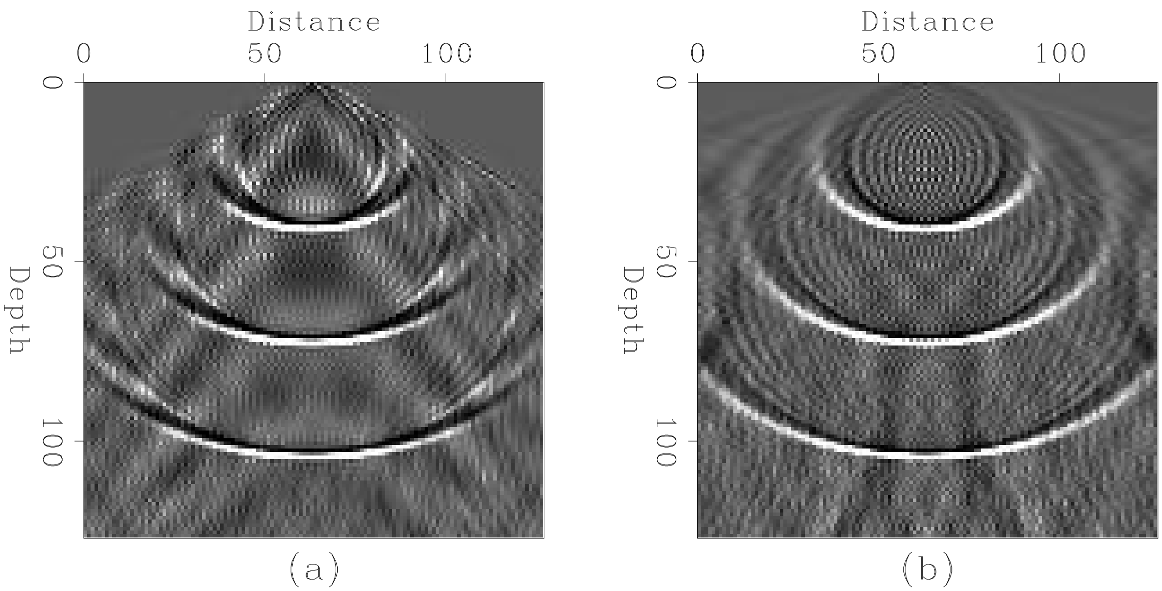

Figure 1: The impulse responses (a) of explicit 15-degree migration with split $M = M_e + M_o$ and (b) of implicit 15-degree migration with Crank-Nicholson approximation. Both extrapolations are performed with the same extrapolation step $\Delta z/\Delta x = 1$. jun-fig1 [ER]

Accuracy

From the series definition of a matrix function, we see that the matrix exponential can be represented as follows:

$$\exp(\Delta z M) = \sum_{k=0}^{\infty} \frac{(\Delta z M)^k}{k!}. \tag{11}$$

Richardson, et al. approximate this matrix exponential by splitting the given matrix into pieces:

$$\exp(\Delta z M) = \exp(\Delta z (M_e + M_o)). \tag{12}$$

Equation (12) can be represented with the series

$$\exp(\Delta z(M_e + M_o)) = I + \Delta z(M_e + M_o) + \frac{\Delta z^2 (M_e + M_o)^2}{2!} + \cdots \tag{13}$$

$$= I + \Delta z(M_e + M_o) + \frac{\Delta z^2 (M_e^2 + M_e M_o + M_o M_e + M_o^2)}{2!} + \cdots . \tag{14}$$

An approximate calculation of equation (13) is

$$\exp(\Delta z M_e) \exp(\Delta z M_o) \tag{15}$$

$$= \left(I + \Delta z(M_e) + \frac{\Delta z^2 (M_e)^2}{2!} + \cdots \right) \left(I + \Delta z(M_o) + \frac{\Delta z^2 (M_o)^2}{2!} + \cdots \right) \tag{16}$$

$$= I + \Delta z(M_e + M_o) + \frac{\Delta z^2 (M_e^2 + 2 M_e M_o + M_o^2)}{2!} + \cdots . \tag{17}$$

