*Stanford Exploration Project, Report SERGEY, November 9, 2000, pages 301–**??***

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Helical preconditioning and splines in tension

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ABSTRACT

Splines in tension are smooth interpolation surfaces whose behavior in unconstrained regions is controlled by the tension parameter. I show that such surfaces can be efficiently constructed with recursive filter preconditioning and introduce a family of corresponding two-dimensional minimum-phase filters. The filters are created by spectral factorization on a helix.

INTRODUCTION

The method of minimum curvature is an old and ever-popular approach for constructing smooth surfaces from irregularly spaced data (Briggs, 1974). The surface of minimum curvature corresponds to the minimum of the Laplacian power or, in an alternative formulation, satisfies the biharmonic differential equation. Physically, it models the behavior of an elastic plane. In the one-dimensional case, the minimum curvature method leads to the natural cubic spline interpolation (de Boor, 1978). In the two-dimensional case, a surface can be interpolated with biharmonic splines (Sandwell, 1987) or gridded with an iterative finite-difference scheme (Swain, 1976). Claerbout (1999) suggests a straightforward least-squares optimization approach employing an iterative conjugate-gradient algorithm.

In most of the practical cases, the minimum curvature method produces a visually pleasing smooth surface. However, in cases of large changes in the surface gradient, the method can create strong artificial oscillations in the unconstrained regions. Switching to lower-order methods, such as minimizing the power of the gradient, solves the problem of extraneous inflections, but also removes the smoothness constraint and leads to gradient discontinuities (Fomel and Claerbout, 1995). A remedy, suggested by Schweikert (1966), is known as *splines in tension*. Splines in tension are constructed by minimizing a modified quadratic form that includes a tension term. Physically, the additional term corresponds to tension in elastic plates (Timoshenko and Woinowsky-Krieger, 1968). Smith and Wessel (1990) developed a practical algorithm of 2-D gridding with splines in tension and implemented it in the GMT software package.[2]

Fomel et al. (1997) have recently shown that an iterative interpolation algorithm can be greatly accelerated by preconditioning with recursive multidimensional filters defined on a

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helix (Claerbout, 1998a,b). To construct a minimum-phase filter suitable for recursive filtering, one can apply an efficient spectral factorization method (Sava et al., 1998).

In this paper, I develop an application of helical preconditioning to gridding with splines in tension. I introduce a family of 2-D minimum-phase filters for different degrees of tension. The filters are constructed by spectral factorization of the corresponding finite-difference forms. In the case of zero tension (the original minimum-curvature formulation), we obtain a minimum-phase version of the Laplacian filter. The case of infinite tension leads to spectral factorization of the Laplacian and produces the known *helical derivative* filter (Claerbout, 1999; Zhao, 1999).

The tension filters can be applied not only for interpolation but also for preconditioning in any estimation problems with smooth models. Tomographic velocity estimation is an obvious example of such an application (Woodward et al., 1998).

MATHEMATICAL THEORY OF SPLINES IN TENSION

The traditional minimum-curvature criterion implies seeking a two-dimensional surface $f(x,y)$ in region D, which corresponds to the minimum of the Laplacian power:

$$\iint\limits_D \nabla^2 f(x,y)\, dx\, dy \; , \tag{1}$$

where ∇^2 denotes the Laplacian operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Alternatively, we can seek $f(x,y)$ as the solution of the biharmonic differential equation

$$(\nabla^2)^2 f(x,y) = 0 \; . \tag{2}$$

Equation (2) corresponds to the normal system of equations in the least-square optimization problem. Briggs (1974) derives it directly from (1) with the help of Gauss's theorem.

Formula (1) approximates the strain energy of a thin elastic plate (Timoshenko and Woinowsky-Krieger, 1968). Taking tension into account modifies both the energy formula (1) and the corresponding equation (2). Smith and Wessel (1990) suggest the following form of the modified equation:

$$\left[(1-t)(\nabla^2)^2 - t(\nabla^2)\right] f(x,y) = 0 \; , \tag{3}$$

where the tension parameter t ranges from 0 to 1. Zero tension leads to the biharmonic equation (2) and corresponds to the minimum curvature construction. The case of $t = 1$ corresponds to infinite tension. Although infinite tension is physically impossible, the resulting Laplace equation does have a physical interpretation of a steady-state temperature distribution. An important property of harmonic functions (solutions of the Laplace equation) is that they cannot have local minima and maxima in the free regions. With respect to interpolation, this means that, in the case of $t = 1$, the interpolation surface will be constrained to have its local extrema only at the input locations.

