*Stanford Exploration Project, Report SERGEY, November 9, 2000, pages 671–**??***

Residual migration in VTI media using anisotropy continuation

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ABSTRACT

We introduce anisotropy continuation as a process which relates changes in seismic images to perturbations in the anisotropic medium parameters. This process is constrained by two kinematic equations, one for perturbations in the normal-moveout (NMO) velocity and the other for perturbations in the dimensionless anisotropy parameter η. We consider separately the case of post-stack migration and show that the kinematic equations in this case can be solved explicitly by converting them to ordinary differential equations by the method of characteristics. Comparing the results of kinematic computations with synthetic numerical experiments confirms the theoretical accuracy of the method.

INTRODUCTION

A well-known paradox in seismic imaging is that the detailed information about the subsurface velocity is required before a reliable image can be obtained. In practice, this paradox leads to an iterative approach to building the image. It looks attractive to relate small changes in velocity parameters to inexpensive operators perturbing the image. This approach has been long known as *residual migration*. A classic result is the theory of residual post-stack migration (Rothman et al., 1985), extended to the prestack case by Etgen (1990). In a recent paper, Fomel (1996) introduced the concept of velocity continuation as the continuous model of the residual migration process. All these results were based on the assumption of the isotropic velocity model.

Recently, emphasis has been put on the importance of considering anisotropy and its influence on data. Alkhalifah and Tsvankin (1995) demonstrated that, for TI media with vertical symmetry axis (VTI media), just two parameters are sufficient for performing all time-related processing, such as normal moveout (NMO) correction (including non-hyperbolic moveout correction, if necessary), dip-moveout (DMO) correction, and prestack and poststack time migration. One of these two parameters, the short-spread NMO velocity for a horizontal reflector, is given by

$$v_{\mathrm{nmo}}(0) = v_v \sqrt{1+2\delta}\,, \tag{1}$$

where v_v is the vertical P-wave velocity, and δ is one of Thomsen's anisotropy parameters

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(Thomsen, 1986). Taking v_h to be the P-wave velocity in the horizontal direction, the other parameter, η, is given by

$$\eta \equiv 0.5(\frac{v_h^2}{v_{\rm nmo}^2(0)} - 1) = \frac{\epsilon - \delta}{1+2\delta}, \tag{2}$$

where ϵ is another of Thomsen's parameters. In addition, Alkhalifah (1997) has showed that the dependency on just two parameters becomes exact when the vertical shear wave velocity (V_{S0}) is set to zero. Setting $V_{S0} = 0$ leads to remarkably accurate kinematic representations. It also results in much simpler equations that describe P-wave propagation in VTI media. Throughout this paper, we use these simplified, yet accurate, equations, based on setting $V_{S0} = 0$, to derive the continuation equations. Because we are only considering time sections, and for the sake of simplicity, we denote $v_{\rm nmo}$ by v. Thus, time processing in VTI media, depends on two parameters (v and η), whereas in isotropic media only v counts.

In this paper, we generalize the velocity continuation concept to handle VTI media. We define anisotropy continuation as the process of seismic image perturbation when either v or η change as migration parameters. This approach is especially attractive, when the initial image is obtained with isotropic migration (that is with $\eta = 0$). In this case, anisotropy continuation is equivalent to introducing anisotropy in the model without the need of repeating the migration step.

For the sake of simplicity, we start from the post-stack case and purely kinematic description. We define however the guidelines for moving to the more complicated and interesting cases of prestack migration and dynamic equations. The results are preliminary, but they open promising opportunities for seismic data processing in presence of anisotropy.

THE GENERAL THEORY

In the case of zero-offset reflection, the ray travel distance, l, from the source to the reflection point is related to the two-way zero-offset time, t, by the simple equation

$$l = v_g t, \tag{3}$$

where v_g is the half of the group velocity, best expressed in terms of its components, as follows:

$$v_g = \sqrt{v_{gx}^2 + v_v^2 v_{g\tau}^2}.$$

Here v_{gx} denotes the horizontal component of group velocity, v_v is the vertical P-wave velocity, and $v_{g\tau}$ is the v_v-normalized vertical component of the group velocity. Under the assumption of zero shear-wave velocity in VTI media, these components have the following analytic expressions:

