Homework 9: Numerical Integration (due on April 16)

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1. (a) Suppose that the function
$$f(x) = \frac{2}{1+x^2} \tag{1}$$
is known at three points: $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$. Interpolate the function with a natural cubic spline and approximate the integral
$$\int\limits_{-1}^{1} \frac{2\,dx}{1+x^2} = \pi \tag{2}$$
by the integral of the spline. Is the result more accurate than the result of Simpson's rule?

   (b) Let $Sp[f]$ be the approximation of the integral
$$\int\limits_{a}^{b} f(x)\,dx \tag{3}$$
by the integral of the natural cubic spline defined at the knots $a = x_1 < x_2 < \ldots < x_n = b$. Prove that
$$Sp[f] = CT[f] - \sum_{k=2}^{n-1} c_k \, \frac{h_{k-1}^3 + h_k^3}{12} \,, \tag{4}$$
where $h_k = x_{k+1} - x_k$, c_k are the coefficients in the spline equation
$$S_k(x) = f(x_k) + b_k\,(x - x_k) + c_k\,(x - x_k)^2 + d_k\,(x - x_k)^3 \quad \text{for} \quad x_k \le x < x_{k+1} \,, \tag{5}$$
and $CT[f]$ is the composite trapezoidal rule.

2. (a) Suppose that function (1) is known at three points: $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$ together with its derivatives. Interpolate the function using Hermite interpolation and approximate integral (2) by the integral of the interpolation polynomial. Is the result more accurate than the result of Simpson's rule?

   (b) Let $CH[f]$ be the approximation of integral (3) by the integral of the piece-wise cubic polynomial defined by applying Hermite interpolation at each of the intervals $[x_k, x_{k+1}]$, $a = x_1 < x_2 < \ldots < x_n = b$.
Prove that
$$CH[f] = CT[f] + b_1\,\frac{h_1^2}{12} - b_n\,\frac{h_{n-1}^2}{12} + \sum_{k=2}^{n-1} b_k\,\frac{h_k^2 - h_{k-1}^2}{12} \,, \tag{6}$$
where $h_k = x_{k+1} - x_k$, $b_k = f'(x_k)$, and $CT[f]$ is the composite trapezoidal rule.

3. (a) The *Gauss-Lobatto* quadrature rule has the form

$$\int_{-1}^{1} f(x)\,dx \approx w_1\, f(-1) + w_n\, f(1) + \sum_{k=2}^{n-1} w_k\, f(x_k) = Lo[f]\,, \tag{7}$$

where the abscissas x_k and weights w_k are chosen so that $f(x)$ is integrated exactly if it is a polynomial of order $2n-3$ or less.

Derive the abscissas and weights for $n=3$ and $n=4$ and apply the Gauss-Lobatto rule to integral (2). Are the results more accurate than the result of Simpson's rule?

Hint: Use the symmetry of the interval to constrain the abscissas and weights.

(b) The *Gauss-Laguerre* quadrature rule has the form

$$\int_{0}^{\infty} e^{-x}\, f(x)\,dx \approx \sum_{k=1}^{n} w_k\, f(x_k) = La[f]\,, \tag{8}$$

where the abscissas x_k and weights w_k are chosen so that $f(x)$ is integrated exactly if it is a polynomial of order $2n-1$ or less.

Derive the abscissas and weights for $n=1$ and $n=2$. Test your formulas by approximating π with

$$\pi = \left[2\,\Gamma(3/2)\right]^2 \approx \left(2\,La[\sqrt{x}]\right)^2\,, \tag{9}$$

where

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1}\, e^{-t}\,dt \tag{10}$$

4. (Programming) Implement the adaptive quadrature method. Test your program by computing integral (2) with the precision of six significant decimal digits. Plot or tabulate the values of x and $f(x)$ that were involved in the computation.

The algorithm of adaptive integration can be defined either recursively

ADAPTIVE RECURSIVE($f(x), a, b, h, I$)

if $h < xtol$
  then
    WARNING('did not converge')
    return I
$c \leftarrow a + h/2$
$I_1 \leftarrow R(f, a, c)$
$I_2 \leftarrow R(f, c, b)$
$D \leftarrow I_2 + I_1 - I$
if $|D| < itol$
  then $J \leftarrow I + \alpha\, D$
  else $J \leftarrow$ ADAPTIVE RECURSIVE($f, a, c, h/2, I_1$) + ADAPTIVE RECURSIVE($f, c, b, h/2, I_2$)
return J

or sequentially

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ADAPTIVE NONRECURSIVE(f(x), a, b, h, I)
J ← 0
PUSH(f, a, b, h, I)
while POP(f, a, b, h, I)
do
    if h < xtol
      then
            WARNING( 'did not converge' )
            return I
    c ← a + h/2
    I_1 ← R(f, a, c)
    I_2 ← R(f, c, b)
    D ← I_2 + I_1 − I
    if |D| < itol
      then J ← J + I + α D
      else
            PUSH(f, a, c, h/2, I_1)
            PUSH(f, c, b, h/2, I_2)
return J
```

Both algorithms involve the rule $R(f,a,b)$ for computing integral (3), the minimum allowed interval $xtol$ and the requested precision $itol$. They are initialized with $I = R(f,a,b)$ and $h = b - a$. The sequential algorithm operates a queue of intervals using a pair of functions PUSH and POP.

Run your program taking R to be the trapezoidal rule. What is the appropriate value of constant α?

5. (Programming) The length of a parametric curve $\{x(t), y(t)\}$ is given by the integral

$$\int_a^b \sqrt{[x'(t)]^2 + \left[y'(t)\right]^2}\, dt \tag{11}$$

The curve that you interpolated in Homework 6 is close to the *hypotrochoid*

$$x(t) = \frac{3}{2}\cos t + \cos 3t\,; \tag{12}$$

$$y(t) = \frac{3}{2}\sin t - \sin 3t\,, \tag{13}$$

defined on the interval $0 \le t \le 2\pi$.

Estimate the length of this curve to six significant decimal digits applying a numerical method of your choice.

Identify the method and plot the $\{x, y\}$ points involved in the computation.