Descartes' Rule of Signs - How hard can it be?

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Descartes' Rule of Signs states that the number of positive roots of a polynomial $p(x)$ with real coefficients does not exceed the number of sign changes of the nonzero coefficients of $p(x)$. More precisely, the number of sign changes minus the number of positive roots is a multiple of two.[1]

Back in high school, I was introduced to Descartes' Rule of Signs as a mysterious, almost magical, aid in polynomial root finding and factoring of ploynomials with real coefficients—an otherwise tedious occupation when computers were unheard of in the classroom. Even today Descartes' Rule continues to be of interest to both mathematicians and computer scientists [2].

Descartes' rule is plausible when we consider that each power of x dominates in a different region of $x > 0$. When x is very large, then the highest power of x in $p(x)$, say x^n, dominates and the sign of $p(x)$ is that of the leading coefficient p_n. When x is very small, then the lowest power of x, typically x^0, rules. As we move along from the origin, each successive power of x comes into play. If the sign of the coefficient of the new power of x does not change, then the function continues the trend set by the previous power, trending towards negative values if the coefficient is negative or positive values if the coefficient is positive. If there is to be a zero crossing, then there needs to be a sign change. If there is a sign change but there isn't a zero crossing, then we must have turned away from the x axis due to another sign change and will need to switch signs again to head back towards the x axis. This explains why signs need to be dropped in pairs when counting roots.

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[1]In keeping with historical usage, the word "root" refers herein to be a real root unless explicitly stated otherwise.

A Simple Example — Three Terms

A concrete example helps illustrate these ideas nicely:

Lemma 1 *For arbitrary powers $n > m > 0$, examples of polynomials of the form $1 - ax^m + bx^n$ with real coefficients having 0, 1, or 2 positive roots are given by the following table*[2]

	Coefficient Inequalities	*Number of Positive Roots*
i)	$a < 0$, $b > 0$	*0*
ii)	$b < 0$	*1*
iii)	$\min(b, 1) > a > 0$	*0*
iv)	$a - 1 > b > 0$	*2*

.

Proof. i) is immediate since all terms are positive. For ii) we rewrite the equation

$$1 - ax^m + bx^n = 0$$

with $b < 0$ as

$$|b|x^{n-m} = x^{-m} - a$$

and notice that the left hand side continually increases and the right hand side continually decreases in $x > 0$ so that there is at most one positive root. As x nears 0, the left hand side approaches zero and the right hand side is arbitrarily large. As $1/x$ nears 0, the right hand side approaches $-a$ and the left hand side is arbitrarily large. Therefore, by continuity, the two curves cross in $x > 0$ and there is exactly one root.

For iii) and iv), where $a > 0$ and $b > 0$, we write

$$1 = \frac{1}{a}x^{-m} + \frac{b}{a}x^{n-m}$$

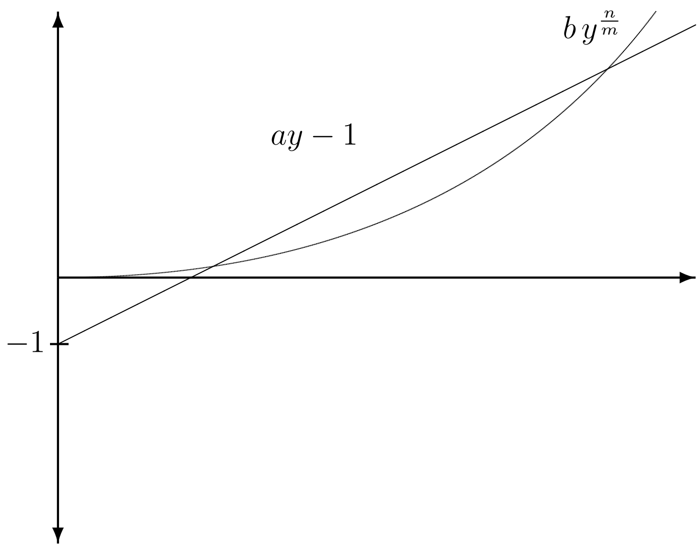
and note that x^{-m} is greater than 1 and x^{n-m} is less than 1 for $0 < x < 1$ while the opposite it true for $x > 1$. In particular if both coefficient ratios are greater than 1, i.e. condition iii), then the equality cannot be satisfied for $x > 0$. On the other hand if the coefficient ratios are positive but sum to less than 1, i.e. condition iv), then the right hand side is less than 1 for $x = 1$ and, as before, greater than 1 for both x approaching 0 and ∞ and so

[2]Appendix A derives exact conditions, not just the sufficient conditions given here.

has at least two positive roots, one less than 1 and one greater than 1. To see that there are no more than two roots, let $y = x^m$. Then the equation becomes

$$b\, y^{\frac{n}{m}} = ay - 1$$

that is, the intersection of a line with the strictly convex function $y^{\frac{n}{m}}$ which can occur at most two times.



Hence these are the only two positive roots.[3] ▌

Descartes' Rule

Despite its intuitive plausibility, Descartes' Rule of Signs was not directly proven until over a century after its original statement[4] in 1637 [3]. In this

[3]Technically, one should prove that this function is strictly convex. This follows, by a marvelous elementary demonstration [8] too long to fit in this footnote (c.f. Appendix A), from the ancient result that the arithmetic mean of positive numbers is strictly greater than their geometric mean except when all the numbers are equal. However, Corollary 5 will independently show that no polynomial with more than 2 positive roots can have only three nonzero terms.