$$v_{gx} = \frac{v^2\, p_x \left(-1 - 2\,\eta + 2\,\eta\, {p_\tau}^2\right)}{-2 + v^2\,(1+2\,\eta)\,{p_x}^2 + {p_\tau}^2}, \tag{4}$$

and

$$v_{g\tau} = \frac{\left(1-2\,v^2\,\eta\,{p_x}^2\right)\,p_\tau}{2-v^2\,(1+2\,\eta)\,{p_x}^2+{p_\tau}^2}, \tag{5}$$

where p_x is the horizontal component of slowness, and p_τ is the normalized (again by the vertical P-wave velocity v_v) vertical component of slowness. The two components of the slowness vector are related by the following eikonal-type equation (Alkhalifah, 1997):

$$p_\tau = \sqrt{1-\frac{v^2\,{p_x}^2}{1-2\,v^2\,\eta\,{p_x}^2}}. \tag{6}$$

Equation (6) corresponds to a normalized version of the dispersion relation in VTI media.

If we consider v and η as imaging parameters (migration velocity and migration anisotropy coefficient), the ray length l can be taken as an imaging invariant. This implies that the partial derivatives of l with respect to the imaging parameters are zero. Therefore,

$$\frac{\partial l}{\partial v} = \frac{\partial v_g}{\partial v} t + v_g \frac{\partial t}{\partial v} = 0, \tag{7}$$

and

$$\frac{\partial l}{\partial \eta} = \frac{\partial v_g}{\partial \eta} t + v_g \frac{\partial t}{\partial \eta} = 0. \tag{8}$$

Applying the simple chain rule to equations (7) and (8), we obtain

$$\frac{\partial t}{\partial v} = \frac{\partial t}{\partial \tau}\frac{\partial \tau}{\partial v}, \;\; \frac{\partial t}{\partial \eta} = \frac{\partial t}{\partial \tau}\frac{\partial \tau}{\partial \eta}, \tag{9}$$

where $\frac{\partial t}{\partial \tau} = -p_\tau$, and the two-way vertical traveltime is given by

$$\tau = v_{g\tau} t.$$

Combining equations (7-9) eliminates the two-way zero-offset time t, which leads to the equations

$$\frac{\partial \tau}{\partial v} = \frac{\partial v_g}{\partial v} \frac{\tau}{p_\tau\, v_{g\tau} v_g}, \tag{10}$$

and

$$\frac{\partial \tau}{\partial \eta} = \frac{\partial v_g}{\partial \eta} \frac{\tau}{p_\tau\, v_{g\tau} v_g}. \tag{11}$$

After some tedious algebraic manipulation, we can transform equations (5) and (6) to the general form

$$\frac{\partial \tau}{\partial v} = \tau\, F_v\,(p_x, v, \eta), \tag{12}$$

and

$$\frac{\partial \tau}{\partial \eta} = \tau\, F_\eta\,(p_x, v, \eta)\,. \tag{13}$$

Since the residual migration is applied to migrated data, with the time axis given by τ and the reflection slope given by $\frac{\partial \tau}{\partial x}$, instead of t and p_x, respectively, we need to eliminate p_x from equations (12) and (13). This task can be achieved with the help of the following explicit relation, derived in Appendix A,

$$p_x^2 = \frac{2\,{\tau_x}^2}{1 + v^2\,(1 + 2\,\eta)\,{\tau_x}^2 + S}, \tag{14}$$

where τ_x=$\frac{\partial \tau}{\partial x}$, and

$$S = \sqrt{-8\,v^2\,\eta\,{\tau_x}^2 + \left(1 + v^2\,(1 + 2\,\eta)\,{\tau_x}^2\right)^2}.$$

Inserting equation (14) into equations (12) and (13) yields exact, yet complicated equations, describing the continuation process for v and η. In summary, these equations have the form

$$\frac{\partial \tau}{\partial v} = \tau\, f_v\left(\frac{\partial \tau}{\partial x}, v, \eta\right) \tag{15}$$

and

$$\frac{\partial \tau}{\partial \eta} = \tau\, f_\eta\left(\frac{\partial \tau}{\partial x}, v, \eta\right). \tag{16}$$

Equations of the form (15) and (16) contain all the necessary information about the kinematic laws of anisotropy continuation in the domain of zero-offset migration.