Equation (14) and (15) are identical if M_e and M_o are commutative, i.e., $2M_eM_o = M_eM_o + M_oM_e$. However, in our case, M_e and M_o are *not* commutative, thus approximation (15) will have errors on the order of $\epsilon(\Delta z^2)$. By careful arrangement of two different split matrices, we can improve the accuracy as follows. We define

$$M = \frac{M_e}{2} + M_o + \frac{M_e}{2} \tag{18}$$

which has an error on the order of $\epsilon(\Delta z^3)$, and

$$M = \frac{M_e}{4} + \frac{M_o}{2} + \frac{M_e}{2} + \frac{M_o}{2} + \frac{M_e}{4} \tag{19}$$

with an error on the order of $\epsilon(\Delta z^4)$.

Figure **??** shows the improvement in accuracy when we use a higher order approximation. On the left impulse response, we used the first order approximation with split as equation (18) and on the right impulse response, we used the third order approximation as equation (19). We can see a decrease in the dispersion at higher order approximation. Comparing with the impulse response of implicit scheme in Fig **??**, the higher order explicit scheme shows the comparable accuracy as we can see in Fig **??**.

Lateral velocity variation

When the velocity varies laterally, the matrix M in the extrapolation operator at equation (7) has coefficients which varies along the diagonal, but we can still derive a symmetric matrix (Godfrey et al., 1979). The symmetry property, which will guarantee the unconditional stability, can be obtained by putting the velocity term on both sides as follows

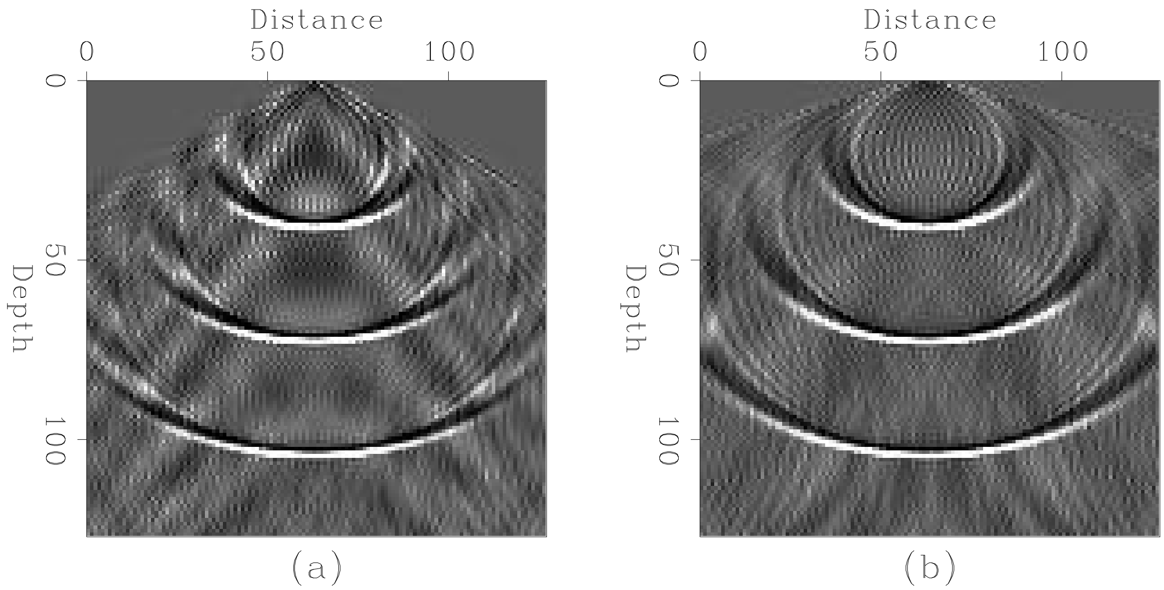

Figure 2: Impulse responses of explicit 15-degree migration with the split matrices (a) $M = M_e + M_o$, (b) $M = M_e/4 + M_o/2 + M_e/2 + M_o/2 + M_e/4$. jun-fig2 [ER]

$$\frac{\partial}{\partial z} P = -\frac{1}{\sqrt{v}} [-\omega^2 - \frac{\partial}{\partial x} v^2 \frac{\partial}{\partial x}]^{1/2} \frac{1}{\sqrt{v}} P \tag{20}$$