To interpolate an irregular set of data values, f_k at points (x_k, y_k), we need to solve equation (3) under the constraint

$$f(x_k, y_k) = f_k \; . \tag{4}$$

An iterative solution of this problem can be greatly accelerated by preconditioning (Fomel, 1997; Fomel et al., 1997). If **A** is the discrete filter representation of the differential operator in equation (3), and we can find a minimum-phase filter **D** whose autocorrelation is equal to **A**, then an appropriate preconditioning operator is a recursive inverse filtering with the filter **D**. Formulating the problem in helical coordinates (Claerbout, 1998a,b) allows us to perform both the spectral factorization of **A** and inverse filtering with **D**.

FINITE DIFFERENCES AND SPECTRAL FACTORIZATION

In the one-dimensional case, a finite-difference representation of the squared Laplacian can be defined as a centered 5-point filter with coefficients $(1, -4, 6, -4, 1)$. On the same grid, the Laplacian operator can be approximated to the same order of accuracy with the filter $(1/12, -4/3, 5/2, -4/3, 1/12)$. Combining the two filters in accordance with equation (3) and performing a spectral factorization with one of the standard methods (Claerbout, 1976, 1992), we can obtain a 3-point minimum-phase filter, suitable for inverse filtering. Figure 1 shows a family of one-dimensional minimum-phase filters for different values of the parameter t. Figure 2 demonstrates the interpolation results obtained with these filters on a simple one-dimensional synthetic. As expected, a small tension value ($t = 0.01$) produces a smooth interpolation, but creates artificial oscillations in the unconstrained regions around sharp changes in the gradient. The value of $t = 1$ leads to linear interpolation with no extraneous inflections, but with discontinuous derivatives. Intermediate values of t allow us to achieve a compromise: a smooth surface with constrained oscillations.

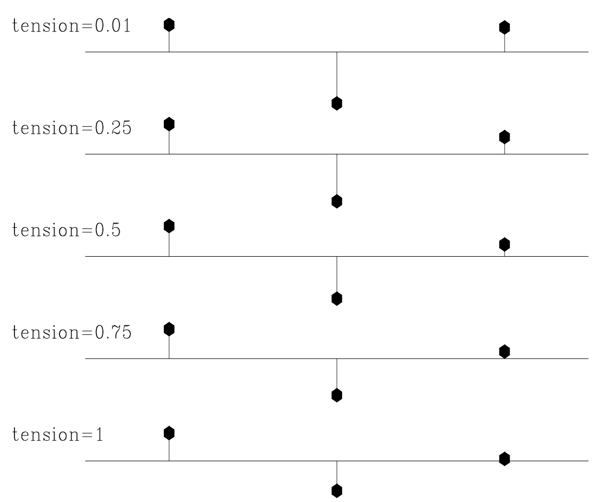

Figure 1: One-dimensional minimum-phase filters for different values of the tension parameter t. The filters range from the second derivative for $t = 0$ to the first derivative for $t = 1$. tension-otens [ER]

To design the corresponding filters in two dimensions, I define the finite-difference representation of operator (3) on a 5-by-5 stencil. The filters coefficients are chosen with the help

