Stanford Exploration Project, Report SERGEY, November 9, 2000, pages 337–??

Short Note

Traveltime computation with the linearized eikonal equation

Sergey Fomel[1]

INTRODUCTION

Traveltime computation is an important part of seismic imaging algorithms. Conventional implementations of Kirchhoff migration require precomputing traveltime tables or include traveltime calculation in the innermost computational loop . The cost of traveltime computations is especially noticeable in the case of 3-D prestack imaging where the input data size increases the level of nesting in computational loops.

The eikonal differential equation is the basic mathematical model, describing the traveltime (eikonal) propagation in a given velocity model. Finite-difference solutions of the eikonal equation have been recognized as one of the most efficient means of traveltime computations (Vidale, 1990; van Trier and Symes, 1991; Popovici, 1991). The major advantages of this method in comparison with ray tracing techniques include an ability to work on regular model grids, a complete coverage of the receiver space, and a fair numerical robustness. The most common implementations of the finite-difference eikonal equation compute the *first-arrival* traveltimes, though frequency-dependent enhancements (Biondi, 1992; Nichols, 1994) can extend the method to computing the most energetic arrivals. The major numerical complexity of the finite-difference eikonal computations arises from the fundamental non-linearity of the eikonal equation. The numerical complexity is related not only to the direct cost of the computation, but also to the accuracy and stability of finite-difference schemes.

It is important to note that the current practice of seismic imaging is not limited to a single migration. Moreover, it is repeated migrations, with velocity analysis and refinement of the velocity model at each step, that take most of the computational effort. When the changes in the velocity model at each step are small compared to the initial model, it is appropriate to linearize the eikonal equation with respect to the slowness and traveltime perturbations. Mathematically, the linearized eikonal equation corresponds precisely to the linearization assumption, commonly used in traveltime tomography.

In this paper, I propose an algorithm of finite-difference traveltime computations, based on an iterative linearization of the eikonal equation. The algorithm takes advantage of an implicit

[1]**email:** sergey@sep.stanford.edu

finite-difference scheme with superior stability and accuracy properties. I test the algorithm on a simple synthetic example and discuss its possible applications in residual traveltime computation, interpolation, and tomography.

THE LINEARIZED EIKONAL EQUATION

The eikonal equation, describing the traveltime propagation in an isotropic medium, has the form

$$(\nabla\tau)^2 = n^2(x,y,z)\;, \tag{1}$$

where $\tau(x,y,z)$ is the traveltime (eikonal) from the source to the point with the coordinates (x,y,z), and n is the slowness at that point (the velocity v equals $1/n$.) In Appendix A, I review a basic derivation of the eikonal and transport equations. To formulate a well-posed initial-value problem on equation (1), it is sufficient to specify τ at some closed surface and to choose one of the two branches of the solution (the wave going from or to the source.)

Equation (1) is nonlinear. The nonlinearity is essential for producing multiple branches of the solution. Multi-valued eikonal solutions can include different types of waves (direct, reflected, diffracted, head, etc.) as well as different branches of caustics. To linearize equation (1), we need to assume that an initial estimate τ_0 of the eikonal τ is available. The traveltime τ_0 corresponds to some slowness n_0, which can be computed from equation (1) as

$$n_0 = |\nabla\tau_0|\;. \tag{2}$$

Let us denote the residual traveltime $\tau-\tau_0$ by τ_1 and the residual slowness $n-n_0$ by n_1. With these definitions, we can rewrite equation (1) in the form

$$(\nabla\tau_0+\nabla\tau_1)^2 = (\nabla\tau_0)^2+2\,\nabla\tau_0\cdot\nabla\tau_1+(\nabla\tau_1)^2 = (n_0+n_1)^2 = n_0^2+2\,n_0\,n_1+n_1^2\;, \tag{3}$$

or, taking into account equality (2),

$$2\,\nabla\tau_0\cdot\nabla\tau_1+(\nabla\tau_1)^2 = 2\,n_0\,n_1+n_1^2\;. \tag{4}$$

Neglecting the squared terms, we arrive at the equation

$$\nabla\tau_0\cdot\nabla\tau_1 = n_0\,n_1\;, \tag{5}$$

which is the linearized version of the eikonal equation (1). The accuracy of the linearization depends on the relative ratio of the slowness perturbation n_1 and the true slowness model n. Though it is difficult to give a quantitative estimate, the ratio of 10% is generally assumed to be a safe upper bound.