[4]Isaac Newton restated the theorem in 1707, but apparently considered it too obvious to merit proof. DeGua is generally considered the first mathematician to publish an adequate proof in 1740.

exposition I reacquaint the mathematical public with the proof by elementary means first presented by the Prince of Mathematicians, Carl Friedrich Gauss [6]. I believe the following development is very clean and accessible, capturing the essence of Gauss' insight without obscuring it in unnecessary formalism[5]. I then enhance Gauss' proof with the addition of parity as noted in Dickson [4, §67] or Albert [1]. Finally I resolve some questions about Descartes' Rule left open in a recent Monthly article [2].

Although the actual proof of Descartes' Rule is brief—Lemma 2 and Theorem 2 cover less than a page—it is instructive to warm up to some special cases, starting with all positive or all negative roots.

Remark 1 *We may take the leading coefficient p_n of $p(x)$ to be unity without loss of generality.*

Multiplying or dividing $p(x)$ by any nonzero real number affects neither the location and number of sign changes in its coefficients nor the location and number of its roots. We will continue to employ the symbol p_n when it will simplify notation.

Remark 2 *We can safely assume the constant term p_0 is nonzero, i.e. the polynomial has no zero roots.*

Removing any common factors of x does not change the number of positive (or negative) roots, just the number of zero roots.

Proposition 1 *If all the coefficients of $p(x)$ are positive, then $p(x)$ has no positive roots.*

Proof. If all the coefficients are positive, $p(x)$ is a sum of positive terms for any $x > 0$ and so cannot equal zero there. ▌

Corollary 1 *If all the coefficients of $p(x)$ are nonzero and alternate in sign, then $p(x)$ has no negative roots.*

Proof. $p_0 \cdot p(-x)/|\,p_0|$ has all positive coefficients, hence Proposition 1 applies. Therefore $-x$ is not positive and x is not negative at a root[6]. ▌

[5] Admittedly I had not seen that work before I came up with this version of the proof . . .

[6] The reader should note that the substitution of $-x$ to deal with negative roots would not have simply changed every other coefficient sign if some of the polynomial coefficients were zero. In such cases, the signs are flipped only if an even number (including zero) of consecutive coefficients are missing; c.f. Corollary 7.

Proposition 2 *If a polynomial $p(x)$ of degree n has n positive roots, then its coefficients are all nonzero and the signs of the coefficients of $p(x)$ alternate.*

Proof. We proceed by induction on n. For $n = 0$ there are no roots and no sign changes. For $n = 1$, there is one sign alternation and the coefficients of $p_1 x - p_0$ are nonzero. Suppose now that $n > 1$ and the proposition holds for polynomials of degree up to $n - 1$ and consider a polynomial of degree n having n positive roots. By induction it may be written as the product

$$(x - \alpha) \sum_{j=0}^{n-1} (-1)^{n-1-j} p_j x^j$$

with α and all p_j positive. Expanding, we get

$$\begin{aligned} & (-1)^n \alpha p_0 + \sum_{j=1}^{n-1} [(-1)^{n-j} \alpha p_j + (-1)^{n-j-2} p_{j-1}] x^j + p_{n-1} x^n \\ = \quad & (-1)^n \alpha p_0 + \sum_{j=1}^{n-1} (-1)^{n-j} (\alpha p_j + p_{j-1}) x^j + p_{n-1} x^n \end{aligned}$$

which also has nonzero coefficients with alternating signs. ▌

Corollary 2 *If a polynomial of degree n has n negative roots, then its coefficients are all nonzero and the signs of the coefficients of $p(x)$ all agree.*

Proof. Apply the previous proposition to $(-1)^n p(-x)$.[7] ▌

The previous observations yielded somewhat stronger results for the special cases of all positive and all negative roots.

We next show that if there is exactly one sign change in the coefficients, there is at least one positive root. As Dickson [4, §22] notes, this follows directly from a result due to Lagrange[8]:

Theorem 1 *If in $p(x)$ the first negative coefficient is preceded by k coefficients which are positive or zero, and if G denotes the greatest of the magnitudes of the negative coefficients, then $p(x)$ is always positive for $x \geq 1 + \sqrt[k]{G/p_n}$ and so all real roots are less than that value.*

[7] While it's shorter to trivially prove this corollary directly and apply it to $(-1)^n p(-x)$ to prove Proposition 2, we will later find the present induction argument useful.

[8] This derivation is even more direct than Dickson's.

Proof. We zero out all but the first of the leading k nonnegative coefficients and replace all following coefficients with $-G$. Then for $x > 1$ we have

$$\begin{aligned} p(x) &\geq p_n x^n - G \sum_{j=0}^{n-k} x^j \\ &> p_n x^{k-1}(x^{n-k+1} - 1) - G \frac{x^{n-k+1} - 1}{x - 1} \\ &> \frac{x^{n-k+1} - 1}{x - 1} (p_n (x-1)^k - G) \end{aligned}$$

Thus $p(x)$ is positive for $x - 1 \geq \sqrt[k]{G/p_n}$ and all real roots must be less than $1 + \sqrt[k]{G/p_n}$. ▌

Theorem 1 confirms much of our intuition about the dominance of certain powers of x in certain ranges of $x > 0$. In particular:

Corollary 3 *For all sufficiently large $x > 0$, the sign of a polynomial matches the sign of its leading coefficient.*

Proof. Divide the polynomial by its leading coefficient. Either all coefficients are positive and the result is immediate or some later coefficient is negative and Theorem 1 applies. ▌

Corollary 4 *For all sufficiently small $x > 0$, the sign of a polynomial $p(x)$ matches the sign of its trailing coefficient.*

Proof. Apply Corollary 3 to $x^n p(1/x)$. ▌

Returning from our small digression, we now show that

Proposition 3 *If there is one sign change in the coefficients of $p(x)$, then it has at least one positive root.*

Proof. By our hypothesis, $p(0) = p_0 < 0$. On the other hand, Theorem 1 says $p(x)$ is positive for sufficiently large x. Therefore, by continuity, $p(x) = 0$ for some $x > 0$. ▌

Using an argument from the days before calculus was invented, we now show that there is exactly one positive root if there is one sign change[9]. We start with a simple observation about the function $\sum_{j=0}^{k-1} x^j$:

Observation 1 *Let* $\phi_0(x) = 0$, $\phi_k(x) = \phi_{k-1}(x) + x^{k-1}$ *for* $k = 1, \ldots$. *Then* $\phi_k(x)$ *satisfies the three relations:*

- $\phi_k(1) = k$,
- $\phi_k(y) \geq \phi_k(x)$ *for* $y \geq x \geq 1$, *and*
- $\phi_m(x) \geq \phi_k(x)$ *for* $m \geq k$ *when* $x \geq 0$.