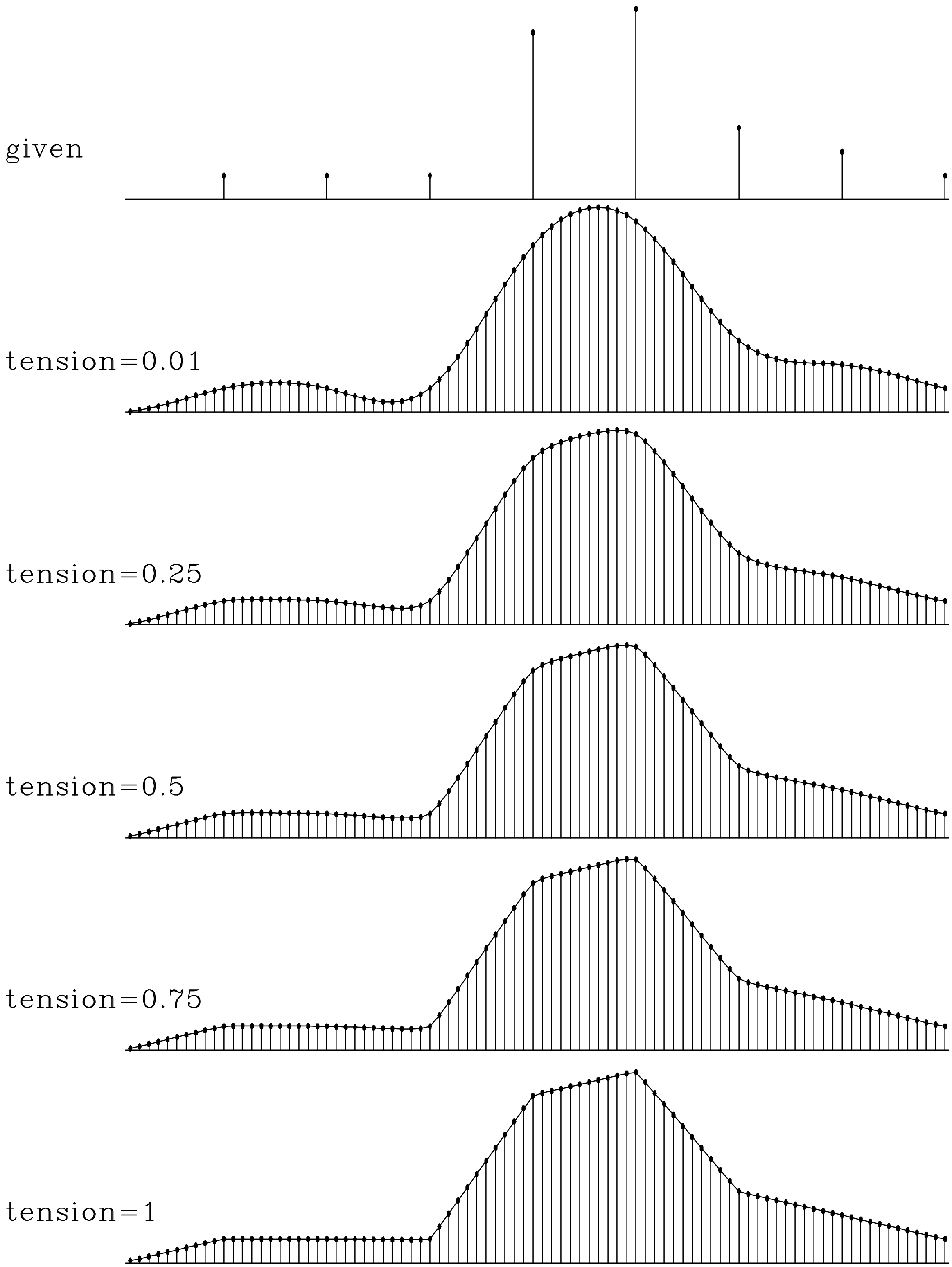

Figure 2: Interpolating a simple one-dimensional synthetic with recursive filter preconditioning for different values of the tension parameter t. The input data is shown on the top. The interpolation results range from a natural cubic spline interpolation for $t = 0$ to linear interpolation for $t = 1$. tension-int [ER,M]

of the Taylor expansion to match the desired spectrum of the operator around the zero spatial frequency. The matching conditions lead to the following set of coefficients for the squared Laplacian:

-1/60	2/5	7/30	2/5	-1/60
2/5	-14/15	-44/15	-14/15	2/5
7/30	-44/15	57/5	-44/15	7/30
2/5	-14/15	-44/15	-14/15	2/5
-1/60	2/5	7/30	2/5	-1/60

= 1/60

-1	24	14	24	-1
24	-56	-176	-56	24
14	-176	684	-176	14
24	-56	-176	-56	24
-1	24	14	24	-1

Laplacian representation with the same order of accuracy has the coefficients

-1/360	2/45	0	2/45	-1/360
2/45	-14/45	-4/5	-14/45	2/45
0	-4/5	41/10	-4/5	0
2/45	-14/45	-4/5	-14/45	2/45
-1/360	2/45	0	2/45	-1/360

= 1/360

-1	16	0	16	-1
16	-112	-288	-112	16
0	-288	1476	-288	0
16	-112	-288	-112	16
-1	16	0	16	-1

For the sake of simplicity, I assumed an equal physical spacing in x and y directions. The coefficients can be easily adjusted for anisotropic spacing. Figures 3 and 4 show the spectra of the finite-difference representations of operator (3) for the different values of the tension parameter. The finite-different spectra appear as fairly isotropic. They match the exact expressions at small frequencies.