The intimate connection of the linearized eikonal equation and traveltime tomography is discussed in Appendix B.

ALGORITHM

Linearization of the eikonal equation suggests the following algorithm of traveltime computation:

1. Start with an initial traveltime field τ_0. The initial traveltime may be the result of a previous computation or (for simple models) the result of an approximate analytic evaluation.

2. Compute the finite-difference gradient $\nabla\tau_0$ and the corresponding slowness model n_0 with equation (2).

3. Compute the slowness perturbation n_1 as the difference between the true slowness model n and n_0. Exit the computation if the perturbation is smaller than the desired accuracy.

4. Solve numerically equation (5) for the traveltime perturbation τ_1.

5. Update the traveltime field τ_0 by adding τ_1 to it.

6. Repeat the loop.

Equation (5) can be solved numerically with a simple explicit upwind finite-difference method. For a numerical test of the algorithm, I chose to solve it by a less efficient but more robust "brute-force" implicit method, applying one of the generic linear solvers. The gradient operator ∇ was computed with centered finite differences. The implicit method is unconditionally stable. Its accuracy corresponds to the accuracy of the finite-difference gradient approximation. I found it helpful to regularize the linear solver with a smoothing preconditioner. The regularization assures that the traveltime remains a smooth function of the spatial coordinates.

An important feature of the suggested algorithm is that it does not require an iterative solver to iterate until the full convergence. A few iteration steps of the estimation process can be interlaced with re-linearization in the main loop of the algorithm.

Theoretically, a global convergence of the described procedure cannot be guaranteed for all cases. However, I observed a stable convergence in the preliminary numerical tests.

NUMERICAL TEST

For the first numerical test, I used a model with a smooth anomaly inside a constant slowness background. The initial traveltime was computed analytically, using the background slowness. The result of the computation is shown in Figure 1. The computation involved 3 re-linearization cycles with 10 linear inversion iterations in each cycle.