Now the small block matrices which are located along the diagonal in split matrices M_e and M_o will have a symmetric form

$$E = \Delta z \begin{pmatrix} a & c \\ c & b \end{pmatrix},$$

where

$$a = -\frac{i}{2}(\frac{\omega}{v(x)} - \frac{v^2(x) + v^2(x-1)}{2v(x)\omega\Delta x^2}),$$

$$b = -\frac{i}{2}(\frac{\omega}{v(x+1)} - \frac{v^2(x+1) + v^2(x)}{2v(x+1)\omega\Delta x^2}),$$

and

$$c = -i\frac{v(x)^2}{\sqrt{v(x+1)v(x)}2\omega\Delta x^2}.$$

The eigenvalues of $\exp(E)$ are given by

$$\lambda = \exp\left(\frac{a + b \pm \sqrt{(a-b)^2 + 4c^2}}{2}\right)$$

and lie on the unit circle since a and b are imaginary, and $(a-b)^2 + 4c^2 \le 0$. It follows that the matrix norms $\| \exp(\Delta z M_e) \| = 1$ and $\| \exp(\Delta z M_o) \| = 1$. To prove the unconditional stability of the algorithm we need only show that $\|B\| \leq 1$. This follows immediately since $\|B\| = \|\exp(\Delta z M_e)\exp(\Delta z M_o)\| \leq \|\exp(\Delta z M_e)\|\|\exp(\Delta z M_o)\|$, each of which is unity according to the preceding argument.

WIDE-ANGLE DEPTH MIGRATION

For increasing the accuracy at higher dips, we can use more terms in the Taylor-series expansion. Resulting in a matrix with thicker bands than the tridiagonal matrix. Using the second-order term in the expansion, we see that the approximated square-root operator takes the form

$$M = -i\frac{\omega}{v}\left(\mathbf{I} + \frac{v^2}{2\omega^2\Delta x^2}\mathbf{T} - \frac{v^4}{8\omega^4\Delta x^4}\mathbf{T}^2\right), \tag{21}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{T}$ represents the tridiagonal matrix that approximates the second partial derivative. Let

$$M = \begin{pmatrix} 2a & b & c & 0 & \cdots & 0 & 0 & 0 \\ b & 2a & b & c & 0 & \cdots & 0 & 0 \\ c & b & 2a & b & c & 0 & \cdots & 0 \\ 0 & c & b & 2a & b & c & \cdots & 0 \\ \cdots \end{pmatrix},$$

with

$$a = -\frac{i}{2}\left(\frac{\omega}{v} - \frac{v}{\omega\Delta x^2} - \frac{6v^3}{8\omega^3\Delta x^4}\right),$$

$$b = -i\left(\frac{v}{2\omega\Delta x^2} + \frac{v^3}{2\omega^3\Delta x^4}\right),$$

and

$$c = i\frac{v^3}{8\omega^3\Delta x^4}.$$

We split the matrix into five pieces, $M = M_e + M_o + M_{t1} + M_{t2} + M_{t3}$, where

$$M_e = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 \\ b & a & 0 & 0 & \cdots & 0 \\ 0 & 0 & a & b & \cdots & 0 \\ 0 & 0 & b & a & \cdots & 0 \\ \cdots \end{pmatrix},$$

$$M_o = \begin{pmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & a & b & 0 & \cdots & 0 \\ 0 & b & a & 0 & \cdots & 0 \\ 0 & 0 & 0 & a & \cdots & 0 \\ \cdots \end{pmatrix},$$