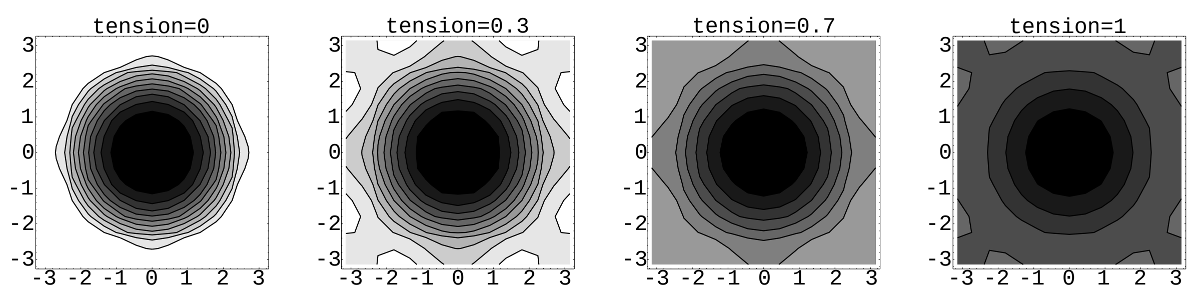

Figure 3: Spectra of the finite-difference splines-in-tension schemes for different values of the tension parameter (contour plots). tension-specc [CR]

Regarding the finite-difference operators as two-dimensional auto-correlations and applying the efficient Wilson-Burg method of spectral factorization (Claerbout, 1999; Sava et al., 1998), I obtain two-dimensional minimum-phase filters suitable for inverse filtering. The exact filters contain many coefficients, which rapidly decrease in magnitude at a distance from the first coefficient. For reasons of efficiency, it is advisable to restrict the shape of the filter so that it contains only the valuable coefficients. Keeping all the coefficients that are 1000 times smaller in magnitude than the leading coefficient creates a 53-point filter for $t = 0$ and a 35-point filter for $t = 1$, with intermediate filter lengths for intermediate values of t. When

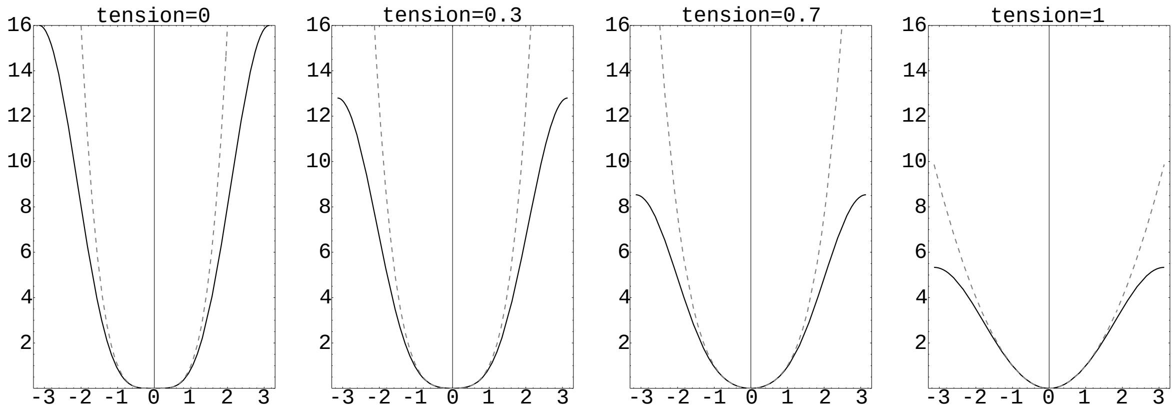

Figure 4: Spectra of the finite-difference splines-in-tension schemes for different values of the tension parameter (cross-section plots). Dashed lines show the exact spectra for continuous operators. tension-specp [CR]

the ratio is changed to 200, we obtain 25- and 16-point filters, respectively. The restricted filters don't factor the autocorrelation exactly, but provide an effective approximation of the exact factors. As outputs of the Wilson-Burg spectral factorization process, they obey the minimum-phase condition.