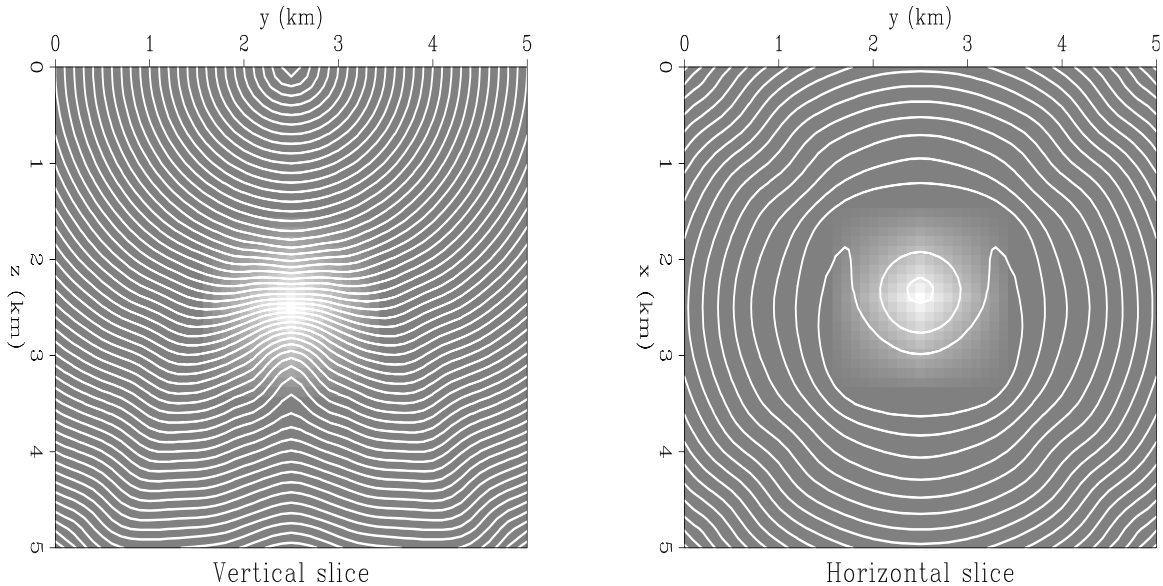

Figure 1: The traveltime contours for a smooth anomaly, computed by the linearized eikonal solver. The background slowness is 1 s/km. The maximum anomaly slowness is 2.3 s/km. The wave source is in the middle of the top plane of the model. The left plot shows a vertical slice. The right plot shows a horizontal slice, taken at 2.5 km depth. lineiko-linear-0.01 [ER]

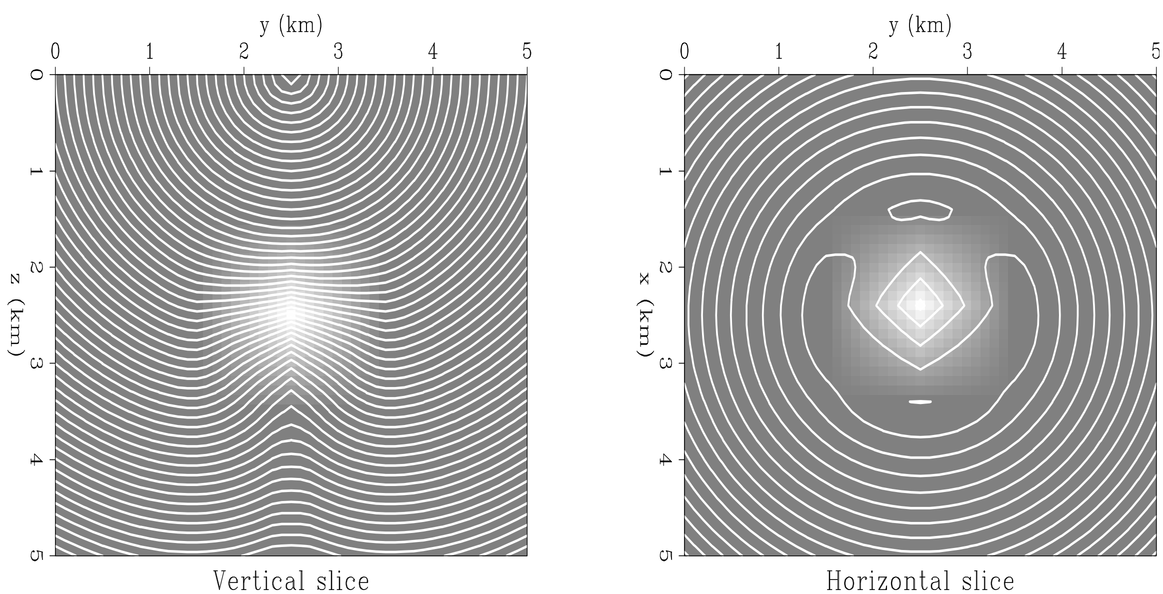

Figure 2: The traveltime contours for a smooth anomaly, computed by the exact eikonal solver. The input and plotting parameters are the same as in the preceeding figure. lineiko-mihai-0.01 [ER]

The result shows the expected behavior of the wavefronts. It agrees with the result of a direct eikonal computation, shown in Figure 2. The direct computation was done with Mihai Popovici's TTGES eikonal solver, which has outstanding efficiency and stability properties. Obviously, more tests are required to evaluate the comparative performance of the algorithm and the limits of its practical applicability. The discussion section contains some speculations about the perspective usage of the linearized algorithm.