Proposition 4 *If there is one sign change in the coefficients of* $p(x)$, *then it has exactly one positive root.*

Proof. By Proposition 3, there is at least one positive root. Let $\alpha > 0$ be the smallest and form the polynomial $\hat{p}(x) = \alpha^{-n} p(\alpha x)$. The coefficients of this new polynomial have the same signs as the original and the smallest positive root is shifted to $x = 1$. We now show that $\hat{p}$ is positive for $x > 1$ and that $x = 1$ is a simple root.

Splitting the positive and negative terms out, we write $\hat{p}(x) = q(x) - r(x)$, where the coefficients of q and r are nonnegative, and let k be as in Theorem 1 so that q has k coefficients. Factoring we have

$$\begin{aligned}
\hat{p}(x) &= \hat{p}(x) - \hat{p}(1) \\
&= \sum_{j=n-k+1}^{n} q_j (x^j - 1) - \sum_{j=0}^{n-k} r_j (x^j - 1) \\
&= (x-1) \left[\sum_{j=n-k+1}^{n} q_j \phi_j(x) - \sum_{j=0}^{n-k} r_j \phi_j(x) \right] \\
&= (x-1) s(x) \ .
\end{aligned}$$

Note that the $j = 0$ term in the second term is identically zero. Writing $0 = r_0 - r_0$, we now show that $s(x)$ is positive for $x \geq 1$, and thus $x = 1$ is a

[9] Dickson [4, §67, problem 9] outlines a related approach which ignores the possibility of a repeated root.

simple root and $\hat{p}(x)$ is positive for $x > 1$.

$$
\begin{aligned}
s(x) &\geq \phi_{n-k+1}(x) \left[r_0 + \sum_{j=n-k+1}^{n} q_j - \sum_{j=0}^{n-k} r_j \right] \\
&\geq \phi_{n-k+1}(1) \; [r_0 + \hat{p}(1)] \\
&= (n-k+1) \cdot r_0 \\
&> 0 \; .
\end{aligned}
$$

We conclude therefore that $\hat{p}(x)$ and hence $p(x)$ has exactly one positive root. ▌

Looking back, Propositions 1, 3 and 4 made Descartes-like statements about the number and location of roots given information about the signs of coefficients and were proven along the lines suggested by our initial plausibility arguments. When multiple coefficient sign changes arise, the number of possibilities grows combinatorially, making direct analysis by such methods quite daunting. Interestingly, our one result in the opposite direction, taking roots and making statements about coefficients (Proposition 2), was proven using a very different argument that foreshadows Lemma 2:

Lemma 2 *If a polynomial $q(x)$ with real coefficients exhibits m sign changes, then for any $\alpha > 0$, the polynomial $p(x) = (x - \alpha)q(x)$ exhibits at least $m+1$ sign changes.*

Proof. Let the degree of $q(x)$ be n. Then forming $p(x) = (x-\alpha)q(x)$ we get

$$
p(x) = -\alpha q_0 + \sum_{j=1}^{n} (q_{j-1} - \alpha q_j) x^j + q_n x^{n+1}
$$

This says that $p_{n+1} = q_n \, (= 1)$ and hence has the same sign. Furthermore, as we scan from $j = n$ down to $j = 1$ we have that at every sign transition between q_j and q_{j-1} the value of $p_j = q_{j-1} - \alpha q_j$ has the same sign as q_{j-1}. Thus, starting with p_{n+1}, there is a subsequence of p_j, call it p_{j_k}, that has the same coefficient signs as the corresponding subsequence $q_{j_k - 1}$ of coefficients of $q(x)$. Since the number of sign changes in the full sequence p_j is no less than the number of sign changes in any subsequence, we have accounted for at least m sign changes in $p(x)$. Finally, p_0 has a sign opposite to that of q_0

and hence opposite to that of p_{j_m}. Therefore $p(x)$ has at least $m+1$ sign changes.[10] ▌

Corollary 5 *A polynomial with m positive roots has more than m nonzero coefficients.*

Proof. By Lemma 2 there are at least m sign changes in the coefficients of a polynomial with m positive roots, hence at least $m+1$ coefficients for the sign changes to occur between. ▌

Theorem 2 [Descartes' Rule of Signs—I] *The number of positive roots of a polynomial $p(x)$ with real coefficients does not exceed the number of sign changes of its coefficients. A zero coefficient is not counted as a sign change.*

Proof. We proceed by induction on the number of positive roots of $p(x)$. If $p(x)$ has no positive roots, the result is immediate. Suppose now that it holds true for less than k positive roots and that we have a polynomial $p(x)$ with k positive roots. Then for any root $\alpha > 0$,

$$p(x) = (x - \alpha)q(x)$$

for some polynomial $q(x)$ with $k-1$ positive roots. By induction, $q(x)$ has at least $k-1$ sign changes. Therefore, by Lemma 2, $p(x)$ has at least $(k-1)+1$ $= k$ sign changes. ▌

To show further that the difference between the number of sign changes and the numbers of roots is even, we employ a pretty observation on parity:

Proposition 5 [Parity] *The parity, i.e. the remainder upon division by 2, of the number of sign changes in a sequence of nonzero real numbers s_j, $j = 0, \ldots, n$ is equal to the number of sign changes in the two element subsequence $s_0\, s_n$.*

Proof. Let σ_j be the sign of s_j. Then the ratio σ_j/σ_{j+1} is -1 at each sign change and 1 otherwise. Therefore

$$(-1)^{\#\text{ sign changes}} = \prod_{j=0}^{j=n-1} \frac{\sigma_j}{\sigma_{j+1}} = \frac{\sigma_0}{\sigma_n}$$

[10]For infinite series one does not necessarily increase the number of sign changes, but the number does not decrease. If the series converges at $x = \alpha$, however, the number of sign changes does increase. [10]

which says that difference in the number of sign changes in the whole sequence and the number of sign changes (i.e. 0 or 1) in the subsequence $s_0\, s_n$ is even. ▌

Using this result, we can immediately expand Lemma 2 to reflect that any additional sign changes must come in pairs between existing sign changes:

Lemma 3 *If a polynomial $q(x)$ with real coefficients exhibits m sign changes, then for any $\alpha > 0$, the polynomial $p(x) = (x - \alpha)q(x)$ exhibits $m + 1 + 2l$ sign changes for some integer $l \geq 0$.*

Finally, when we include parity in Descartes' Rule, the $n = 0$ case of the induction is no longer immediate, but is, fortunately, readily established:

Proposition 6 *If $p(x)$ has no positive roots then its coefficients have an even number of sign changes.*

Proof. Since p_n is positive, by parity we need only show p_0 is also positive. Suppose it is negative. Then $p(0)$ is negative. By Theorem 1, $p(x)$ is positive for sufficiently large x, so $p(x)$ is zero for some $x > 0$, contradicting the hypothesis that $p(x)$ has no positive roots. ▌

Theorem 3 [Descartes' Rule of Signs—II] *The number of positive roots of a polynomial $p(x)$ with real coefficients does not exceed the number of sign changes of its coefficients and differs from it by a multiple of two. A zero coefficient is not counted as a sign change.*

For the record, Propositions 1 and 4 are one-liners:

Corollary 6 *If all the coefficients of $p(x)$ are positive, then $p(x)$ has no positive roots. If there is one sign change in the coefficients of $p(x)$, then it has at exactly one positive root.*

Proof. 0 is the only nonnegative even number ≤ 0 and 1 is the only nonnegative odd number ≤ 1. ▌

To find out about negative roots, we look at the sign changes in $p(-x)$. In terms of of $p(x)$ itself, sign changes in $p(-x)$ correspond to one of two sign behaviors:

- No sign change (a *sign series* or *permanence*) between powers of x separated by an even number, including zero, of missing terms, or

- A sign change between powers of x separated by an odd number of missing terms.

Corollary 7 *The number of negative roots of $p(x)$ does not exceed the number of sign series separated by an even number of missing terms added to the number of sign changes separated by an odd number of missing terms in $p(x)$. Furthermore, the difference is an even number.*

Corollary 8 *The number of complex roots*[11] *of $p(x)$ exceeds by a nonnegative even integer the number of missing terms plus the number of sign series separated by an odd number of terms less the number of sign changes separated by an odd number of missing terms.*

The proof of this last corollary, also from Gauss [6], is left to the reader.

Going the Other Way

Quite recently this Monthly [2] addressed the question of whether given any particular sign sequence all positive root combinations allowed by Descartes' Rule were possible. With the stipulation that all coefficients be nonzero, the answer given was yes. Left open was the question of whether their theorem holds in the presence of missing terms. Grabiner [7] proves that the extension to missing coefficients is possible by a very pretty direct construction of such polynomials[12].

In this section, we will shortly develop, and later generalize, an elementary inductive proof of this extension, first tackling the easier question of whether given any particular list of powers of x there are examples of polynomials with those nonzero terms that have any given numbers of positive roots and sign changes allowable by Theorem 3. We can show this is the case using our original simple three term constructions:

Theorem 4 *Given any specified sequence of exponents $0 = m_0 < m_1 < \ldots < m_n$, there exist polynomials $p_0x^{m_0} + p_1x^{m_1} \ldots + p_nx^{m_n}$ whose number*

[11]As per Remark 2 we assume we have previously factored out all zero roots. Drucker [5] provides a variant formula that includes the number of zero roots.

[12]Laguerre [9] has shown that Descartes' Rule holds for arbitrary real exponents of x, positive or negative, rational or irrational, which would suggest that the "missing terms" result holds in this more general setting and this is indeed the case.

of positive roots equals the upper bound given by Descartes' Rule of Signs. Furthermore there also exist such polynomials having a (nonnegative) number of positive roots differing from this upper bound by every possible multiple of two.

Proof. We proceed by induction and show how to construct suitable examples featuring only simple (or no) positive roots. The first problem here is to figure out what to induct on. The degree of the polynomial? The number of nonzero coefficients? The trick is to induct on the number of *sign changes*, $\mathcal{N} \leq n$, and, because of parity, to work even and odd numbers separately. For $\mathcal{N} = 0$ and $\mathcal{N} = 1$ the result follows immediately from Propositions 1 and 4. Suppose now that the theorem holds for $\mathcal{N} = k - 2 \leq n - 2$. We now construct polynomials with the desired n nonzero coefficients having k sign changes and either 0 or 2 additional roots.