Figure 5 shows the two-dimensional filters for different values of t and illustrates inverse recursive filtering, which is the essence of the helix method (Claerbout, 1999, 1998a,b). The case of $t = 1$ leads to the filter known as *helix derivative* (Claerbout, 1999; Zhao, 1999). The filter values are spread mostly on two columns. The other boundary case of $t = 0$ leads to a three-column filter, which serves as the minimum-phase version of the Laplacian. As expected from the theory, the inverse impulse response of this filter is noticeably smoother and wider than the inverse response of the helix derivative. Filters corresponding to intermediate values of t exhibit intermediate properties. Theoretically, the inverse impulse response of the filter corresponds to the Green function of equation (3). The theoretical Green function for the case of $t = 1$ is

$$G = \frac{1}{2\pi} \ln r \; , \tag{5}$$

where r is the distance from the impulse: $r = \sqrt{(x-x_k)^2 + (y-y_k)}$. In the case of $t = 0$, the Green function is smoother at the origin:

$$G = \frac{1}{8\pi} r^2 \ln r \; . \tag{6}$$

The theoretical Green function expression for an arbitrary value of t is not known, but we can assume that its smoothness lies between the two boundary conditions.

In the next section, I illustrate an application of helical inverse filtering to a two-dimensional interpolation problem.