Linearization

A useful approximation of equations (15) and (16) can be obtained by simply setting η equal to zero in the right-hand side of the equations. Under this approximation, equation (15) leads to the kinematic velocity-continuation equation for elliptically anisotropic media, which has the following relatively simple form:

$$\frac{\partial \tau}{\partial v} = \frac{v\,\tau\,\left(2\,v^2 - {v_v}^2\right)\,{\tau_x}^2\,\left(1 + v^2\,{\tau_x}^2\right)}{{v_v}^2 + v^4\,{\tau_x}^2}. \tag{17}$$

It is interesting to note that setting $v = v_v$, yields Fomel's expression for isotropic media (Fomel, 1996) given by

$$\frac{\partial \tau}{\partial v} = v\,\tau\,{\tau_x}^2. \tag{18}$$

Alkhalifah and Tsvankin (1995) have shown that time-domain processing algorithms for elliptically anisotropic media should be the same as those for isotropic media. However, in anisotropic continuation, elliptical anisotropy and isotropy differ by a vertical scaling factor that is related to the difference between the vertical and NMO velocities. In isotropic media, when velocity is continued, both the vertical and NMO velocities (which are the same) are continued together, whereas in anisotropic media (including elliptically anisotropic) the NMO-velocity continuation is separated from the vertical-velocity one, and equation (19) corresponds to continuation only in the NMO velocity. This also implies that equation (19) is more flexible than equation (18), in that we can isolate the vertical-velocity continuation (a parameter that is usually ambiguous in surface processing) from the rest of the continuation process. Using $\tau = \frac{z}{v_v}$, where z is depth, we immediately obtain the equation

$$\frac{\partial \tau}{\partial v_v} = -\frac{\tau}{v_v},$$

which represents the vertical-velocity continuation.

Setting $\eta = 0$ and $v = v_v$ in equation (16) leads to the following kinematic equation for η-continuation:

$$\frac{\partial \tau}{\partial \eta} = \frac{\tau v^4 {\tau_x}^4}{1 + v^2 {\tau_x}^2}. \tag{19}$$

We include more discussion about different aspects of linearization in Appendix B. The next section presents the analytic solution of equation (19). Later in this paper, we compare the analytic solution with a numerical synthetic example.

ORDINARY DIFFERENTIAL EQUATION REPRESENTATION: ANISOTROPY RAYS

According to the classic rules of mathematical physics, the solution of the kinematic equations (15) and (16) can be obtained by solving the following system of ordinary differential equations:

$$\frac{dx}{dm} = -\tau \frac{\partial f_m}{\partial \tau_x} \quad , \quad \frac{d\tau}{dm} = -\tau \tau_x \frac{\partial f_m}{\partial \tau_x} + \tau_m,$$
$$\frac{d\tau_m}{dm} = \tau \frac{\partial f_m}{\partial m} + \tau_m f_m \quad , \quad \frac{d\tau_x}{dm} = \tau_x f_m \, . \tag{20}$$

Here m stands for either v or η, $\tau_x = \frac{\partial \tau}{\partial x}$, $f = \frac{\partial \tau}{\partial m}$. To trace the v and η rays, we must first identify the initial values x_0, τ_0, τ_{x0}, and τ_{m0} from the boundary conditions. The variables x_0 and τ_0 describe the initial position of a reflector in a time-migrated section, τ_{x0} describes its migrated slope, and τ_{m0} is simply $f(m_0, \tau_0, \tau_{x0})$.

Using the exact kinematic expressions for f results in rather complicated representations of the ordinary differential equations. The linearized expressions, on the other hand, are simple and allow for a straightforward analytical solution.