By hypothesis, there is a polynomial

$$q(x) = \sum_{j=0}^{n-2} q_j x^{m_{n-2} - m_j}$$

having $k - 2$ sign changes and $n - 1$ nonzero coefficients, where we may take the constant term q_{n-2} to have the value 1. We now write the polynomial

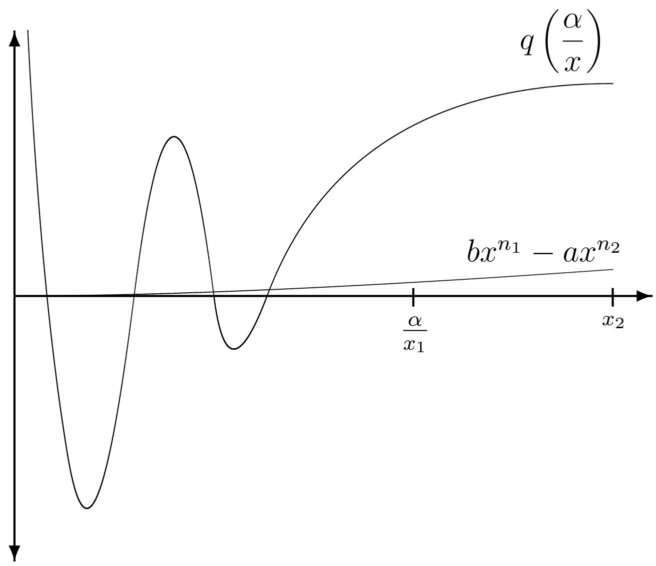
$$\begin{aligned} p(x) &= bx^{m_n} - ax^{m_{n-1}} + x^{m_{n-2}} q\left(\frac{\alpha}{x}\right) \\ &= x^{m_{n-2}}\left[(bx^{n_1} - ax^{n_2}) + q\left(\frac{\alpha}{x}\right)\right] \end{aligned}$$

with k sign changes, where a, b and α are to-be-chosen positive numbers and the integers $n_1 = m_n - m_{n-2}$ and $n_2 = m_{n-1} - m_{n-2}$ satisfy $n_1 > n_2 > 0$.

If the positive roots of $q(x)$ are greater than x_1, then the roots of $q(\alpha/x)$ lie strictly between 0 and α/x_1. By taking α sufficiently small, we can guarantee two key conditions hold:

- $bx^{n_1} - ax^{n_2}$ is as small as we please within some larger range, say for $x \leq x_2$ with all the positive roots (if any) of $bx^{n_1} - ax^{n_2} + 1$ greater than x_2, and
- the slope of $q(\alpha/x)$ at its (simple) zeros is as large as we please.

This ensures that $p(x)$ has the same number of roots as q for $x \leq x_2$. (For the purposes of exposition, I omit some technical continuity and finiteness arguments. Appendix A contains elementary machinery to fill in the details without calculus.)



Similarly, substituting $\hat{x} = \alpha/x$ and possibly making α even smaller, we interchange the roles of $bx^{n_1} - ax^{n_2}$ and q and can ensure that $p(x)$ and $bx^{n_1} - ax^{n_2} + 1$ have the same number of roots for $x \geq x_2$ which, by Lemma 1, can be made to number either 0 or 2. ▌

The argument we have employed to construct new roots without destroying existing roots is a special case of a more general (ancient) lemma whose proof is left to the interested reader:

Lemma 4 *Let $p(x)$ be a polynomial of degree m with $p(0) = 1$ having only simple real roots with M^+ of them positive and M^- of them negative. Let $q(x)$ be a polynomial of degree n with leading coefficient 1 having only simple real roots with N^+ of them positive and N^- of them negative. Then the polynomial*

$$x^n(p(x) - 1) + x^n q\left(\frac{\alpha}{x}\right)$$

has exactly $M^+ + N^+$ positive roots and $M^- + N^-$ negative roots for sufficiently small α and these real roots are all simple.

We now generalize Lemma 1 a bit so that the approach used to prove the last theorem allows us to specify the exact sequence of coefficient signs, not just the number of sign changes:

Lemma 5 *The coefficients of a polynomial with exactly two sign changes can be modified without changing its coefficient signs to produce two other polynomials, one with zero and the other with two positive roots.*

Proof. Taking the leading coefficient to be 1 as before, write the given polynomial $p(x)$ as

$$p(x) = x^n q(x) - x^m r(x) + s(x)$$

where the coefficients of q, r and s are positive and n and m are the appropriate positive integers. We note (c.f. Corollary 3 and 4) that $p(x)$ is positive for all sufficiently small and all sufficiently large positive x. Choose any two arbitrary positive values $x_1 < x_2$. Then

$$\begin{aligned} \min_{x_1 \le x \le x_2} [x^n q(x) + s(x)] &= x_1^n q(x_1) + s(x_1) \text{ , and} \\ \max_{x_1 \le x \le x_2} [x^n q(x) + s(x)] &= x_2^n q(x_2) + s(x_2) \text{ .} \end{aligned}$$

We now construct the polynomial

$$x^n q(x) - \gamma x^m r(x) + s(x)$$

and adjust the parameter $\gamma > 0$ to ensure either 0 or 2 roots respectively. Since

$$\begin{aligned} \min_{x_1 \le x \le x_2} [x^m r(x)] &= x_1^m r(x_1) \text{ , and} \\ \max_{x_1 \le x \le x_2} [x^m r(x)] &= x_2^m r(x_2) \text{ ,} \end{aligned}$$

the ratio of $x^n q(x) + s(x)$ to $x^m r(x)$ satisfies

$$\begin{aligned} \min_{x_1 \le x \le x_2} \frac{x^n q(x) + s(x)}{x^m r(x)} &\ge \frac{x_1^n q(x_1) + s(x_1)}{x_2^m r(x_2)} \text{ , and} \\ \max_{x_1 \le x \le x_2} \frac{x^n q(x) + s(x)}{x^m r(x)} &\le \frac{x_2^n q(x_2) + s(x_2)}{x_1^m r(x_1)} \text{ .} \end{aligned}$$