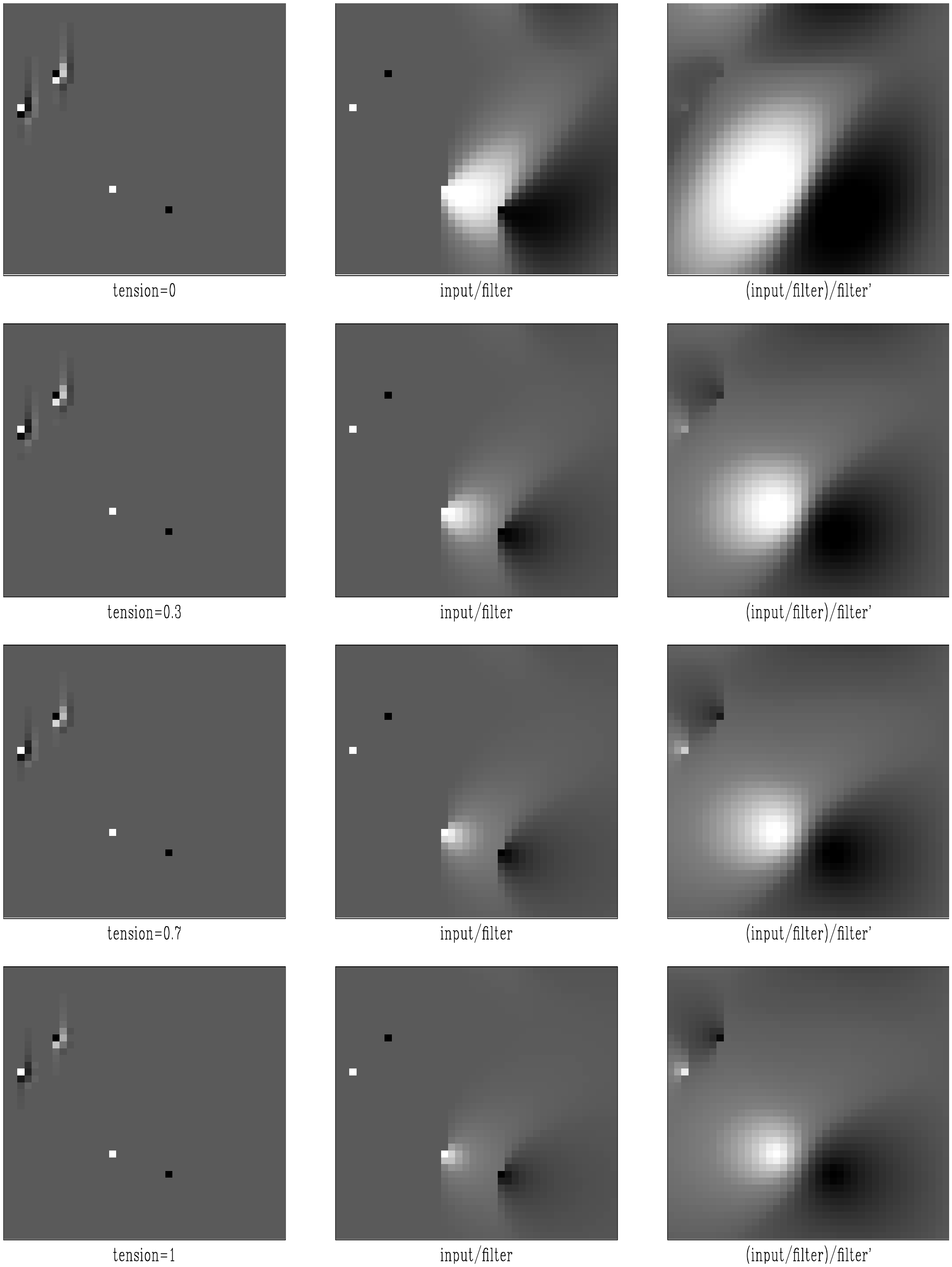

Figure 5: Inverse filtering with the tension filters. The left plots show the inputs composed of filters and spikes. Inverse filtering turns filters into impulses and turns spikes into inverse filter responses (middle plots). Adjoint filtering creates smooth isotropic shapes (right plots). The tension parameter takes values 0, 0.3, 0.7, and 1 (from top to bottom). tension-splin [ER,M]

INTERPOLATION EXAMPLE

I chose the familiar Galilee dataset (Fomel and Claerbout, 1995; Claerbout, 1999) for a simple interpolation illustration. The data was collected on a bottom sounding survey of the Sea of Galilee in Israel (Ben-Avraham et al., 1990). The data contain a number of noisy, erroneous and inconsistent measurements, which present a challenge for the traditional estimation methods. Addressing this challenge completely goes beyond the scope of this paper.

Figure 6 shows the data after a nearest-neighbor binning to a regular grid. The data was then passed to an interpolation program to fill the empty bins. The results (for different values of t) are shown in Figures 7 and 8. Interpolation with the minimum-phase Laplacian ($t = 0$) creates a relatively smooth interpolation surface but plants artificial little mountains around the edge of the sea. This effect is caused by large gradient changes and is similar to the sidelobe effect in the one-dimensional example (Figure 2). It is clearly seen in the cross-section plots in Figure 8. Interpolation with the helix derivative ($t = 1$) is free from the sidelobe artifacts, but it also produces an undesirable non-smooth behavior in the middle part of the image. As in the one-dimensional example, intermediate tension allows us to achieve a compromise: smooth interpolation in the middle and constrained behavior at the sides of the sea bottom.

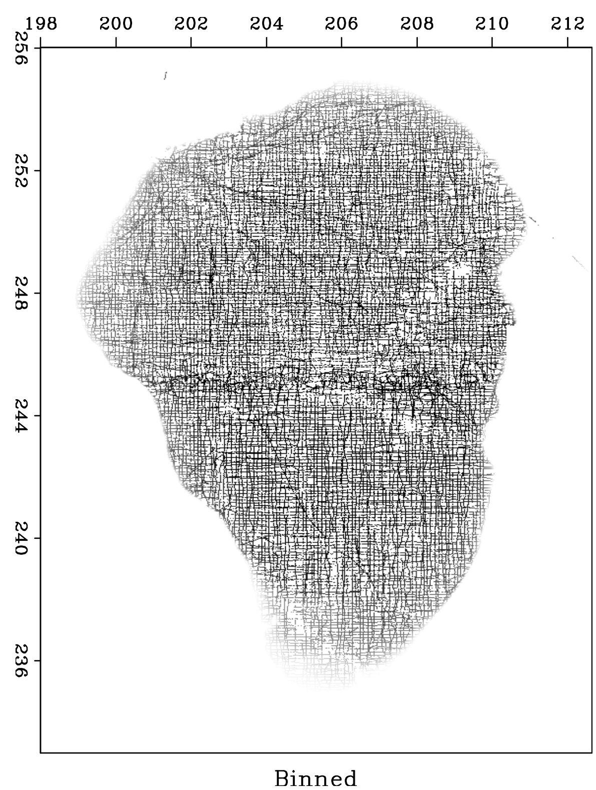

Figure 6: The Sea of Galilee dataset after a nearest-neighbor binning. The binned data is used as an input for the missing data interpolation program. tension-mesh [ER]