$$
M_{t1} = \left(\begin{array}{ccccccccccc}
0 & 0 & c & . & . & . & . & . & . & \cdots & 0 \\
0 & 0 & 0 & . & . & . & . & . & . & \cdots & 0 \\
c & 0 & 0 & . & . & . & . & . & . & \cdots & 0 \\
. & . & . & 0 & 0 & c & . & . & . & \cdots & 0 \\
. & . & . & 0 & 0 & 0 & . & . & . & \cdots & 0 \\
. & . & . & c & 0 & 0 & . & . & . & \cdots & 0 \\
. & . & . & . & . & . & 0 & 0 & c & \cdots & 0 \\
. & . & . & . & . & . & 0 & 0 & 0 & \cdots & 0 \\
\cdots & & & & & & & & & &
\end{array} \right),
$$

$$
M_{t2} = \left(\begin{array}{ccccccccccc}
0 & . & . & . & . & . & . & . & . & \cdots & 0 \\
. & 0 & 0 & c & . & . & . & . & . & \cdots & 0 \\
. & 0 & 0 & 0 & . & . & . & . & . & \cdots & 0 \\
. & c & 0 & 0 & . & . & . & . & . & \cdots & 0 \\
. & . & . & . & 0 & 0 & c & . & . & \cdots & 0 \\
. & . & . & . & 0 & 0 & 0 & . & . & \cdots & 0 \\
. & . & . & . & c & 0 & 0 & . & . & \cdots & 0 \\
. & . & . & . & . & . & . & 0 & 0 & \cdots & 0 \\
\cdots & & & & & & & & & &
\end{array} \right),
$$

and

$$
M_{t3} = \left(\begin{array}{ccccccccccc}
0 & 0 & . & . & . & . & . & . & . & \cdots & 0 \\
0 & 0 & . & . & . & . & . & . & . & \cdots & 0 \\
. & . & 0 & 0 & c & . & . & . & . & \cdots & 0 \\
. & . & 0 & 0 & 0 & . & . & . & . & \cdots & 0 \\
. & . & c & 0 & 0 & . & . & . & . & \cdots & 0 \\
. & . & . & . & . & 0 & 0 & c & . & \cdots & 0 \\
. & . & . & . & . & 0 & 0 & 0 & . & \cdots & 0 \\
. & . & . & . & . & c & 0 & 0 & . & \cdots & 0 \\
\cdots & & & & & & & & & &
\end{array} \right).
$$

In the same manner as the tridiagonal matrix, we can approximate the time-stepping operator as

$$
B = \exp(\Delta z M_e) \exp(\Delta z M_o) \exp(\Delta z M_{t1}) \exp(\Delta z M_{t2}) \exp(\Delta z M_{t3}). \qquad (22)
$$

M_{t1}, M_{t2}, and M_{t3} are block diagonal, and the small block matrix F along the diagonal is defined as

$$
F = \Delta z \left(\begin{array}{ccc} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{array} \right).
$$

Using the series definition of the exponential function, we see that

$$
\exp(F) = \frac{1}{2} \left(\begin{array}{ccc}
\exp(c\Delta z) + \exp(-c\Delta z) & 0 & \exp(c\Delta z) - \exp(-c\Delta z) \\
0 & 1 & 0 \\
\exp(c\Delta z) - \exp(-c\Delta z) & 0 & \exp(c\Delta z) + \exp(-c\Delta z)
\end{array} \right).
$$

Exponentiating M_{t1}, M_{t2}, and M_{t3} amounts to exponentiating F. The terms $\exp(\Delta z M_{t1})$, $\exp(\Delta z M_{t2})$ and $exp(\Delta z M_{t3})$ are also block diagonal. Since the eigenvalues of $\exp(F)$

are 1 and $\exp(\pm c)$ with imaginary c, it follows that $\|\exp(\Delta z M_{ti})\| = 1$. As before, to prove the unconditional stability of the algorithm, we need only show that $\|B\| \leq 1$. This follows immediately since

$$\|B\| = \|\exp(\Delta z M_e)\exp(\Delta z M_o)\exp(\Delta z M_{t1})\exp(\Delta z M_{t2})\exp(\Delta z M_{t3})\|$$

$$\leq \|\exp(\Delta z M_e)\|\|\exp(\Delta z M_o)\|\|\exp(\Delta z M_{t1})\|\|\exp(\Delta z M_{t2})\|\|\exp(\Delta z M_{t3})\|$$

each of which is equal to 1 according to the preceding argument.