Therefore the following choices for γ

	γ	Number of Positive Roots
a)	$\frac{1}{2}\dfrac{x_1^n q(x_1) + s(x_1)}{x_2^m r(x_2)}$	0
b)	$2\dfrac{x_2^n q(x_2) + s(x_2)}{x_1^m r(x_1)}$	2

meet our need. If p originally had 2 positive roots, then let x_1 and x_2 be their locations. Using choice a) above for γ shifts p upwards everywhere in $x > 0$ and makes it also now positive in $x_1 \leq x \leq x_2$, eliminating the two roots. On the other hand, if p originally had no positive roots, choose any convenient x_1 and x_2 and choice b) above for γ shifts p downwards everywhere in $x > 0$ and makes both $p(x_1)$ and $p(x_2)$ negative, ensuring p has two positive roots. ▌

Theorem 5 *Given any specified sequence of coefficient signs,* $\sigma_0, \ldots, \sigma_n$ *and corresponding exponents* $0 = m_0 < m_1 < \ldots < m_n$*, there exist polynomials* $p_0 x^{m_0} + p_1 x^{m_1} \ldots + p_n x^{m_n}$ *whose number of roots equals the upper bound given by Descartes' Rule of Signs. Furthermore there also exists such polynomials having a (nonnegative) number of positive roots differing from this upper bound by every possible multiple of two.*

Proof. We again proceed by induction on the number of sign changes, $\mathcal{N} \leq n$. As before, for $\mathcal{N} = 0$ and $\mathcal{N} = 1$ the result follows immediately from Corollary 6. Suppose now that the theorem holds for $\mathcal{N} \leq l - 2$ and let $p(x)$ be a polynomial with the given exponents and $l \geq 2$ prescribed sign changes.

Since $p(x)$ has at least 2 sign changes, we can split off the first two sign changes and write

$$p(x) = x^{k_1} q(x) - x^{k_2} r(x) + s(x)$$

where the coefficients of q and r are positive. Writing

$$s(x) = \sum_{j=0}^{k} s_j x^{m_j} \ ,$$

by induction there is a corresponding polynomial $\hat{s}$ taking the form

$$\hat{s}(x) = \sum_{j=0}^{k} s_j x^{k - m_j}$$

having $l - 2$ sign changes and any number of allowable roots with its leading coefficient s_k positive.

By Lemma 5 the polynomial

$$x^{k_1} q(x) - \gamma x^{k_2} r(x) + s_k$$

can be made to have either zero or two positive roots simply by adjusting $\gamma > 0$ appropriately. Thus, using Lemma 4, we may construct a polynomial

$$\hat{p}(x) = x^{k_1} q(x) - \gamma x^{k_2} r(x) + x^k \hat{s}\left(\frac{\alpha}{x}\right)$$

for which we may make α sufficiently small so as to both preserve the number of roots from s and add to that the 0 or 2 additional roots determined by selection of γ. ▌

In addition to the question we have just reanswered affirmatively of whether missing terms are allowable, the Monthly article [2] concludes with the question: *Given a sign sequence* (*which may include some zeros*), *do there exist polynomials containing positive and negative roots numbering each of the possible combinations allowed by Descartes' Rule of Signs?*

In general, the answer is no and Grabiner [7] provides nice quartic counterexamples. However, when there are no missing terms, which we'll term a *complete* polynomial, we *can* extend Theorem 4. We start with a simple observation:

Proposition 7 *When a polynomial $p(x)$ is complete, then any sign change in $p(x)$ is not a sign change in $p(-x)$ and vice versa.*

Proof. Let any two consecutive terms be ax^m and bx^{m-1}. Substituting $-x$ transforms them into $(-1)^m ax^m$ and $(-1)^{m-1} bx^{m-1}$. The ratio of the coefficients signs σ_m/σ_{m-1} thus transforms to $-\sigma_m/\sigma_{m-1}$ demonstrating the result. ▌

Corollary 9 *The sum of the number of sign changes in $p(x)$ and the number of sign changes in $p(-x)$ is n for a complete polynomial of degree n.*

Theorem 6 *For any specified number of coefficient sign changes, there exist complete polynomials having any possible combination of numbers of positive and negative roots allowable by Descartes' Rule of Signs.*

Proof. We mimic Theorem 4 by inducting on the number of sign changes $\mathcal{N}$. Because we must worry about an indefinite number of negative roots, the cases $\mathcal{N} = 0$ and $\mathcal{N} = 1$ are not trivial. Fortunately, Theorem 4 guarantees examples of $p(x)$ with all permissible combinations of negative roots and

Corollary 6 guarantees that any such example must have exactly 0 or 1 positive root respectively. Therefore the theorem is true for these initial cases.

Having established these initial cases, following the induction on the number of sign changes from $k-2 \rightarrow k$ in Theorem 4 is clear sailing. By adding two *consecutive* leading sign changes, with 0 or 2 accompanying positive roots, Proposition 7 guarantees we add no further sign changes to $p(-x)$ and hence introduce no new possible combinations of the numbers of signs and negative roots. Thus we can construct a polynomial with $\mathcal{N} = k$ sign changes and which has any allowable numbers of positive and negative roots. ▌

References

[1] Albert, G. E., 1943, An Inductive Proof of Descartes' Rule of Signs: Amer. Math. Monthly v. 50, no. 3 (March), p. 178–180.

[2] Anderson, B., Jackson, J. and Sitharam, M., 1998, Descartes' Rule of Signs Revisited: Amer. Math. Monthly, v. 105, no. 5 (May), p. 447–451.