DISCUSSION

Although the first numerical experiments have been too incomplete for drawing any solid conclusions, it is interesting to discuss the possible applications of the linearized eikonal.

Multi-valued traveltimes Conventional eikonal solvers usually force the choice of a particular branch of the multi-valued traveltime, most commonly the first-arrival branch. However, in some cases other branches may in fact be more useful for imaging or velocity estimation (Gray and May, 1994). When the linearization assumption is correct, the linearized eikonal should follow the branch of the initial traveltime. This branch does not have to be the first arrival. It can correspond to any other arrival, such as reflected waves or multiple reflections.

Spherical Coordinates Though the eikonal equation itself does not favor any particular direction, its solution for the case of a point source lands more naturally into a spherical coordinate system. van Trier and Symes (1991), Popovici (1991), Fowler (1994), and Schneider (1995) presented upwind finite-difference eikonal schemes based on a spherical computational grid. To use the linearized equation (5) on such a grid, it is necessary to rewrite the gradient operator in the spherical coordinates, as follows:

$$\nabla\tau = \left\{\frac{\partial\tau}{\partial r}\,,\ \frac{1}{r}\,\frac{\partial\tau}{\partial\theta}\,,\ \frac{1}{r\sin^2\theta}\,\frac{\partial\tau}{\partial\phi}\right\}\ .$$

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Interpolation One of the most natural applications for the linearized eikonal is interpolation of traveltimes. Interpolating regularly gridded input (such as subsampled traveltime tables) reduces to *masked* inversion of equation (5). Interpolating irregular input (such as the result of a ray tracing procedure) reduces to *regularized* inversion. In both cases, a simpler way of traveltime binning would be required to initiate the linearization.

Tomography Tomographic velocity estimation is possible when the input traveltime data corresponds to a collection of sources. In this case, we can reduce the linearized traveltime inversion to the system of equations

$$n_0^{(1)} \cdot \nabla\tau_1^{(1)} = n_0^{(2)} \cdot \nabla\tau_1^{(2)} = \cdots = s_1\ . \tag{6}$$

Here $\tau_1^{(i)}$ stands for the traveltime from source i. Equations (6) are additionally constrained by the known values of the traveltime fields at the receiver locations.

Amplitudes The amplitude transport equation, briefly reviewed in Appendix A, has the form (A-4). Introducing the logarithmic amplitude $J = -ln(A/A_0)$, where A_0 is the constant reference, we can rewrite this equation in the form

$$2\nabla\tau \cdot \nabla J = \Delta\tau \; . \tag{7}$$

The left-hand side of equation (7) has exactly the same form as the left-hand side part of the linearized eikonal equation (5). This suggests reusing the traveltime computation scheme for amplitude calculations. The amplitude transport equation is linear. However, it explicitly depends on the traveltime. Therefore, the amplitude computation needs to be coupled with the eikonal solution.

Anisotropy In a recent paper, Alkhalifah (1997) proposed a simple eikonal-type equation for seismic imaging in vertically transversally-isotropic media. Alkhalifah's equation should be suitable for linearization, either in the normal moveout velocity V_{NMO} or in the dimensionless anisotropy parameter η. This untested opportunity looks promising because of the validity of the weak anisotropy assumption in many regions of the world.

CONCLUSIONS

I have presented a finite-difference method of traveltime computations, based on the linearized eikonal equation. Preliminary numerical experiments show that the method is as simple and robust as can be expected from the theory. The required assumption is that a reasonable estimate of the traveltime is available prior to linearization. Such an estimate may result from the computation in a different velocity model, with a different method (e.g., ray tracing), or by an analytic evaluation.

In the situations where the underlying assumption is valid, the linearized approach may allow us

- to employ unconditionally stable implicit finite-difference schemes with an easy control of the numerical stability,
- to parallelize the essential parts of the algorithm with minimum effort,
- to compute branches of the multi-valued traveltime other than the first arrival,
- to connect traveltime computations with tomographic velocity estimation,
- to couple traveltime and amplitude computations.