From kinematics to dynamics

The kinematic η-continuation equation (19) corresponds to the following linear fourth-order dynamic equation

$$\frac{\partial^4 P}{\partial t^3 \, \partial \eta} + v^2 \frac{\partial^4 P}{\partial x^2 \, \partial t \, \partial \eta} + t v^4 \frac{\partial^4 P}{\partial x^4} = 0 \,, \qquad (21)$$

where the t coordinate refers to the vertical traveltime τ, and $P(t,x,\eta)$ is the migrated image, parameterized in the anisotropy parameter η. To find the correspondence between equations (19) and (21), it is sufficient to apply a ray-theoretical model of the image

$$P(t,x,\eta) = A(x,\eta) \, f(t - \tau(x,\eta)) \qquad (22)$$

as a trial solution to (21). Here the surface $t = \tau(x,\eta)$ is the anisotropy continuation "wavefront" - the image of a reflector for the corresponding value of η, and the function A is the amplitude. Substituting the trial solution into the partial differential equation (21) and considering only the terms with the highest asymptotic order (those containing the fourth-order derivative of the wavelet f), we arrive at the kinematic equation (19). The next asymptotic order (the third-order derivatives of f) gives us the linear partial differential equation of the amplitude transport, as follows:

$$\left(1 + v^2 \tau_x^2\right) \frac{\partial A}{\partial \eta} + 2 v^2 \tau_x \left(\tau_\eta - 2 v^2 \tau \tau_x^2\right) \frac{\partial A}{\partial x} + v^2 A \left(2 \tau_x \tau_{x\eta} + \tau_\eta \tau_{xx} - 6 v^2 \tau \tau_x^2 \tau_{xx}\right) = 0 \,. \qquad (23)$$

We can see that when the reflector is flat ($\tau_x = 0$ and $\tau_{xx} = 0$), equation (23) reduces to the equality

$$\frac{\partial A}{\partial \eta} = 0 \,,$$

and the amplitude remains unchanged for different η. This is of course a reasonable behavior in the case of a flat reflector. It doesn't guarantee though that the amplitudes, defined by (23), behave equally well for dipping and curved reflectors. The amplitude behavior may be altered by adding low-order terms to equation (21). According to the ray theory, such terms can influence the amplitude behavior, but do not change the kinematics of the wave propagation.

An appropriate initial-value condition for equation (21) is the result of isotropic migration that corresponds to the $\eta = 0$ section in the (t,x,η) domain. In practice, the initial-value problem can be solved by a finite-difference technique.

SYNTHETIC TEST

Residual post-stack migration operators can be obtained by generating synthetic data for a model consisting of diffractors for given medium parameters and then migrating the same data with different medium parameters. For example, we can generate diffractions for isotropic media and migrate those diffractions using an anisotropic migration. The resultant operator

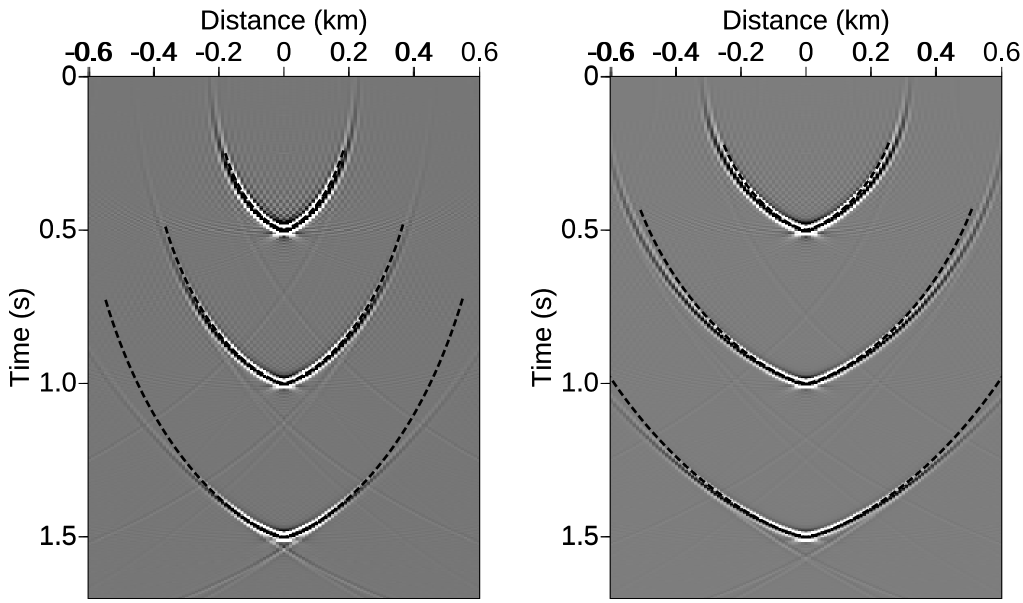

Figure 1: Residual post-stack migration operators calculated by solving equation (19), overlaid above synthetic operators. The synthetic operators are obtained by applying TI post-stack migration with $\eta = 0.1$ (left) and $\eta = 0.2$ (right) to three diffractions generated considering isotropic media. The NMO velocity for the modeling and migration is 2.0 km/s. anico-impres [NR]