In Figure **??**, we compared the impulse responses of first-order approximation with the second-order approximation in the Taylor-series expansion of the square-root operator. We also superposed semicircles which is the theoretical solution of the extrapolation operator on Fig **??** and the higher order approximation shows better fitting to semicircles than the first-first order approximation.

With the same manner as showed in this section, we can get more accurate operator by taking more terms in a Taylor series expansion of the square-root operator. However, it will produce a matrix with increasing width of the band and thus will cause to an increasing in computation cost.

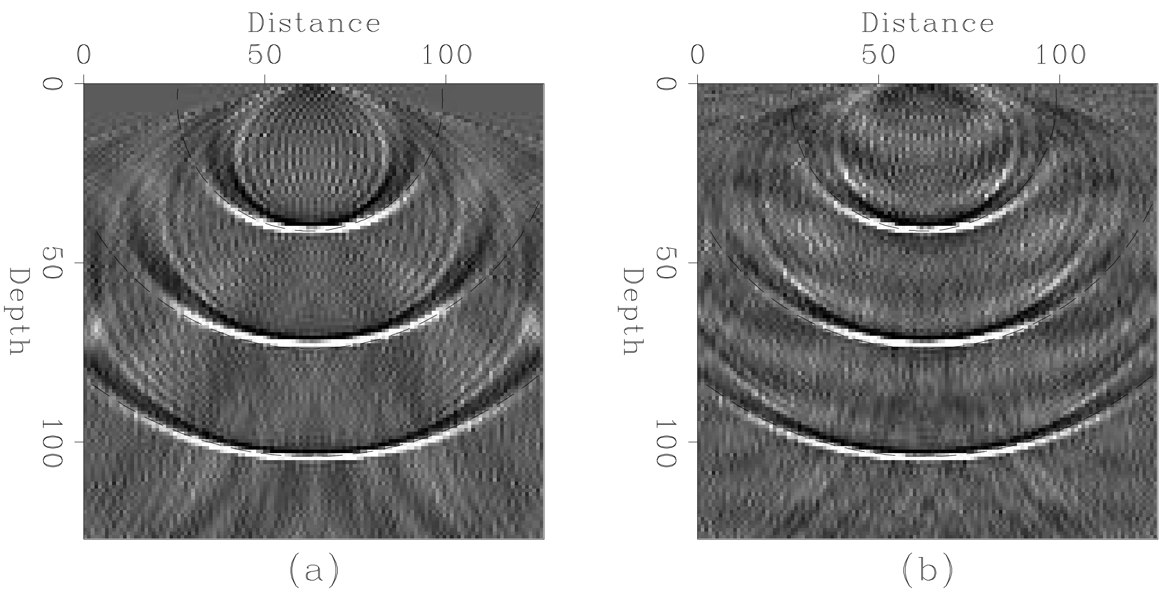

Figure 3: Impulse response of (a) explicit depth migration with the first-order approximation for the square-root operator and of (b) explicit depth migration with the second-order approximation for the square-root operator. jun-fig3 [ER]

CONCLUSION

We presented an explicit depth migration method which is unconditionally stable. This explicit depth-migration scheme is stable even in the case of lateral velocity variation. For 15-degree depth migration, its impulse responses are comparable in accuracy to those produced by the implicit scheme of Crank-Nicholson approximation. To get a more accurate operator at steep dip we can add more terms in the Taylor expansion of the square-root operator and it results in a matrix with thicker bands than a tridiagonal matrix. In such case, we can still use the unconditionally stable, explicit algorithm but it requires expensive computation to get a good accuracy.

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