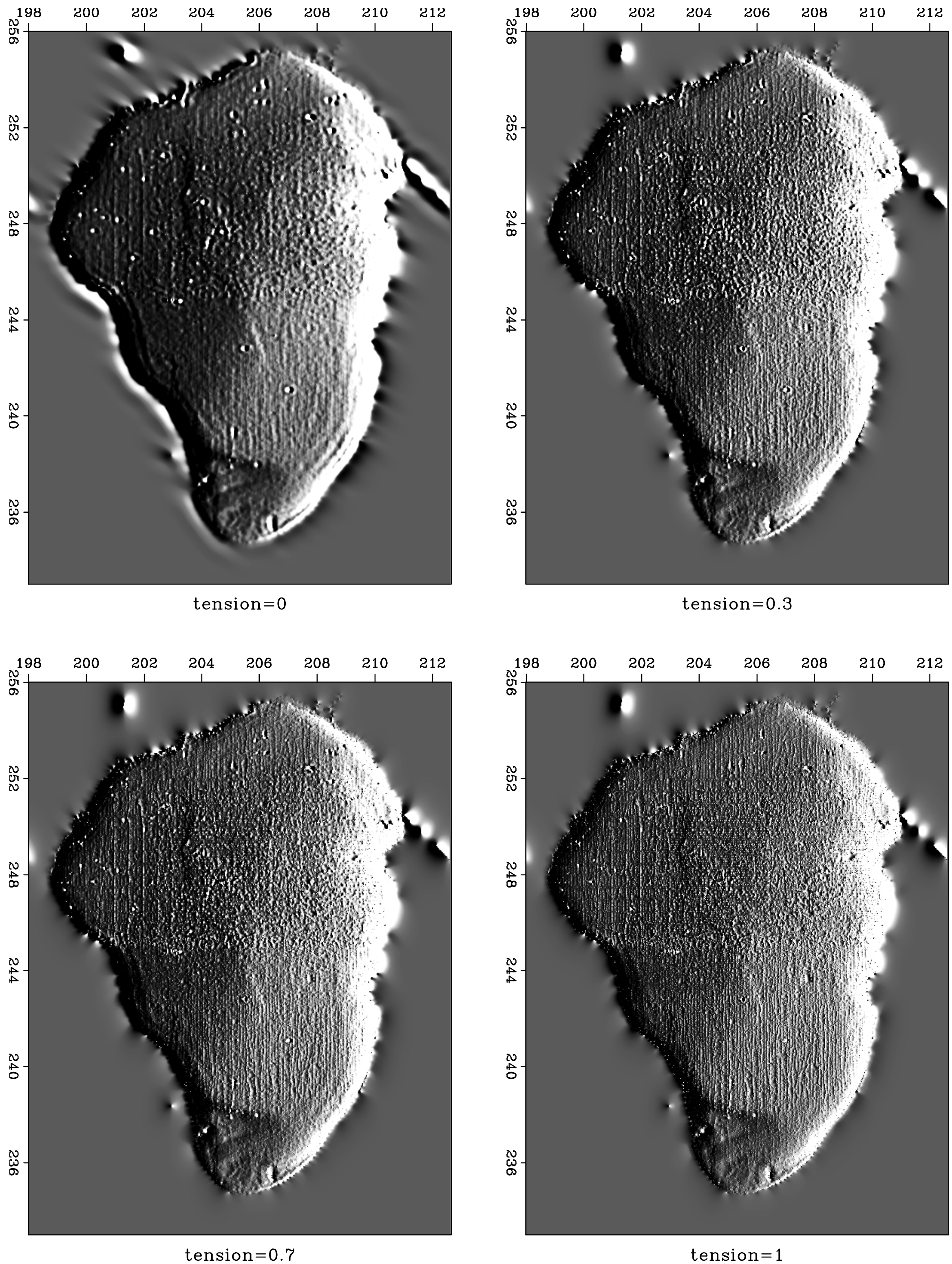

Figure 7: The Sea of Galilee dataset after missing data interpolation with helical preconditioning. Different plots correspond to different values of the tension parameter. An east-west derivative filter was applied to illuminate the surface. tension-gal [ER,M]