describes the correction needed to transform an isotropically migrated section to an anisotropic one, that is the *anisotropic residual migration* operator.

Figure 1 shows such synthetic operators overlaid by kinematically calculated operators that were computed with the help of equation (19) (the continuation equations for the case of small η). Despite the inherent accuracy of the synthetic operators, they suffer from the lack of aperture in modeling the diffractions, and therefore, beyond a certain angle the operators vanish. The agreement between the synthetic and calculated operators for small angles, especially for the $\eta = 0.1$ case, promises reasonable results in future dynamic implementations.

CONCLUSIONS

We have extended the concept of velocity continuation in isotropic media to continuations in both the NMO velocity and the anisotropy parameter η for VTI media. Despite the fact that we have considered the simple case of post-stack migration separately, the exact kinematic equations describing the continuation process are anything but simple. However, useful insights into this problem are deduced from linearized approximations of the continuation equations. These insights include the following conclusions:

- The leading order behavior of the velocity continuation is proportional to τ_x^2, which corresponds to small or moderate dips.
- The leading order behavior of the η continuation is proportional to τ_x^4, which corresponds to moderate or steep dips.
- Both leading terms are independent of the strength of anisotropy (η).

In practical applications, the initial migrated section is obtained by isotropic migration, and, therefore, the residual process is used to correct for anisotropy. Setting $\eta = 0$ in the continuation equations for this type of an application is a reasonable approximation, given that η=0 is the starting point and we consider only weak to moderate degrees of anisotropy ($\eta \approx 0.1$). Numerical experiments with synthetically generated operators confirm this conclusion. Future directions of research will focus on a practical implementation of anisotropy continuation as well as on extending the theory to the case of residual prestack migration.

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APPENDIX A

RELATING THE ZERO-OFFSET AND MIGRATION SLOPES

The chain rule of differentiation leads to the equality

$$p_x = \frac{\partial t}{\partial x} = -p_\tau \frac{\partial \tau}{\partial x}, \tag{A-1}$$

where $p_\tau = -\frac{\partial t}{\partial \tau}$. It is convenient to transform equality (A-1) to the form

$$\frac{\partial \tau}{\partial x} = -\frac{p_x}{p_\tau}. \tag{A-2}$$

Using the expression for p_τ from the main text, we can write equation (A-2) as a quadratic polynomial in p_x^2 as follows

$$a p_x^4 + b p_x^2 + c = 0, \tag{A-3}$$

where

$$a = -2v^2\eta,$$

$$b = (\frac{\partial \tau}{\partial x})^2 v^2 (1+2\eta) + 1,$$

and

$$c = -(\frac{\partial \tau}{\partial x})^2.$$

Since η can be small (as small as zero for isotropic media), we use the following form of solution to the quadratic equation

$$p_x^2 = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

(Press et al., 1992). This form does not go to infinity as η approaches 0. We choose the solution with the negative sign in front of the square root, because this solution complies with the isotropic result when η is equal to zero.

APPENDIX B

LINEARIZED APPROXIMATIONS

Although the exact expressions might be sufficiently constructive for actual residual migration applications, linearized forms are still useful, because they give us valuable insights into the problem. The degree of parameter dependency for different reflector dips is one of the most obvious insights in the anisotropy continuation problem. Perturbation of a small parameter provides a general mechanism to simplify functions by recasting them into power-series expansion over a parameter that has small values. Two variables can satisfy the small perturbation criterion in this problem: The anisotropy parameter η ($\eta << 1$) and the reflection dip τ_x ($\tau_x v << 1$ or $p_x v << 1$).