[3] Bartolozzi, M. and Franci, R., 1993, La Regola dei Segni dall'Enunciato di R. Descartes (1637) alla Dimostrazione di C. F. Gauss (1828) (*The Rule of Signs, from its statement by R. Descartes (1637) to its proof by C.F. Gauss (1828)*): Arch. Hist. Exact Sci., v. 45, no. 4, p. 335–374 (Italian).

[4] Dickson, L. E., 1922, First Course in the Theory of Equations, Wiley, New York.

[5] Drucker, D. S., 1979, A Second Look at Descartes' Rule of Signs: Mathematics Magazine, v. 52, no. 5 (Sept), p. 237–238.

[6] Gauss, C. F., 1828, Beweis eines algebraischen Lehrsatzes (*Proof of certain algebraic Theorems*): Crelle's Journal für die reine und angewandte Mathematik, v. 3, no. 1, p. 1–4 (German).

[7] Grabiner, D. J., 1999, Descartes' Rule of Signs: Another construction: *to appear in* Amer. Math. Monthly, v. 106.

[8] Korovkin, P. P., 1961, Inequalities, Blaisdell, New York. (Translation of 1952 *Neravenstva*, Gostekhizdat, Moscow-Leningrad.)

[9] Laguerre, E. N., 1883, Sur la Théorie des Équations Numériques (*On the theory of numeric equations*), *in* Œuvres de Laguerre, p. 3–47, v. I, Gauthier-Villars, Paris. (Reprinted in http://sepwww.stanford.edu /oldsep/stew/laguerre.pdf)

[10] Pólya, G., 1914, Über einige Verallgemeinerungen der Descartesschen Zeichenregel (*On some generalizations of Descartes' Rule of Signs*): Arch. Math. Phys., v. 23, no. 4, p. 22-32 (German).

APPENDIX—Elementary Convexity

The following arguments are adapted from Korovkin [8].

Theorem A *If the product of n positive numbers equals 1 their sum is not less than n. Furthermore equality holds only when all the numbers are equal to 1.*

Proof. We proceed by induction. The case $n = 1$ is immediate. Suppose now that it holds for all $n \leq k$ and consider the product of $k+1$ positive numbers[13]

$$x_1 x_2 \ldots x_k x_{k+1} = 1 \ .$$

Two cases may arise:

- All the numbers are identical, i.e.

$$x_1 = x_2 = \ldots = x_k = x_{k+1} \, , \text{ or}$$

- Some factors are different.

In the first case all the terms are equal to 1 and their sum is $k+1$.

In the second case at least one of the terms is less than 1 and one greater than 1. After suitable renumbering, we may suppose that $x_1 < 1$ and $x_{k+1} > 1$. Putting $y_1 = x_1 x_{k+1}$ we have

$$y_1 x_2 \ldots x_k = 1 \, .$$

By our induction hypothesis, we have

$$y_1 + x_2 + \ldots + x_k \geq k \, .$$

But

$$\begin{aligned} & x_1 + x_2 + \ldots + x_k + x_{k+1} \\ & \quad = \; (y_1 + x_2 + \ldots + x_k) + x_{k+1} - y_1 + x_1 \\ & \quad \geq \; (k+1) + x_{k+1} - y_1 + x_1 - 1 \\ & \quad = \; (k+1) + (x_{k+1} - 1)(1 - x_1) \, , \end{aligned}$$

[13] The famous classical proof of this result by Cauchy is done using induction from $n \to 2n$. The interested reader should easily reconstruct that proof after understanding the present one.

remembering that $y_1 = x_1 x_{k+1}$. As $x_1 < 1$ and $x_{k+1} > 1$, the product $(x_{k+1} - 1)(1 - x_1)$ is positive and so

$$x_1 + x_2 + \ldots + x_k + x_{k+1} > k + 1\,,$$

proving Theorem A. ▌

Recalling the definitions of the *geometric mean*

$$\mathcal{G} = \sqrt[n]{x_1 x_2 \ldots x_n}$$

and the *arithmetic mean*

$$\mathcal{A} = \frac{x_1 + x_2 + \ldots + x_n}{n}\,,$$

we now show

Theorem B *The geometric mean of positive numbers is not greater than the arithmetic mean of the same numbers and equality holds only when all the numbers are equal.*

Proof. From the definition of $\mathcal{G}$ we have

$$\frac{x_1}{\mathcal{G}} \frac{x_2}{\mathcal{G}} \cdots \frac{x_n}{\mathcal{G}} = 1$$

and hence by Theorem A

$$\frac{x_1}{\mathcal{G}} + \frac{x_2}{\mathcal{G}} + \ldots + \frac{x_n}{\mathcal{G}} \geq n\,.$$

Multiplying both sides by $\mathcal{G}$ and dividing by n we have

$$\mathcal{A} = \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \mathcal{G}$$

with equality holding only when all the x_j's are equal. ▌

We now use this inequality to study $(1 + x)^\alpha$.

Theorem C *For rational* α *and* $x \geq -1$ *if* $0 < \alpha < 1$, *then*

$$(1 + x)^\alpha \leq 1 + \alpha x\ .$$

But if $\alpha > 1$,

$$(1 + x)^\alpha \geq 1 + \alpha x\ .$$

Proof. Write $\alpha = \frac{m}{n}$. If $1 \le m < n$, then using Theorem B we have

$$\begin{aligned}(1+x)^\alpha &= \sqrt[n]{(1+x)^m \cdot 1^{n-m}} \\ &\le \frac{m(1+x)+(n-m)\cdot 1}{n} \\ &= 1 + \frac{m}{n}x = 1 + \alpha x\end{aligned}$$

with equality holding only when $x = 0$, proving the first part of the theorem.