Future research is necessary to confirm these expectations.

ACKNOWLEDGMENTS

I thank Biondo Biondi for interesting discussions on the traveltime computation problem and its applications. The TTGES eikonal solver that I used for testing was kindly made available to SEP by Mihai Popovici of 3DGeo.

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APPENDIX A

A SIMPLE DERIVATION OF THE EIKONAL AND TRANSPORT EQUATIONS

In this Appendix, I remind the reader how the eikonal equation is derived from the wave equation. The derivation is classic and can be found in many popular textbooks. See, for example, (Červeny et al., 1977).

Starting from the wave equation,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = n^2(x,y,z)\,\frac{\partial^2 P}{\partial t^2}\;, \tag{A-1}$$

we introduce a trial solution of the form

$$P(x,y,t) = A(x,y,z)\, f(t-\tau(x,y,z))\;, \tag{A-2}$$

where τ is the eikonal, and A is the wave amplitude. The waveform function f is assumed to be a high frequency (discontinuous) signal. Substituting solution (A-2) into equation (A-1), we arrive at the constraint

$$\Delta A\, f - 2\nabla A\cdot\nabla\tau\, f' - A\Delta\tau\, f' + A\,(\nabla\tau)^2\, f'' = n^2 A f''\;. \tag{A-3}$$

Here $\Delta \equiv \nabla^2$ denotes the Laplacian operator. Equation (A-3) is as exact as the initial wave equation (A-1) and generally difficult to satisfy. However, we can try to satisfy it asymptotically, considering each of the high-frequency asymptotic components separately. The leading-order component corresponds to the second derivative of the wavelet f''. Isolating this component, we find that it is satisfied if and only if the traveltime function $\tau(x,y,z)$ satisfies the eikonal equation (1).

The next asymptotic order corresponds to the first derivative f'. It leads to the *amplitude transport equation*

$$2\nabla A\cdot\nabla\tau + A\Delta\tau = 0\;. \tag{A-4}$$

The amplitude, defined by equation (A-4), is often referred to as the amplitude of the zero-order term in the ray series. A series expansion of the function f in high-frequency asymptotic components produces recursive differential equations for the terms of higher order. In practice, equation (A-4) is sufficiently accurate for describing the major amplitude trends in most of the cases. It fails, however, in some special cases, such as caustics and diffraction.

APPENDIX B

CONNECTION OF THE LINEARIZED EIKONAL EQUATION AND TRAVELTIME TOMOGRAPHY

The eikonal equation (1) can be rewritten in the form

$$\mathbf{n}\cdot\nabla\tau = n\;, \tag{B-1}$$

where **n** is the unit vector, pointing in the traveltime gradient direction. The integral solution of equation (B-1) takes the form

$$\tau = \int_{\Gamma(\mathbf{n})} n dl \; , \tag{B-2}$$

which states that *the traveltime* τ *can be computed by integrating the slowness* n *along the ray* $\Gamma(\mathbf{n})$, *tangent at every point to the gradient direction* $\mathbf{n}$.

Similarly, we can rewrite the linearized eikonal equation (5) in the form

$$\mathbf{n_0} \cdot \nabla \tau_1 = n_1 \; , \tag{B-3}$$

where $\mathbf{n_0}$ is the unit vector, pointing in gradient direction for the initial traveltime τ_0. The integral solution of equation (B-3) takes the form

$$\tau_1 = \int_{\Gamma(\mathbf{n_0})} n_1 dl \; , \tag{B-4}$$

which states that *the traveltime perturbation* τ_1 *can be computed by integrating the slowness perturbation* n_1 *along the ray* $\Gamma(\mathbf{n_0})$, *defined by the initial slowness model* n_0. This is exactly the basic principle of traveltime tomography.

I have borrowed this proof from Lavrentiev et al. (1970), who used linearization of the eikonal equation as the theoretical basis for traveltime inversion.