Setting $\eta = 0$ yields equation (19) for the velocity continuation in elliptical anisotropic media and

$$\frac{\partial \tau}{\partial \eta} = \frac{v^4\,\tau\,\tau_x{}^4\left(-3\,v_v{}^2 + 2\,v^4\,\tau_x{}^2 + v^2\left(4 - v_v{}^2\,\tau_x{}^2\right)\right)}{\left(1 + v^2\,\tau_x{}^2\right)\left(v_v{}^2 + v^4\,\tau_x{}^2\right)}. \tag{B-1}$$

which represents the case when we initially introduce anisotropy into our model.

Because p_x (the zero-offset slope) is typically lower than τ_x (the migrated slope), we perform initial expansions in terms of $y = p_x v$. Applying the Taylor series expansion of equations (12) and (13) in terms of y and dropping all terms beyond the fourth power in y, we obtain

$$\frac{\partial \tau}{\partial v} = \frac{v\,\tau\,p_x{}^2\left(2\,v^2 - v_v{}^2\right)}{v_v{}^2} - \frac{v^3\,\tau\,p_x{}^4\left(2\,v^2 - v_v{}^2\right)\left(v^2 - 2\,(1+6\,\eta)\,v_v{}^2\right)}{v_v{}^4}, \tag{B-2}$$

and

$$\frac{\partial \tau}{\partial \eta} = \frac{v^4\,\tau\,p_x{}^4\left(4\,v^2 - 3\,v_v{}^2\right)}{v_v{}^2}. \tag{B-3}$$

Although both equations are equal to zero for p_x=0, the leading term in the velocity continuation is proportional to p_x^2, whereas the the leading term in the η continuation is proportional to p_x^4. As a result the velocity continuation has greater influence at lower angles than the η continuation. It is also interesting to note that both leading terms are independent of the size of anisotropy (η).

Despite the typically lower values of p_x, expansions in terms of τ_x are more important, but less accurate. For small τ_x, $p_x \approx \tau_x$, and, therefore, the leading-term behavior of τ_x expansions is the same as that of p_x As a result, we arrive at the equation

$$\frac{\partial \tau}{\partial v} = \frac{v\,\tau\,\left(2\,v^2 - v_v{}^2\right)\,\tau_x{}^2}{v_v{}^2} + v^4\left(-\frac{v\,\tau\,\left(2\,v^2 - v_v{}^2\right)}{v_v{}^4} + \frac{\tau\,\left(2\,v^2 - v_v{}^2\right)}{v\,v_v{}^2} + \frac{12\,\eta\,\tau\,\left(2\,v^2 - v_v{}^2\right)}{v\,v_v{}^2}\right)\tau_x{}^4, \tag{B-4}$$

and

$$\frac{\partial \tau}{\partial \eta} = \frac{v^4 \, \tau \, \left(4\,v^2 - 3\,{v_v}^2\right) \, {\tau_x}^4}{{v_v}^2}. \qquad \text{(B-5)}$$

Most of the terms in equations (B-4) and (B-5) are functions of the difference between the vertical and NMO velocities. Therefore, for simplicity and without a loss of generality, we set $v_v = v$ and keep only the terms up to the eighth power in τ_x. The resultant expressions take the form

$$\frac{\partial \tau}{\partial v} = v\,\tau\,{\tau_x}^2 + 12\,v^3\,\eta\,\tau\,{\tau_x}^4 - 4\,v^5\,\eta\,(4 - 25\,\eta)\,\tau\,{\tau_x}^6 + 4\,v^7\,\eta\,\left(5 - 83\,\eta + 144\,\eta^2\right)\,\tau\,{\tau_x}^8 \qquad \text{(B-6)}$$

and

$$\frac{\partial \tau}{\partial \eta} = v^4\,\tau\,{\tau_x}^4 - v^6\,(1 - 20\,\eta)\,\tau\,{\tau_x}^6 + v^8\,\left(1 - 54\,\eta + 156\,\eta^2\right)\,\tau\,{\tau_x}^8. \qquad \text{(B-7)}$$

Curiously enough, the second term of the η continuation heavily depends on the size of anisotropy ($\sim 20\eta$). The first term of equation (B-6) ($\sim \tau_x^2$) is the isotropic term; all other terms in equations (B-6) and (B-7) are induced by the anisotropy.