For the second part, let $\alpha > 1$. If $1 + \alpha x < 0$ the inequality is immediate as $(1+x)^\alpha$ is nonnegative. For $1 + \alpha x \ge 0$, that is $\alpha x \ge -1$, the first part of the theorem gives us

$$(1+\alpha x)^{\frac{1}{\alpha}} \le 1 + \frac{1}{\alpha}\alpha x = 1 + x \ .$$

Raising both sides to the power α we obtain

$$1 + \alpha x \le (1+x)^\alpha$$

with equality holding only for $x = 0$. ▌

We note that the second inequality applies for $\alpha < 0$, but we will not need that for the main result that follows:

Theorem D *Let $y > x > 0$ and α be rational and greater than 1. Then for $0 < \lambda < 1$ we have*

$$\lambda x^\alpha + (1-\lambda)y^\alpha > (\lambda x + (1-\lambda)y)^\alpha \ .$$

Proof. We divide the left hand side by the right hand side and show the result, $\mathcal{R}$, is greater than 1:

$$\begin{aligned}\frac{\lambda x^\alpha + (1-\lambda)y^\alpha}{(\lambda x + (1-\lambda)y)^\alpha} &= \lambda \left(\frac{x}{\lambda x + (1-\lambda)y}\right)^\alpha + (1-\lambda)\left(\frac{y}{\lambda x + (1-\lambda)y}\right)^\alpha \\ &= \lambda \qquad d_1^\alpha \qquad + (1-\lambda) \qquad d_2^\alpha\end{aligned}$$

where we note $\lambda d_1 + (1-\lambda)d_2 = 1$ and both d_1 and d_2 are positive. Writing

$$\begin{aligned}d_1 &= 1 + \frac{z_1}{\lambda} \\ d_2 &= 1 + \frac{z_2}{1-\lambda} \ ,\end{aligned}$$

we have

$$z_1 + z_2 = \lambda(d_1 - 1) + (1 - \lambda)(d_2 - 1) = 0 \ .$$

Invoking Theorem C, we can write

$$\mathcal{R} > \lambda(1 + \alpha\frac{z_1}{\lambda}) + (1 - \lambda)(1 + \alpha\frac{z_2}{1 - \lambda}) = 1 + \alpha(z_1 + z_2) = 1 \ ,$$

completing the proof. ▌

From Theorem C, we can precisely determine the tangent line to $y = x^{\frac{n}{m}}$ and sharpen Lemma 1.

Lemma A *Let $\alpha > 1$ be rational. Then the tangent line to $y = x^\alpha$ at $x = x_0 \geq 0$ is given by the formula*

$$y = x_0^\alpha + \alpha x_0^{\alpha-1}(x - x_0) \ .$$

Proof. For $x \geq 0$, Theorem C yields the formula

$$\left(1 + \alpha\frac{x - x_0}{x_0}\right) \leq \left(1 + \frac{x - x_0}{x_0}\right)^\alpha$$

with equality exactly when $x = x_0$. Multiplying through by x_0^α yields

$$x_0^\alpha + \alpha x_0^{\alpha-1}(x - x_0) \leq x^\alpha$$

again with equality only for $x = x_0$. Therefore

$$y = x_0^\alpha + \alpha x_0^{\alpha-1}(x - x_0)$$

touches the curve $y = x^\alpha$ exactly at the point $y = x_0^\alpha$ and hence is the tangent line there. ▌

Proposition A *For arbitrary integer powers $n > m > 0$, polynomials of the form $1 - ax^m + bx^n$ with nonvanishing real coefficients a and b, have either 0, 1, or 2 positive roots according the following table*

	Coefficient Inequalities	*Number of Positive Roots*
i)	$a < 0$, $b > 0$	*0*
ii)	$b < 0$	*1*
iii)	$1 \geq a > 0$	*0*
iv)	$a > 1$, $(a/\alpha)^{\alpha}/(a-1)^{\alpha-1} \geq b > 0$	*2*
v)	$a > 1$, $b > (a/\alpha)^{\alpha}/(a-1)^{\alpha-1} > 0$	*0*

where $\alpha = \frac{n}{m}$.

Proof. i) and ii) were shown in Lemma 1. For iii) through v), where $a > 0$ and $b > 0$, we repeat the substitution $y = x^m$ of Lemma 1 to transform the problem to the determination of the roots of

$$ay - 1 = by^{\alpha} \ .$$

From Lemma A, the line $ay - 1$ will be tangent to by^{α} if both slope and intercepts match

$$\begin{aligned} a &= b\alpha y_0^{\alpha-1} \\ 1 &= b(\alpha - y_0)y_0^{\alpha-1} \end{aligned}$$

for some abscissa $y_0 > 0$.

Solving the slope equation for y_0, we have

$$\begin{aligned} by_0^{\alpha-1} &= \frac{a}{\alpha} \\ y_0 &= \left(\frac{a}{\alpha b}\right)^{\frac{1}{\alpha-1}} \end{aligned}$$

which we can plug into the intercept condition to find

$$1 = \left(\alpha - \left(\frac{a}{\alpha b}\right)^{\frac{1}{\alpha-1}}\right) \cdot \left(\frac{a}{\alpha}\right)$$

whence a and b must satisfy

$$\alpha\left(1 - \frac{1}{a}\right) = \left(\frac{a}{\alpha b}\right)^{\frac{1}{\alpha-1}}$$

to achieve tangency.

When $0 < a \leq 1$, i.e. condition iii), there is no solution as the left hand side is less than or equal to zero and the right hand side is positive. When $a > 1$, we solve for b to find

$$b = \frac{\left(\frac{a}{\alpha}\right)^{\alpha}}{(a-1)^{\alpha-1}} \ .$$

When b is less than or equal to this value, the line $-1 + ay$ intersects by^{α} in two positive roots (or a root of multiplicity two if exactly equal), while if b is greater than this value the line falls entirely below the curve and there are no positive roots. This establishes cases iv) and v), completing the proof. ▌