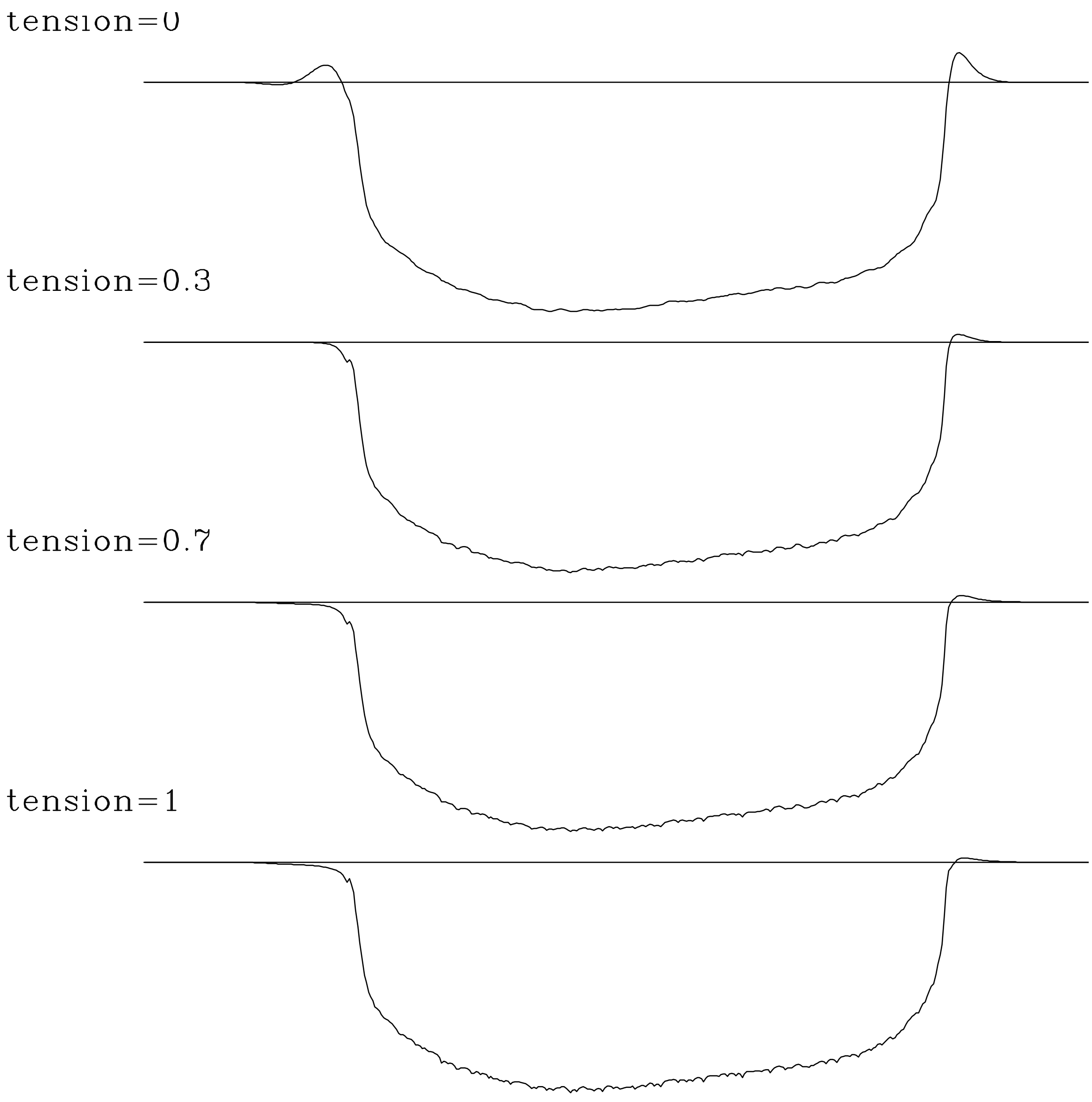

Figure 8: Cross-sections of the Sea of Galilee dataset after missing data interpolation with helical preconditioning. Different plots correspond to different values of the tension parameter. tension-cross [ER]

CONCLUSIONS

Splines in tension represent an approach to constrained interpolation of smooth surfaces. The constraint is embedded in a user-specified tension parameter. The two boundary values of tension correspond to cubic and linear interpolation.

By applying the method of spectral factorization on a helix, I have been able to define a family of two-dimensional minimum-phase filters, which correspond to the spline interpolation problem with different values of tension. These filters contribute to our collection of useful helical filters. They can be used for preconditioning interpolation problems with smooth surfaces and, in general, for preconditioning geophysical estimation problems with smooth models.

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